Kawamoto, F. and Tomita, K.
Osaka J. Math.
46 (2009), 949–993

CONTINUED FRACTIONS WITH EVEN PERIOD AND AN INFINITE FAMILY OF REAL QUADRATIC FIELDS OF MINIMAL TYPE

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(Received April 3, 2008, revised July 25, 2008)

Abstract

In a previous paper [4], we introduced the notion of real quadratic fields with period l *of minimal type* in terms of continued fractions. As a consequence, we have to examine a construction of real quadratic fields with period ≥ 5 of minimal type in order to find many real quadratic fields of class number 1. When $l \geq 4$, it appears that there exist infinitely many real quadratic fields with period l of minimal type. Indeed, we provided an infinitude of real quadratic fields with period 4 of minimal type in [4]. In this paper, we construct an infinite family of real quadratic fields with large even period of minimal type whose class number is greater than any given positive integer, and whose Yokoi invariant is greater than any given positive integer.

1. Introduction

In [4] we defined real quadratic fields with period l *of minimal type* in terms of continued fractions (see Definition 2.1 for the precise definition), and studied Yokoi invariants introduced by Yokoi [12] (see Definition 3.1 of Section 3.4 for the precise definition) and class numbers (in the wide sense) of real quadratic fields with period $\leq$ 4. Also, as explained there, we have to examine a construction of real quadratic fields with period ≥ 5 of minimal type in order to find many real quadratic fields of class number 1. When $l \geq 4$, it appears that there exist infinitely many real quadratic fields with period l of minimal type. Indeed, we provided an infinitude of real quadratic fields with period 4 of minimal type in [4]. In this paper, we shall show the existence of an infinite family of real quadratic fields with large even period of minimal type:

Theorem 1.1. *Let l be an even integer greater than or equal to* 4 *which is not divisible by* 8, *and h and m any positive integers. Then, there exist infinitely many real quadratic fields with period l of minimal type whose class number is greater than h, and whose Yokoi invariant is greater than m.*

2000 Mathematics Subject Classification. Primary 11R29; Secondary 11A55, 11R11, 11R27.

Many numerical examples show that the Yokoi invariants of real quadratic fields of class number 1 are relatively large. Theorem 1.1 suggests that, in order to find such fields, it is necessary to study more precisely real quadratic fields of minimal type whose Yokoi invariant is relatively large. Mollin [6] and McLaughlin [5] independently constructed non-square positive integers d' such that the simple continued fraction expansion of $\sqrt{d'}$ has the symmetric part of some type (see $(*)$). In the proof of Theorem 1.1 we utilize a generalized form of such a symmetric part. Our family of real quadratic fields thus obtained explicitly has three or four parameters of nonnegative integers. If the period of such fields is fixed then we see that the values of Yokoi invariants are bounded, and then by using a theorem of Siegel concerning the approximate behavior of the product of class number and regulator, we see by the same argument in [4] that the class numbers are relatively large. We use a theorem of Nagell to show that our family contains infinite ones.

This paper is organized as follows. In Section 2 we state basic properties of continued fractions. In particular, a theorem of Friesen and Halter-Koch (Theorem 2.4) is our basic tool. Let d and $\omega = \sqrt{d}$ (or $(1+\sqrt{d})/2$) be, respectively, a non-square positive integer and a quadratic irrational > 1 constructed in Theorem 2.4. We start with assuming that the (minimal) period l of the continued fraction expansion $\omega = [a_0, \overline{a_1, \dots, a_{l-1}, a_l}]$ is even: $l = 2L$. In [4] we give quadratic irrationals ω with period 2, 4 (and real quadratic fields $\mathbb{Q}(\sqrt{d})$ with period 4 of minimal type whose Yokoi invariant is relatively large) by using Theorem 2.4 (see Section 4). We begin with such quadratic irrationals ω and, following an idea of Mollin [6], consider the following new symmetric string of positive integers. If we put

$$\overrightarrow{\mathbf{v}} := a_1, \dots, a_{L-1}, \quad \overleftarrow{\mathbf{v}} := a_{L-1}, \dots, a_1,$$

then the symmetric part $a_1, \dots, a_{l-1}$ of the continued fraction expansion of ω can be written as

$$\overrightarrow{\mathbf{v}},\ a_L,\ \overleftarrow{\mathbf{v}}.$$

For any integer $e \geq 0$, $\overrightarrow{\mathbf{w}}_e$ denotes e iterations of the periodic part $a_1, \dots, a_l$, and we denote by $\overleftarrow{\mathbf{w}}_e$ the reverse of $\overrightarrow{\mathbf{w}}_e$, which is e iterations of a string of l positive integers $a_l, \dots, a_1$. Also, we let b be any positive integer and consider a symmetric string of $(2e+1)l - 1$ positive integers

$$(*) \qquad \overrightarrow{\mathbf{w}}_e,\ \overrightarrow{\mathbf{v}},\ b + a_L,\ \overleftarrow{\mathbf{v}},\ \overleftarrow{\mathbf{w}}_e.$$

In Section 3.1 we investigate basic properties of such symmetric strings, which is induced by "symmetric properties of recurrence equations" (Lemma 2.1). In Section 3.2, by using them, we choose a suitable positive integer b depending on the integer e (Lemma 3.4), and give special quadratic irrationals $\omega' = \sqrt{d'}$ or $(1+\sqrt{d'})/2$ by using such symmetric strings of positive integers and Theorem 2.4 (Theorem 3.6 of Section 3.3). This is a generalization of results of Mollin and of McLaughlin (Proposition 3.7). Furthermore, we give a necessary and sufficient condition for the positive integer d' with period $(2e+1)l$ to be of minimal type (Proposition 3.3, Remark 3.2). In particular, we see that d' always becomes of minimal type when e is sufficiently large (Remark 3.1). In Section 3.4, we extend the Yokoi invariant of real quadratic field to that of non-square positive integer $d' \not\equiv 0 \bmod 4$ (Definition 3.1), and give an estimate for its value (Proposition 3.10, Remark 3.3). In Section 4, we construct real quadratic fields $\mathbb{Q}(\sqrt{d'})$ with even period of minimal type whose Yokoi invariant is relatively large, and prove Theorem 1.1 by investigating the class numbers. In Section 5, some numerical examples are calculated by using PARI-GP [1].

For a real number x, $[x]$ denotes the largest integer $\leq x$. We denote by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ the set of positive integers, the ring of rational integers and the field of rational numbers, respectively.

2. Preparations on continued fractions

In this section we collect basic properties of continued fractions, and refer the reader to excellent books of Ono [9] and Rosen [10] for them. We first state Lemma 2.1 which is of central importance in the present paper, and may call it "symmetric properties of recurrence equations".

2.1. Symmetric properties of recurrence equations. If a_0 is any positive integer and $\{a_n\}_{n\geq 1}$ is a sequence of positive integers, then we define nonnegative integers p_n, q_n, r_n by using the recurrence equation:

$$\begin{cases} p_0 = 1,\ p_1 = a_0,\ p_n = a_{n-1}p_{n-1} + p_{n-2}, \\ q_0 = 0,\ q_1 = 1,\ q_n = a_{n-1}q_{n-1} + q_{n-2}, \qquad n \geq 2. \\ r_0 = 1,\ r_1 = 0,\ r_n = a_{n-1}r_{n-1} + r_{n-2}, \end{cases} \tag{2.1}$$

Let λ be a variable. Then the following are known:

$$[a_0, \dots, a_n, \lambda] = \frac{\lambda p_{n+1} + p_n}{\lambda q_{n+1} + q_n}, \quad [a_0, \dots, a_n] = \frac{p_{n+1}}{q_{n+1}}, \quad n \geq 0, \tag{2.2}$$

$$q_n r_{n-1} - q_{n-1} r_n = (-1)^{n-1}, \quad n \geq 1, \tag{2.3}$$

$$p_n = a_0 q_n + r_n, \quad n \geq 0. \tag{2.4}$$

(Recurrence equations and partial quotients of a continued fraction are both numbered beginning with 0.)

We let $a_0, a_1, \dots, a_l$ be any $l+1$ positive integers, and assume that $l-1$ positive integers $a_1, \dots, a_{l-1}$ satisfy the symmetric property: $a_n = a_{l-n}$, $1 \leq n \leq l-1$ if $l \geq 2$. Then we define a sequence $\{a_n\}_{n\geq 1}$ of positive integers: for each integer $n \geq 1$, we put $a_n := a_r$ if $r > 0$, and otherwise $a_n := a_l$ where r is the remainder of the division of

n by l. Thus, we construct periodically $\{a_n\}_{n\geq 1}$ from $a_1, \ldots, a_l$ in what follows and throughout this paper. We shall see that the symmetric string of $l-1$ positive integers $a_1, \ldots, a_{l-1}$ induces symmetric properties of the recurrence equation (2.1). Let $M_0 := E$ be the unit matrix of degree 2, and put for each integer $n \geq 1$,

$$M_n := \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}.$$

We easily see that

$$M_n = \begin{pmatrix} q_{n+1} & q_n \\ r_{n+1} & r_n \end{pmatrix}, \quad n \geq 0 \tag{2.5}$$

by induction. Let k be a positive integer. Since $a_1, \ldots, a_{l-1}$ have the symmetric property, M_{l-1} is a symmetric matrix. Furthermore, M_{kl-1} is also a symmetric matrix by the definition of the sequence $\{a_n\}_{n\geq 1}$. As $a_1, \ldots, a_{kl-1}$ also have the symmetric property, we have for $n \neq 0, kl-1$,

$$\begin{aligned} M_{kl-1} &= \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{kl-1} & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} a_{kl-n-1} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} = M_n \times {}^t M_{kl-n-1}. \end{aligned}$$

Here, ${}^tM$ denotes the transpose of a matrix M. Since M_{kl-1} is a symmetric matrix and $M_0 = E$, this equation also holds for $n = 0, kl-1$:

$$M_{kl-1} = M_n {}^t M_{kl-n-1}, \quad 0 \leq n \leq kl-1. \tag{2.6}$$

Lemma 2.1. *Let k be a positive integer and $0 \leq n \leq kl-1$. Under the above setting, the following hold.*

$$q_{kl-1} = r_{kl}, \tag{2.7}$$

$$q_{kl-1}^2 - (-1)^{kl} = q_{kl} r_{kl-1}, \tag{2.8}$$

$$q_{kl} = q_{n+1} q_{kl-n} + q_n q_{kl-n-1}, \tag{2.9}$$

$$r_{kl} = q_{kl-n} r_{n+1} + q_{kl-n-1} r_n, \tag{2.10}$$

$$r_{kl-1} = r_{n+1} r_{kl-n} + r_n r_{kl-n-1}, \tag{2.11}$$

$$p_{kl} = p_{n+1} q_{kl-n} + p_n q_{kl-n-1}. \tag{2.12}$$

Proof. In [4, Lemma 2.1], we have shown that (2.7) and (2.8) hold. By (2.5) and (2.6), we have

$$\begin{aligned}\begin{pmatrix} q_{kl} & q_{kl-1} \\ r_{kl} & r_{kl-1} \end{pmatrix} &= \begin{pmatrix} q_{n+1} & q_n \\ r_{n+1} & r_n \end{pmatrix}\begin{pmatrix} q_{kl-n} & r_{kl-n} \\ q_{kl-n-1} & r_{kl-n-1} \end{pmatrix} \\ &= \begin{pmatrix} q_{n+1}q_{kl-n} + q_nq_{kl-n-1} & q_{n+1}r_{kl-n} + q_nr_{kl-n-1} \\ q_{kl-n}r_{n+1} + q_{kl-n-1}r_n & r_{n+1}r_{kl-n} + r_nr_{kl-n-1} \end{pmatrix}.\end{aligned}$$

Comparing with corresponded components of both sides of it yields that (2.9), (2.10) and (2.11) hold, and (2.12) follows from (2.4), (2.9) and (2.10). □

2.2. A theorem of Friesen and Halter-Koch. To describe Theorem 2.4, we consider three cases separately for a given symmetric string of $l - 1$ positive integers, and explain what case arises from it. From now on, we let $a_1, \dots, a_L$ be any string of L (≥ 1) positive integers.

(A). The even period case. First, let $l := 2L$. By placing a_L at the center and folding back $a_1, \dots, a_{L-1}$ as

$$a_{L+1} := a_{L-1}, a_{L+2} := a_{L-2}, \dots, a_{2L-1} := a_1,$$

we construct a string of $l - 1$ positive integers $a_1, \dots, a_{l-1}$. The string satisfies the symmetric property. By using the recurrence equation (2.1), we define nonnegative integers $q_0, \dots, q_l, r_0, \dots, r_{l-1}$. For brevity, we put $A := q_l$, $B := q_{l-1}$, $C := r_{l-1}$, and consider three cases separately:

(I) $A \equiv 1 \bmod 2$,

(II) $(A, C) \equiv (0, 0) \bmod 2$,

(III) $(A, C) \equiv (0, 1) \bmod 2$.

Lemma 2.2. *Under the above setting, the following hold.*

(i) $A = (q_{L+1} + q_{L-1})q_L$, $B = (q_{L+1} + q_{L-1})r_L - (-1)^L$, $C = (r_{L+1} + r_{L-1})r_L$.

(ii) *If a_L is even then* Case (II) *occurs.*

(iii) *If $(a_L, q_L) \equiv (1, 1) \bmod 2$ then* Case (I) *occurs, and if $(a_L, q_L) \equiv (1, 0) \bmod 2$ then* Case (III) *occurs.*

Proof. It follows from $(2.9)_{k=1,n=L}$ that

$$A = (q_{L+1} + q_{L-1})q_L = (a_Lq_L + 2q_{L-1})q_L \equiv a_Lq_L \mod 2,$$

and $(2.11)_{k=1,n=L}$ yields that

$$C = (r_{L+1} + r_{L-1})r_L = (a_Lr_L + 2r_{L-1})r_L \equiv a_Lr_L \mod 2.$$

Consequently, if a_L is even then Case (II) occurs for $a_1, \ldots, a_{l-1}$. On the other hand, if a_L is odd then $(A, C) \equiv (q_L, r_L) \bmod 2$. Hence, when q_L is odd, Case (I) occurs. When q_L is even, $(2.3)_{n=L}$ implies that r_L is odd, therefore, Case (III) occurs. By $(2.7)_{k=1}$, $(2.10)_{k=1,n=L}$, and $(2.3)_{n=L+1}$, we have

$$B = q_L r_{L+1} + q_{L-1} r_L = (q_{L+1} r_L - (-1)^L) + q_{L-1} r_L = (q_{L+1} + q_{L-1}) r_L - (-1)^L.$$

This proves our lemma. □

The above lemma shall be used in the proof of Lemma 3.5.

(B). The odd period case. Next, let $l := 2L + 1$. By folding back $a_1, \ldots, a_L$ as

$$a_{L+1} := a_L,\ a_{L+2} := a_{L-1}, \ldots, a_{2L} := a_1,$$

we construct a symmetric string of $l - 1$ positive integers $a_1, \ldots, a_{l-1}$, and consider the above three cases separately.

Lemma 2.3. *Under the above setting, the following hold.*
(i) $A = q_{L+1}^2 + q_L^2$, $B = q_{L+1} r_{L+1} + q_L r_L$, $C = r_{L+1}^2 + r_L^2$.
(ii) *If* $(a_L, q_L + q_{L-1}) \equiv (0, 1) \bmod 2$ *then* Case (I) *occurs, and if* $(a_L, q_L + q_{L-1}) \equiv (0, 0) \bmod 2$ *then* Case (III) *occurs.*
(iii) *If* $(a_L, q_{L-1}) \equiv (1, 1) \bmod 2$ *then* Case (I) *occurs, and if* $(a_L, q_{L-1}) \equiv (1, 0) \bmod 2$ *then* Case (III) *occurs.*

Proof. It follows from $(2.9)_{k=1,n=L}$ that

$$A = q_{L+1}^2 + q_L^2 = (a_L q_L + q_{L-1})^2 + q_L^2 \equiv (a_L + 1) q_L + q_{L-1} \mod 2,$$

and $(2.11)_{k=1,n=L}$ yields that

$$C = r_{L+1}^2 + r_L^2 = (a_L r_L + r_{L-1})^2 + r_L^2 \equiv (a_L + 1) r_L + r_{L-1} \mod 2.$$

Consequently, if a_L is even then $(A, C) \equiv (q_L + q_{L-1}, r_L + r_{L-1}) \bmod 2$. Hence, when $q_L + q_{L-1}$ is odd, Case (I) occurs for $a_1, \ldots, a_{l-1}$. When $q_L + q_{L-1}$ is even, we have $q_L \equiv q_{L-1} \bmod 2$. Since $q_L r_{L-1} + q_{L-1} r_L \equiv 1 \bmod 2$ by $(2.3)_{n=L}$, we see that $q_L(r_{L-1} + r_L) \equiv 1 \bmod 2$. As $r_{L-1} + r_L$ is odd, Case (III) occurs. On the other hand, if a_L is odd then $(A, C) \equiv (q_{L-1}, r_{L-1}) \bmod 2$. Hence, when q_{L-1} is odd, Case (I) occurs. When q_{L-1} is even, (2.3) implies that r_{L-1} is odd, therefore, Case (III) occurs. By $(2.7)_{k=1}$, and $(2.10)_{k=1,n=L}$, we have $B = q_{L+1} r_{L+1} + q_L r_L$, and our lemma is proved. □

REMARK 2.1. If l and a_L are both even, then Lemma 2.2 (ii) implies that Case (II) occurs for $a_1, \ldots, a_{l-1}$. Also, if "l is even and a_L is odd", or l is odd, then Lemmas 2.2 and 2.3 imply that Case (I) or Case (III) occurs.

We define polynomials $g(x)$, $h(x)$ of degree 1 and a quadratic polynomial $f(x)$ in $\mathbb{Z}[x]$ by putting

$$g(x) := Ax - (-1)^l BC, \quad h(x) := Bx - (-1)^l C^2,$$
$$f(x) := g(x)^2 + 4h(x) = A^2x^2 + 2(2B - (-1)^l ABC)x + (B^2 - (-1)^l 4)C^2.$$

Furthermore, we let s_0 be the least integer s for which $g(s) > 0$, that is, $s > (-1)^l BC/A$. The quadratic function $f(x)$ becomes strictly, monotonously increasing in the interval $[s_0, \infty)$. Under the above setting, Theorem 2.4 is shown in Friesen [2, Theorem] and Halter-Koch [3, Theorem 1A and Corollary 1A], which is improved in [4, Theorem 3.1] and is our basic tool.

Theorem 2.4 (Friesen, Halter-Koch). *Let l be a fixed positive integer ≥ 2 and $a_1, \ldots, a_{l-1}$ any symmetric string of $l-1$ positive integers.*
(i) *When* Case (I) *or* Case (II) *occurs, we let s be any integer with $s \geq s_0$, and put $d := f(s)/4$ and $a_0 := g(s)/2$. Here, we choose an even integer s in* Case (I), *and assume that*

$$g(s) > a_1, \ldots, a_{l-1}. \tag{2.13}$$

Then, d and a_0 are positive integers, d is non-square,

$$a_0 = [\sqrt{d}], \quad \textit{and} \quad \omega := \sqrt{d} = [a_0, \overline{a_1, \ldots, a_{l-1}, 2a_0}] \tag{2.14}$$

is the continued fraction expansion with period l of $\sqrt{d}$. Also, in Case (III), *there is no positive integer d such that* (2.14) *is the continued fraction expansion of $\sqrt{d}$.*
(ii) *When* Case (I) *or* Case (III) *occurs, we let s be any integer with $s \geq s_0$, and put $d := f(s)$ and $a_0 := (g(s)+1)/2$. Here, we choose an odd integer s in* Case (I), *and assume that* (2.13) *holds. Then, d and a_0 are positive integers, d is non-square,* $d \equiv 1 \bmod 4$,

$$a_0 = [(1+\sqrt{d})/2], \quad \textit{and} \quad \omega := (1+\sqrt{d})/2 = [a_0, \overline{a_1, \ldots, a_{l-1}, 2a_0 - 1}] \tag{2.15}$$

is the continued fraction expansion with period l of $(1+\sqrt{d})/2$. Also, in Case (II), *there is no positive integer d such that $d \equiv 1 \bmod 4$ and* (2.15) *is the continued fraction expansion of $(1+\sqrt{d})/2$.*
Conversely, we let d be any non-square positive integer. By using a quadratic polynomial $f(x)$ obtained as above from the symmetric part of the continued fraction expansion of $\sqrt{d}$, d becomes uniquely of the form $d = f(s)/4$ with some integer $s \geq s_0$, and (2.13) *holds. If $d \equiv 1 \bmod 4$ in addition then the same thing is true for $(1+\sqrt{d})/2$.*

DEFINITION 2.1. As we have seen in the above, the symmetric part $a_1, \ldots, a_{l-1}$ can be obtained from a string of L positive integers $a_1, \ldots, a_L$. We call such a string *the primary symmetric part*.

Let d be any non-square positive integer. We see by Theorem 2.4 that d is uniquely of the form $d = f(s)/4$ with some integer $s \geq s_0$. Here, the quadratic polynomial $f(x)$ and the integer s_0 are obtained as above from the symmetric part of the continued fraction expansion with period l of $\sqrt{d}$. If $s = s_0$ then we say that d is *a positive integer with period l of minimal type for $\sqrt{d}$*. When $d \equiv 1 \bmod 4$ in addition, we see that d is uniquely of the form $d = f(s)$ with some integer $s \geq s_0$. Here, the quadratic polynomial $f(x)$ and the integer s_0 are obtained as above from the symmetric part of the continued fraction expansion with period l of $(1+\sqrt{d})/2$. If $s = s_0$ then we say that d is *a positive integer with period l of minimal type for* $(1+\sqrt{d})/2$.

Let $\mathbb{Q}(\sqrt{d})$ be a real quadratic field. Here, d is a square-free positive integer. We say that $\mathbb{Q}(\sqrt{d})$ is *a real quadratic field with period l of minimal type*, if d is a positive integer with period l of minimal type for $\sqrt{d}$ when $d \equiv 2, 3 \bmod 4$, and if d is a positive integer with period l of minimal type for $(1+\sqrt{d})/2$ when $d \equiv 1 \bmod 4$.

We mention an important supplement to Theorem 2.4.

REMARK 2.2. Under the setting of Theorem 2.4 (i) or (ii), if $s > s_0$ then the condition (2.13) holds.

Proof. Let $1 \leq n \leq L$. As $A > 0$, the linear function $g(x)$ is strictly, monotonously increasing. By the definition of s_0, we have $g(s_0) > 0$. Therefore it follows from $s > s_0$ that $g(s) \geq g(s_0+1) = g(s_0) + A > A = q_l$. On the other hand, as $l \geq L+1$, we see that $q_l \geq q_{n+1} \geq a_n q_n \geq a_n$. Hence, $g(s) > q_l \geq a_n$. Thus, our assertion is proved. □

From now on, we let d be a non-square positive integer constructed in Theorem 2.4 (i), or (ii). If we put $a_l := g(s)$ then it holds that $a_l = 2a_0$ in (i), and that $a_l = 2a_0 - 1$ in (ii). For brevity, we write $\omega = (P_0 + \sqrt{d})/Q_0$. Here, $P_0 := 0$, $Q_0 := 1$ in (i) and $P_0 := 1$, $Q_0 := 2$ in (ii). For all integers $n \geq 0$, we put

$$G_n := Q_0 p_n - P_0 q_n.$$

For each integer $n \geq 0$, we determine a quadratic irrational ω_{n+1} such that

$$\omega_0 := \omega, \quad \omega_n = a_n + \frac{1}{\omega_{n+1}}, \quad a_n = [\omega_n].$$

(Note that the sequence $\{a_n\}_{n\geq 1}$ of positive integers is defined periodically.) Then we can write uniquely $\omega_n = (P_n + \sqrt{d})/Q_n$ with some positive integers P_n and Q_n, and

Q_n/Q_0 becomes a positive integer. (Cf. Section 2 and the proof of Lemma 2.2 in [4].) Also, the following are known for any integer $n \geq 0$:

$$P_{n+1} = a_n Q_n - P_n, \tag{2.16}$$

$$Q_{n+1} = Q_{n-1} + a_n(P_n - P_{n+1}), \tag{2.17}$$

$$Q_n Q_{n+1} = d - P_{n+1}^2, \tag{2.18}$$

where we put $Q_{-1} := (d - P_0^2)/Q_0$. (Also, $0 < P_{n+1} < \sqrt{d}$, $0 < Q_{n+1} < 2\sqrt{d}$.) We describe properties of recurrence equations in Lemmas 2.5 and 2.6 which are widely used in Section 3.

Lemma 2.5. *Let k be an integer. Under the above setting, the following hold.*

$$G_n = P_n q_n + Q_n q_{n-1}, \quad n \geq 1, \tag{2.19}$$

$$a_l q_n = 2((G_n/Q_0) - r_n), \quad n \geq 0, \tag{2.20}$$

$$q_{kl} r_{l+1} = r_{kl+1} q_l, \quad k \geq 0, \tag{2.21}$$

$$h(s) q_{kl} - g(s) q_{kl-1} = r_{kl-1}, \quad k \geq 1, \tag{2.22}$$

$$h(s) q_{kl-1} - g(s) r_{kl-1} = ((-1)^{kl} a_l + q_{kl-1} r_{kl-1})/q_{kl}, \quad k \geq 1. \tag{2.23}$$

Proof. By putting $\lambda = \omega_n$ in $(2.2)_{n-1}$, we see that

$$\omega = [a_0, \ldots, a_{n-1}, \omega_n] = \frac{\omega_n p_n + p_{n-1}}{\omega_n q_n + q_{n-1}} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \omega_n.$$

Since the inverse of the matrix in the right hand side of it is equal to $(-1)^n \begin{pmatrix} q_{n-1} & -p_{n-1} \\ -q_n & p_n \end{pmatrix}$, we have

$$\omega_n = (-1)^n \begin{pmatrix} q_{n-1} & -p_{n-1} \\ -q_n & p_n \end{pmatrix} \omega = \frac{p_{n-1} - q_{n-1}\omega}{q_n \omega - p_n}, \quad n \geq 1, \tag{2.24}$$

so that $\omega_n(q_n\omega - p_n) = p_{n-1} - q_{n-1}\omega$. Therefore,

$$(P_n + \sqrt{d})(-G_n + q_n\sqrt{d}) = Q_n G_{n-1} - Q_n q_{n-1}\sqrt{d},$$

so that

$$(-P_n G_n + d q_n) + (-G_n + P_n q_n)\sqrt{d} = Q_n G_{n-1} - Q_n q_{n-1}\sqrt{d}.$$

Comparing with coefficients of $\sqrt{d}$ in both sides of it yields that (2.19). First, let $\omega = \sqrt{d}$ to show (2.20). It follows from (2.4), $G_n = p_n$ and $Q_0 = 1$ that

$$a_l q_n = 2a_0 q_n = 2(p_n - r_n) = 2((G_n/Q_0) - r_n).$$

Next, let $\omega = (1+\sqrt{d})/2$. As $G_n = 2p_n - q_n$ and $Q_0 = 2$, we have

$$a_l q_n = (2a_0 - 1)q_n = G_n - 2r_n = 2((G_n/Q_0) - r_n).$$

Thus, (2.20) holds.

The equation (2.21) holds for $k = 0, 1$. We show it by induction in k (and Lemma 2.1), and assume that (2.21) holds for $k \geq 1$. First, we see that

$$\begin{aligned} q_{(k+1)l} r_{l+1} &= (q_{l+1} q_{kl} + q_l q_{kl-1}) r_{l+1} \quad (\text{by } (2.9)_{n=l \text{ for } (k+1)l}) \\ &= (q_{kl} r_{l+1}) q_{l+1} + q_{kl-1} r_{l+1} q_l = (r_{kl+1} q_l) q_{l+1} + r_{kl} r_{l+1} q_l \\ &\qquad\qquad (\text{by the hypothesis of induction and } (2.7)) \\ &= (r_{kl+1} q_{l+1} + r_{kl} r_{l+1}) q_l . \end{aligned} \tag{2.25}$$

Next, we calculate $r_{(k+1)l+1}$ and note that $a_{(k+1)l} = a_l$ by the definition of $\{a_n\}_{n\geq 1}$. Since

$$\begin{cases} r_{(k+1)l} = q_{kl} r_{l+1} + q_{kl-1} r_l , \\ r_{(k+1)l-1} = r_{kl} r_{l+1} + r_{kl-1} r_l \end{cases}$$

by $(2.10)_{n=l}$ and $(2.11)_{n=l}$ for $(k+1)l$, we have

$$\begin{aligned} r_{(k+1)l+1} &= a_l r_{(k+1)l} + r_{(k+1)l-1} = (a_l q_{kl} + r_{kl}) r_{l+1} + (a_l q_{kl-1} + r_{kl-1}) r_l \\ &= a_l (r_{kl+1} q_l) + r_{kl} r_{l+1} + (a_l q_{kl-1} + r_{kl-1}) r_l , \end{aligned}$$

where we use the hypothesis of induction. As $a_l q_l = q_{l+1} - q_{l-1}$, we obtain

$$r_{(k+1)l+1} = r_{kl+1} q_{l+1} + r_{kl} r_{l+1} + (a_l q_{kl-1} + r_{kl-1}) r_l - r_{kl+1} q_{l-1}.$$

Here, since $q_{l-1} = r_l$ by $(2.7)_{k=1}$, we have

$$\begin{aligned} (a_l q_{kl-1} + r_{kl-1}) r_l - r_{kl+1} q_{l-1} &= a_l q_{kl-1} r_l + r_{kl-1} r_l - (a_l r_{kl} + r_{kl-1}) r_l \\ &= a_l (q_{kl-1} - r_{kl}) r_l = 0 \quad (\text{by } (2.7)). \end{aligned}$$

Hence, $r_{(k+1)l+1} = r_{kl+1} q_{l+1} + r_{kl} r_{l+1}$. So, (2.25) implies that $q_{(k+1)l} r_{l+1} = r_{(k+1)l+1} q_l$.

Since we see in the proof of Theorem 2.4 (see [4, (3.10)]) that (2.22) holds for $k = 1$, we may assume that $k \geq 2$. By $(2.9)_{n=l}$ and $(2.10)_{n=l}$, we have

$$\begin{cases} q_{kl} = q_{(k-1)l} q_{l+1} + q_{(k-1)l-1} q_l , \\ r_{kl} = q_{(k-1)l} r_{l+1} + q_{(k-1)l-1} r_l , \end{cases}$$

and note that $r_{kl} = q_{kl-1}$, and $r_l = q_{l-1}$. Then, $(2.22)_{k=1}$ yields that

$$h(s)q_{kl} - g(s)q_{kl-1} = q_{(k-1)l}(h(s)q_{l+1} - g(s)r_{l+1}) + q_{(k-1)l-1}r_{l-1}, \tag{2.26}$$

and also,

$$h(s)q_{l+1} - g(s)r_{l+1} = a_l(h(s)q_l - g(s)r_l) + h(s)q_{l-1} - g(s)r_{l-1}. \tag{2.27}$$

Furthermore we have by $(2.22)_{k=1}$

$$\begin{aligned} h(s)q_{l-1} - g(s)r_{l-1} &= h(s)q_l\frac{q_{l-1}}{q_l} - g(s)r_{l-1} = (g(s)r_l + r_{l-1})\frac{q_{l-1}}{q_l} - g(s)r_{l-1} \\ &= \frac{1}{q_l}\{g(s)(q_{l-1}r_l - q_lr_{l-1}) + q_{l-1}r_{l-1}\}. \end{aligned}$$

As $q_{l-1}r_l - q_lr_{l-1} = -(-1)^{l-1}$ by $(2.3)_{n=l}$ and $a_l = g(s)$, we see $(2.23)_{k=1}$, and it follows from (2.27), $(2.22)_{k=1}$, and this that

$$h(s)q_{l+1} - g(s)r_{l+1} = \frac{1}{q_l}\{a_lq_lr_{l-1} + (-1)^la_l + q_{l-1}r_{l-1}\}.$$

By $(2.8)_{k=1}$, $q_lr_{l-1} = q_{l-1}^2 - (-1)^l$. Also, $q_{l-1} = r_l$. Therefore,

$$h(s)q_{l+1} - g(s)r_{l+1} = \frac{1}{q_l}\{a_lq_{l-1}^2 + q_{l-1}r_{l-1}\} = \frac{1}{q_l}(a_lr_l + r_{l-1})r_l = \frac{1}{q_l}r_{l+1}r_l.$$

As $q_{(k-1)l-1} = r_{(k-1)l}$ by (2.7), hence, we see by (2.26) that

$$\begin{aligned} h(s)q_{kl} - g(s)q_{kl-1} &= \frac{1}{q_l}q_{(k-1)l}r_{l+1}r_l + r_{(k-1)l}r_{l-1} \\ &= r_{(k-1)l+1}r_l + r_{(k-1)l}r_{l-1} \quad (\text{by } (2.21)_{k-1}) \\ &= r_{kl-1} \quad (\text{by } (2.11)_{n=(k-1)l}). \end{aligned}$$

Thus, we obtain (2.22).

We use the same argument in the above proof of $(2.23)_{k=1}$ to show (2.23). By (2.22), we have

$$\begin{aligned} h(s)q_{kl-1} - g(s)r_{kl-1} &= h(s)q_{kl}\frac{q_{kl-1}}{q_{kl}} - g(s)r_{kl-1} \\ &= (g(s)r_{kl} + r_{kl-1})\frac{q_{kl-1}}{q_{kl}} - g(s)r_{kl-1} \\ &= \frac{1}{q_{kl}}\{g(s)(q_{kl-1}r_{kl} - q_{kl}r_{kl-1}) + q_{kl-1}r_{kl-1}\}. \end{aligned}$$

As $q_{kl-1}r_{kl} - q_{kl}r_{kl-1} = -(-1)^{kl-1}$ by $(2.3)_{n=kl}$ and $a_l = g(s)$, we obtain (2.23). This proves our lemma. □

We shall use the following lemma in the proof of Lemma 3.5.

Lemma 2.6. *Under the above setting, let k be a positive integer.*
(i) *When $\omega = \sqrt{d}$, the following hold.*

$$q_{kl} \equiv q_l \quad (\textit{resp.}, \ \equiv 0) \quad \bmod 2, \quad \textit{if} \quad 2 \nmid k \quad (\textit{resp.}\ 2 \mid k), \tag{2.28}$$

$$q_{kl-1} \equiv q_{l-1} \quad (\textit{resp.}, \ \equiv 1) \quad \bmod 2, \quad \textit{if} \quad 2 \nmid k \quad (\textit{resp.}\ 2 \mid k). \tag{2.29}$$

(ii) *When $\omega = (1+\sqrt{d})/2$, the following hold.*

$$q_{kl} \equiv q_l \quad (\textit{resp.}, \ \equiv 0) \quad \bmod 2, \quad \textit{if} \quad 3 \nmid k \quad (\textit{resp.}\ 3 \mid k), \tag{2.30}$$

$$\begin{aligned} q_{kl-1} \equiv q_{l-1} \quad (\textit{resp.}, \ \equiv q_l q_{l-1} + 1, \ 1) \quad \bmod 2, \\ \textit{if} \quad k \equiv 1 \quad (\textit{resp.}, \ \equiv 2, \ 0) \quad \bmod 3. \end{aligned} \tag{2.31}$$

Proof. By (2.7) of Lemma 2.1, we have $q_{kl-1} = r_{kl}$, and $q_{l-1} = r_l$. First, we show (2.30) and (2.31) simultaneously by induction in k. They trivially hold for $k = 1$. Since a_l is odd when $\omega = (1+\sqrt{d})/2$, we have $q_{l+1} + q_{l-1} = a_l q_l + 2q_{l-1} \equiv q_l \bmod 2$, and then $(2.9)_{n=l}$ and $(2.10)_{n=l}$ yield that

$$q_{2l} = q_{l+1}q_l + q_l q_{l-1} \equiv q_l \quad \bmod 2, \tag{2.32}$$

$$\begin{aligned} q_{2l-1} &= q_l r_{l+1} + q_{l-1} r_l = q_l(a_l r_l + r_{l-1}) + q_{l-1} r_l \\ &\equiv q_l r_l + (q_l r_{l-1} - q_{l-1} r_l) \equiv q_l q_{l-1} + 1 \quad \bmod 2 \quad (\text{by } (2.3)_{n=l}). \end{aligned} \tag{2.33}$$

Thus, they hold for $k = 2$. Also, (2.32) and (2.33) imply that

$$\begin{aligned} q_{3l} &= q_{l+1}q_{2l} + q_l q_{2l-1} \equiv q_{l+1}q_l + q_l(q_l q_{l-1} + 1) \\ &\equiv q_l(q_{l+1} + q_{l-1}) + q_l \equiv 2q_l \equiv 0 \quad \bmod 2, \end{aligned} \tag{2.34}$$

$$\begin{aligned} q_{3l-1} &= q_{2l} r_{l+1} + q_{2l-1} r_l \equiv q_l r_{l+1} + (q_l q_{l-1} + 1) r_l \\ &\equiv q_l r_{l+1} + q_l r_l + r_l = q_l\{(a_l + 1) r_l + r_{l-1}\} + r_l \\ &\equiv q_l r_{l-1} + r_l \equiv (q_{l-1} + 1) + q_{l-1} \equiv 1 \quad \bmod 2 \quad (\text{by } (2.8)_{k=1}). \end{aligned} \tag{2.35}$$

Thus, they hold for $k = 3$. We let $n \geq 1$, and assume that both (2.30) and (2.31) hold for $k = 3n-2, 3n-1$, and $3n$. Similarly, $(2.9)_{n=l}$ and $(2.10)_{n=l}$ yield that

$$q_{(3n+1)l} = q_{l+1}q_{3nl} + q_l q_{3nl-1} \equiv q_{l+1}0 + q_l 1 = q_l \quad \bmod 2,$$
$$q_{(3n+1)l-1} = q_{3nl} r_{l+1} + q_{3nl-1} r_l \equiv 0 r_{l+1} + 1 r_l = q_{l-1} \quad \bmod 2.$$

It follows from this, (2.32) and (2.33) that

$$q_{(3n+2)l} = q_{l+1}q_{(3n+1)l} + q_l q_{(3n+1)l-1} \equiv q_{l+1}q_l + q_l q_{l-1} \equiv q_l \quad \bmod 2,$$
$$q_{(3n+2)l-1} = q_{(3n+1)l} r_{l+1} + q_{(3n+1)l-1} r_l \equiv q_l r_{l+1} + q_{l-1} r_l \equiv q_l q_{l-1} + 1 \quad \bmod 2.$$

We see by this, (2.34) and (2.35) that

$$q_{3(n+1)l} = q_{l+1}q_{(3n+2)l} + q_l q_{(3n+2)l-1} \equiv q_{l+1}q_l + q_l(q_l q_{l-1} + 1) \equiv 0 \mod 2,$$
$$q_{3(n+1)l-1} = q_{(3n+2)l}r_{l+1} + q_{(3n+2)l-1}r_l \equiv q_l r_{l+1} + (q_l q_{l-1} + 1)r_l \equiv 1 \mod 2.$$

Thus, both (2.30) and (2.31) hold for $k = 3n+1, 3n+2$, and $3(n+1)$. Next, we show (2.28) and (2.29) simultaneously by induction in k. Note that a_l is even when $\omega = \sqrt{d}$. Then we obtain them by the same argument. This proves our lemma. □

It is known that the following lemma is of central importance in the theory of continued fractions, which is used in the proofs of Propositions 4.4 and 4.5 in Section 4. In the case where $\omega = (1+\sqrt{d})/2$, as no reference for the proof of it is known to the authors, we give it here.

Lemma 2.7. *Under the above setting, we have* $G_n^2 - dq_n^2 = (-1)^n Q_n Q_0$ *for all* $n \geq 0$*. Here, we put* $G_n := Q_0 p_n - P_0 q_n$.

Proof. For any positive integer n, we put $\theta_{n+1} := \prod_{i=1}^{n} \omega_i^{-1}$, and $\theta_1 := 1$ (H.C. Williams and Wunderlich [11, p. 408, (2.7)]). By induction in $n \geq 0$, we show that

$$\theta_{n+1} = (-1)^n(p_n - q_n\omega) \tag{2.36}$$

holds ([11, Theorem 2.1, (2.9)]). This holds for $n = 0, 1$ from the definition of θ_{n+1}. We assume that (2.36) holds for $n \geq 1$. By (2.24) and (2.1), we have

$$\omega_{n+1}^{-1} = \omega_n - a_n = \frac{(a_n p_n + p_{n-1}) - (a_n q_n + q_{n-1})\omega}{q_n\omega - p_n} = \frac{p_{n+1} - q_{n+1}\omega}{q_n\omega - p_n}.$$

Hence the hypothesis of induction implies that

$$\theta_{n+2} = \theta_{n+1}\omega_{n+1}^{-1} = (-1)^n(p_n - q_n\omega)\frac{p_{n+1} - q_{n+1}\omega}{q_n\omega - p_n} = (-1)^{n+1}(p_{n+1} - q_{n+1}\omega).$$

Thus, (2.36) holds for $n+1$. Since $(G_n - q_n\sqrt{d})/Q_0 = p_n - q_n\omega$ by the definition of G_n, we see from (2.36) that

$$\theta_{n+1} = (-1)^n(G_n - q_n\sqrt{d})/Q_0, \quad n \geq 0. \tag{2.37}$$

For any element x in "a real quadratic field $\mathbb{Q}(\sqrt{d})$", x' denotes its non-trivial conjugate over $\mathbb{Q}$. As the definition of ω_i and (2.18) yield that $(\omega_i\omega_i')^{-1} = Q_i^2/(P_i^2 - d) = -Q_i/Q_{i-1}$ for each integer $i \geq 1$, we have

$$\theta_{n+1}\theta_{n+1}' = \prod_{i=1}^{n}(\omega_i\omega_i')^{-1} = (-1)^n\frac{Q_n}{Q_0}, \quad n \geq 0.$$

On the other hand, we see by (2.37) that $\theta_{n+1}\theta'_{n+1} = (G_n^2 - dq_n^2)/Q_0^2$. Hence, we obtain $G_n^2 - dq_n^2 = (-1)^n Q_n Q_0$. Our lemma is proved. □

3. Certain positive integers with even period of minimal type

We let d be a non-square positive integer constructed in Theorem 2.4 (i) (resp. (ii)), and assume that the period l of the continued fraction expansion of $\omega = \sqrt{d}$ (resp., $= (1+\sqrt{d})/2$) is even: $l = 2L$. For any integer $e \geq 0$, $\overrightarrow{\mathbf{w}}_e$ denotes e iterations of the periodic part $a_1, \ldots, a_l$, and we put $\overrightarrow{\mathbf{v}} := a_1, \ldots, a_{L-1}$. Then, $\overrightarrow{\mathbf{w}}_0$ is empty and if $L = 1$ then $\overrightarrow{\mathbf{v}}$ is also empty. The symmetric part $a_1, \ldots, a_{l-1}$ of the continued fraction expansion of ω can be written as $\overrightarrow{\mathbf{v}}, a_L, \overleftarrow{\mathbf{v}}$. Here, $\overleftarrow{\mathbf{v}} := a_{L-1}, \ldots, a_1$ is the reverse of $\overrightarrow{\mathbf{v}}$. Let b be any positive integer. We put

$$a' := b + a_L$$

and consider a symmetric string of $(2e+1)l - 1$ positive integers

$$\overrightarrow{\mathbf{w}}_e,\ \overrightarrow{\mathbf{v}},\ a',\ \overleftarrow{\mathbf{v}},\ \overleftarrow{\mathbf{w}}_e,$$

where $\overleftarrow{\mathbf{w}}_e$ denotes the reverse of $\overrightarrow{\mathbf{w}}_e$, which is e iterations of a string of l positive integers $a_l, \ldots, a_1$. For brevity, we put $L' := (2e+1)L$ and $l' := (2e+1)l = 2L'$. From this symmetric string of $l'-1$ positive integers, we define nonnegative integers $q'_n, r'_n, n \geq 0$ by using the recurrence equation (2.1). Since the former part $\overrightarrow{\mathbf{w}}_e, \overrightarrow{\mathbf{v}}$ of it gives the same integers q_n, r_n, we have

$$q'_n = q_n, \quad r'_n = r_n, \quad 0 \leq n \leq L'. \tag{3.1}$$

We assume this setting throughout this paper. In Section 3.2, we shall choose a suitable positive integer b depending on the integer e (Lemma 3.4) to give positive integers of minimal type.

3.1. Basic properties.

The following hold for the positive integer a_0 (in (2.14) or (2.15)) and the symmetric string of positive integers $\overrightarrow{\mathbf{w}}_e, \overrightarrow{\mathbf{v}}, a_L, \overleftarrow{\mathbf{v}}, \overleftarrow{\mathbf{w}}_e$.

Lemma 3.1.

$$q_{l'} = (q_{L'+1} + q_{L'-1})q_{L'}, \tag{3.2}$$

$$r_{l'} + (-1)^{L'} = (q_{L'+1} + q_{L'-1})r_{L'}, \tag{3.3}$$

$$p_{l'} + (-1)^{L'} = (q_{L'+1} + q_{L'-1})p_{L'}, \tag{3.4}$$

$$G_{L'} = \frac{Q_L}{2}(q_{L'+1} + q_{L'-1}), \tag{3.5}$$

$$2((G_{l'}/Q_0) + (-1)^{L'}) = \frac{Q_L}{Q_0}(q_{L'+1} + q_{L'-1})^2. \tag{3.6}$$

Proof. The equation $(2.9)_{n=L'}$ of Lemma 2.1 for the symmetric string of positive integers $\overrightarrow{\mathbf{w}}_e$, $\overrightarrow{\mathbf{v}}$, a_L, $\overleftarrow{\mathbf{v}}$, $\overleftarrow{\mathbf{w}}_e$ yields that

$$q_{l'} = q_{L'+1}q_{L'} + q_{L'}q_{L'-1} = (q_{L'+1} + q_{L'-1})q_{L'},$$

which gives (3.2). By $(2.3)_{n=L'+1}$, we have $q_{L'}r_{L'+1} = q_{L'+1}r_{L'} - (-1)^{L'}$. Consequently, $(2.10)_{n=L'}$ implies that

$$r_{l'} = q_{L'}r_{L'+1} + q_{L'-1}r_{L'} = (q_{L'+1} + q_{L'-1})r_{L'} - (-1)^{L'},$$

and we see (3.3). By adding (3.3) to (3.2) times a_0, we obtain (3.4) by (2.4). Next, we show (3.5). The periodic part $\omega_1 = [\overline{a_1, \dots, a_l}]$ yields that $\omega_{kl+n} = \omega_n$ for all $k \geq 0$ and all n, $1 \leq n \leq l-1$. Therefore, $P_{kl+n} = P_n$ and $Q_{kl+n} = Q_n$. Also, it is known that

$$P_{n+1} = P_{l-n},\ Q_n = Q_{l-n}, \quad 0 \leq n \leq l-1. \tag{3.7}$$

(By using $\omega_l = a_l + (1/\omega_1)$, (2.16) and (2.18), this is shown by induction in n.) As $L' = el + L$, we see by (3.7) that $P_{L'+1} = P_{L+1} = P_L = P_{L'}$. Hence, (2.16) gives that $P_{L'} = P_{L'+1} = a_{L'}Q_{L'} - P_{L'}$, so that $2P_{L'} = a_{L'}Q_{L'}$. It follows from $(2.19)_{n=L'}$ of Lemma 2.5 and this that

$$\begin{aligned} G_{L'} &= \frac{a_{L'}Q_{L'}}{2}q_{L'} + Q_{L'}q_{L'-1} \\ &= \frac{Q_{L'}}{2}(q_{L'+1} - q_{L'-1}) + Q_{L'}q_{L'-1} = \frac{Q_{L'}}{2}(q_{L'+1} + q_{L'-1}). \end{aligned}$$

As $Q_{L'} = Q_{el+L} = Q_L$, we have (3.5). Finally, we show (3.6). We see by (3.4) and (3.2) that

$$\begin{aligned} G_{l'} &= Q_0 p_{l'} - P_0 q_{l'} = (q_{L'+1} + q_{L'-1})(Q_0 p_{L'} - P_0 q_{L'}) - (-1)^{L'} Q_0 \\ &= (q_{L'+1} + q_{L'-1})G_{L'} - (-1)^{L'} Q_0, \end{aligned}$$

so that

$$\frac{G_{l'}}{Q_0} + (-1)^{L'} = \frac{G_{L'}}{Q_0}(q_{L'+1} + q_{L'-1}).$$

By substituting (3.5) for this equation, we obtain (3.6). This proves our lemma. □

The following hold for the symmetric string of positive integers $\overrightarrow{\mathbf{w}}_e$, $\overrightarrow{\mathbf{v}}$, a', $\overleftarrow{\mathbf{v}}$, $\overleftarrow{\mathbf{w}}_e$.

Lemma 3.2.

$$q'_{l'} = bq^2_{L'} + q_{l'}, \tag{3.8}$$

$$q'_{l'-1} = r'_{l'} = bq_{L'}r_{L'} + r_{l'}, \tag{3.9}$$

(3.10) $$r'_{l'-1} = br_{L'}^2 + r_{l'-1},$$

(3.11) $$r_{L'}q'_{l'} - q_{L'}q'_{l'-1} = (-1)^{L'}q_{L'},$$

(3.12) $$h(s)q'_{l'} - g(s)q'_{l'-1} = r'_{l'-1} - (-1)^{L'}\frac{Q_L}{Q_0}b.$$

Proof. By (3.1), we have

$$q'_{L'+1} = a'q'_{L'} + q'_{L'-1} = (b + a_L)q_{L'} + q_{L'-1} = bq_{L'} + q_{L'+1}.$$

Consequently, $(2.9)_{n=L'}$ of Lemma 2.1 for $\overrightarrow{\mathbf{w}}_e$, $\overrightarrow{\mathbf{v}}$, a', $\overleftarrow{\mathbf{v}}$, $\overleftarrow{\mathbf{w}}_e$ and (3.1) yield that

$$q'_{l'} = q'_{L'+1}q'_{L'} + q'_{L'}q'_{L'-1} = (bq_{L'} + q_{L'+1})q_{L'} + q_{L'}q_{L'-1} = bq_{L'}^2 + q_{l'} \quad \text{(by (3.2))},$$

which gives (3.8). By (2.7), we have $q'_{l'-1} = r'_{l'}$, and by (3.1),

$$r'_{L'+1} = a'r'_{L'} + r'_{L'-1} = (b + a_L)r_{L'} + r_{L'-1} = br_{L'} + r_{L'+1}.$$

Therefore, $(2.10)_{n=L'}$ and (3.1) imply that

$$\begin{aligned} r'_{l'} &= q'_{L'}r'_{L'+1} + q'_{L'-1}r'_{L'} \\ &= q_{L'}(br_{L'} + r_{L'+1}) + q_{L'-1}r_{L'} = bq_{L'}r_{L'} + r_{l'}, \end{aligned}$$

where we use $(2.10)_{n=L'}$ for $\overrightarrow{\mathbf{w}}_e$, $\overrightarrow{\mathbf{v}}$, a_L, $\overleftarrow{\mathbf{v}}$, $\overleftarrow{\mathbf{w}}_e$. Thus, we see (3.9). The equations $(2.11)_{n=L'}$ and (3.1) yield that

$$\begin{aligned} r'_{l'-1} &= r'_{L'+1}r'_{L'} + r'_{L'}r'_{L'-1} \\ &= (br_{L'} + r_{L'+1})r_{L'} + r_{L'}r_{L'-1} = br_{L'}^2 + r_{l'-1} \quad \text{(by } (2.11)_{n=L'}), \end{aligned}$$

which gives (3.10). It follows from (3.8) and (3.9) that

$$r_{L'}q'_{l'} - q_{L'}q'_{l'-1} = bq_{L'}^2r_{L'} + q_{l'}r_{L'} - (bq_{L'}^2r_{L'} + q_{L'}r_{l'}) = q_{l'}r_{L'} - q_{L'}r_{l'},$$

and $(2.9)_{n=L'}$ and $(2.10)_{n=L'}$ for $\overrightarrow{\mathbf{w}}_e$, $\overrightarrow{\mathbf{v}}$, a_L, $\overleftarrow{\mathbf{v}}$, $\overleftarrow{\mathbf{w}}_e$ imply that

(3.13) $$\begin{aligned} q_{l'}r_{L'} - q_{L'}r_{l'} &= (q_{L'+1}q_{L'} + q_{L'}q_{L'-1})r_{L'} - q_{L'}(q_{L'}r_{L'+1} + q_{L'-1}r_{L'}) \\ &= (q_{L'+1}r_{L'} - q_{L'}r_{L'+1})q_{L'} = (-1)^{L'}q_{L'} \quad \text{(by } (2.3)_{n=L'+1}). \end{aligned}$$

Thus, we obtain (3.11). Finally, we show (3.12). We see by (3.8) and (3.9) that

$$\begin{aligned} h(s)q'_{l'} - g(s)q'_{l'-1} &= h(s)(bq_{L'}^2 + q_{l'}) - g(s)(bq_{L'}r_{L'} + r_{l'}) \\ &= bq_{L'}(h(s)q_{L'} - g(s)r_{L'}) + r_{l'-1} \quad \text{(by (2.22))}, \end{aligned}$$

so that

$$h(s)q'_{l'} - g(s)q'_{l'-1} = r'_{l'-1} + bq_{L'}(h(s)q_{L'} - g(s)r_{L'}) - br_{L'}^2 \tag{3.14}$$

by (3.10). We see from $(2.5)_{n=l'-1}$ and $(2.6)_{n=L'}$ that

$$\begin{pmatrix} q_{l'} & q_{l'-1} \\ r_{l'} & r_{l'-1} \end{pmatrix} = \begin{pmatrix} q_{L'+1} & q_{L'} \\ r_{L'+1} & r_{L'} \end{pmatrix} \begin{pmatrix} q_{L'} & r_{L'} \\ q_{L'-1} & r_{L'-1} \end{pmatrix}.$$

Multiplying this equation from the right by the inverse matrix $(-1)^{L'-1}\begin{pmatrix} r_{L'-1} & -r_{L'} \\ -q_{L'-1} & q_{L'} \end{pmatrix}$ gives

$$(-1)^{L'-1}\begin{pmatrix} q_{l'}r_{L'-1} - q_{l'-1}q_{L'-1} & -q_{l'}r_{L'} + q_{l'-1}q_{L'} \\ r_{l'}r_{L'-1} - q_{L'-1}r_{l'-1} & -r_{l'}r_{L'} + q_{L'}r_{l'-1} \end{pmatrix} = \begin{pmatrix} q_{L'+1} & q_{L'} \\ r_{L'+1} & r_{L'} \end{pmatrix}.$$

Furthermore, multiplying the above equation from the left by a row vector $(h(s), -g(s))$ and comparing with the second components of both sides of it yield that

$$\begin{aligned} &(-1)^{L'-1}(-q_{l'}r_{L'} + q_{l'-1}q_{L'})h(s) - (-1)^{L'-1}(-r_{l'}r_{L'} + q_{L'}r_{l'-1})g(s) \\ &= h(s)q_{L'} - g(s)r_{L'}. \end{aligned}$$

Now we use Lemma 2.5 and also note that l' is even. This implies that

$$\begin{aligned} &h(s)q_{L'} - g(s)r_{L'} \\ &= (-1)^{L'}(h(s)q_{l'} - g(s)r_{l'})r_{L'} - (-1)^{L'}(h(s)q_{l'-1} - g(s)r_{l'-1})q_{L'} \\ &= (-1)^{L'}r_{l'-1}r_{L'} - (-1)^{L'}((-1)^{l'}a_l + q_{l'-1}r_{l'-1})\frac{q_{L'}}{q_{l'}} \quad \text{(by (2.22), (2.23))} \\ &= \frac{1}{q_{l'}}\{(-1)^{L'}(q_{l'}r_{L'} - q_{l'-1}q_{L'})r_{l'-1} - (-1)^{L'}a_l q_{L'}\}. \end{aligned}$$

Since $q_{l'-1} = r_{l'}$ by (2.7), we see from (3.13) that

$$\begin{aligned} h(s)q_{L'} - g(s)r_{L'} &= \frac{q_{L'}}{q_{l'}}(r_{l'-1} - (-1)^{L'}a_l) = \frac{q_{L'}}{q_{l'}^2}(q_{l'}r_{l'-1} - (-1)^{L'}a_l q_{l'}) \\ &= \frac{q_{L'}}{q_{l'}^2}(r_{l'}^2 - 1 - (-1)^{L'}a_l q_{l'}) \quad \text{(by (2.8))}. \end{aligned}$$

By substituting this equation for (3.14), we obtain

$$h(s)q'_{l'} - g(s)q'_{l'-1} = r'_{l'-1} + \frac{bq_{L'}^2}{q_{l'}^2}\mathcal{E}.$$

Here, we put $\mathcal{E} := r_{l'}^2 - 1 - (-1)^{L'} a_l q_{l'} - (q_{l'}^2 r_{L'}^2 / q_{L'}^2)$. Then,

$$\begin{aligned}
\mathcal{E} &= r_{l'}^2 - 1 - (-1)^{L'} 2((G_{l'}/Q_0) - r_{l'}) - (q_{l'}^2 r_{L'}^2/q_{L'}^2) \quad \text{(by (2.20))} \\
&= r_{l'}^2 + (-1)^{L'} 2r_{l'} + 1 - (-1)^{L'} 2(G_{l'}/Q_0) - 2 - (q_{l'}^2 r_{L'}^2/q_{L'}^2) \\
&= (r_{l'} + (-1)^{L'})^2 - (-1)^{L'} 2((G_{l'}/Q_0) + (-1)^{L'}) - (q_{l'}^2 r_{L'}^2/q_{L'}^2) \\
&= (q_{L'+1} + q_{L'-1})^2 r_{L'}^2 - (-1)^{L'} (Q_L/Q_0) \frac{q_{l'}^2}{q_{L'}^2} - (q_{L'+1} + q_{L'-1})^2 r_{L'}^2 \\
&\qquad\qquad \text{(by (3.3), (3.6) and (3.2))} \\
&= -(-1)^{L'} (Q_L/Q_0) \frac{q_{l'}^2}{q_{L'}^2}.
\end{aligned}$$

Thus, we have (3.12) and this proves our lemma. □

3.2. Integers s' and suitable positive integers b. To give positive integers of minimal type, we define an integer s' (Proposition 3.3) and choose a suitable positive integer b depending on the integer e (Lemma 3.4). Let k be a positive integer. By $(2.9)_{n=l}$ of Lemma 2.1, we see that $q_{kl} = q_{l+1} q_{(k-1)l} + q_l q_{(k-1)l-1}$. If $q_l \mid q_{(k-1)l}$ holds for $k \geq 2$, then this equation implies that $q_l \mid q_{kl}$. Thus, q_l divides q_{kl} for all $k \geq 1$.

Proposition 3.3. *Let s and s_0 be integers as in* Theorem 2.4. *Under the above setting, the following hold.*
(i) *We assume that the positive integer b is divisible by $q_{L'}$, and put*

$$s' := \frac{g(s) + (Q_L/Q_0)b + q'_{l'-1} r'_{l'-1}}{q'_{l'}}. \tag{3.15}$$

Then, s' is an integer and $s' > q'_{l'-1} r'_{l'-1} / q'_{l'}$ holds.
(ii) *Furthermore, we assume that b is also divisible by q_l, and let s'_0 be the least integer t for which $t > q'_{l'-1} r'_{l'-1}/q'_{l'}$. Then, $s' = s'_0$ if and only if*

$$s - s_0 \leq \frac{b}{q_l}(q_{L'}^2 - (Q_L/Q_0)) + \frac{q_{l'}}{q_l} - 1.$$

Proof. (i) Multiplying both sides of (3.11) in Lemma 3.2 by $(Q_L/Q_0)b/q_{L'}$ yields that

$$\{(Q_L/Q_0) b r_{L'}/q_{L'}\} q'_{l'} - \{(Q_L/Q_0)b\} q'_{l'-1} = (-1)^{L'} (Q_L/Q_0) b. \tag{3.16}$$

By (2.8), we have $r'_{l'-1} q'_{l'} - q'_{l'-1} q'_{l'-1} = -(-1)^{l'} = -1$. Multiplying both sides of this equation by $r'_{l'-1}$ gives

$$({r'_{l'-1}}^2) q'_{l'} - (q'_{l'-1} r'_{l'-1}) q'_{l'-1} = -r'_{l'-1}. \tag{3.17}$$

If we add up both sides of (3.12), (3.16) and (3.17), then the right hand side of it is equal to 0, and we obtain

$$\{h(s) + ((Q_L/Q_0)br_{L'}/q_{L'}) + {r'_{l'-1}}^2\}q'_{l'} = \{g(s) + (Q_L/Q_0)b + q'_{l'-1}r'_{l'-1}\}q'_{l'-1}.$$

By the assumption, $b/q_{L'}$ is an integer and $q'_{l'}$ is co-prime to $q'_{l'-1}$ by $(2.3)_{n=l'}$. Hence, s' is an integer and we have

$$s' = \frac{h(s) + ((Q_L/Q_0)br_{L'}/q_{L'}) + {r'_{l'-1}}^2}{q'_{l'-1}}. \tag{3.18}$$

Also, as $g(s) > 0$, we see by (3.15) that $s' > q'_{l'-1}r'_{l'-1}/q'_{l'}$.

(ii) For brevity, we put

$$E := q_{l-1}r_{l-1}/q_l, \quad E' := q'_{l'-1}r'_{l'-1}/q'_{l'}.$$

Since $s'_0 - 1 \leq E' < s'_0$ by the definition of s'_0, the integer s'_0 is characterized as an integer t satisfying $E' < t \leq E' + 1$. The same thing is true for E. Also,

$$s' = \frac{g(s) + (Q_L/Q_0)b}{q'_{l'}} + E'$$

and the first term of the right hand side of it is positive as $g(s) > 0$. Hence,

$$\begin{aligned} s' = s'_0 &\iff \frac{g(s) + (Q_L/Q_0)b}{q'_{l'}} \leq 1 \\ &\iff q_l s - q_{l-1}r_{l-1} + (Q_L/Q_0)b \leq bq_{L'}^2 + q_{l'} \quad \text{(by (3.8))} \\ &\iff q_l s \leq b(q_{L'}^2 - (Q_L/Q_0)) + q_{l'} + q_{l-1}r_{l-1} \\ &\iff s - s_0 \leq \frac{b}{q_l}(q_{L'}^2 - (Q_L/Q_0)) + \frac{q_{l'}}{q_l} + E - s_0. \end{aligned}$$

We see by the assumption and the remark in the beginning of this section that both b/q_l and $q_{l'}/q_l$ are integers. Also, $-1 \leq E - s_0 < 0$. Therefore,

$$s' = s'_0 \iff s - s_0 \leq \frac{b}{q_l}(q_{L'}^2 - (Q_L/Q_0)) + \frac{q_{l'}}{q_l} - 1.$$

Our proposition is proved. □

REMARK 3.1. As we have seen in [4, Lemma 2.2], $2[\sqrt{d}]/a_L \geq Q_L$ holds. Hence, since a sequence $\{q_n\}_{n\geq 2}$ of positive integers is strictly monotonously increasing, there exists some number e_0 for the constant $s - s_0$ such that

$$e \geq e_0 \implies \frac{1}{q_l}(q_{(2e+1)L}^2 - (Q_L/Q_0)) + \frac{q_{(2e+1)l}}{q_l} - 1 \geq s - s_0.$$

We assume that $e \geq e_0$, and b is divisible by both $q_{L'}$ and q_l. Then, we see by the above and Proposition 3.3 (ii) that $s' = s'_0$ holds for an integer s' determined by (3.15), depending on integers $e \geq e_0$ and b.

For any integer $e \geq 0$ and any positive integer b, we define polynomials $g'(x), h'(x)$ of degree 1 and a quadratic polynomial $f'(x)$ in $\mathbb{Z}[x]$ by putting

$$g'(x) := q'_{l'}x - q'_{l'-1}r'_{l'-1}, \quad h'(x) := q'_{l'-1}x - {r'_{l'-1}}^2, \quad f'(x) := g'(x)^2 + 4h'(x).$$

Lemma 3.4. *We let t be any positive integer and put $b := (q_{L'+1} + q_{L'-1})q_{L'}t$. Then, the assumption of* Proposition 3.3 *for b holds and*

$$s' = \{g(s) + (Q_L/Q_0)(q_{L'+1} + q_{L'-1})q_{L'}t + q'_{l'-1}r'_{l'-1}\}/q'_{l'}, \tag{3.19}$$

$$\begin{aligned} f'(s') = {} & (Q_L/Q_0)^2(q_{L'+1} + q_{L'-1})^2 q_{L'}^2 t^2 \\ & + 2(Q_L/Q_0)^2(q_{L'+1} + q_{L'-1})^2 t + f(s). \end{aligned} \tag{3.20}$$

Proof. By (3.2) of Lemma 3.1, we have $b = q_{l'}t$. Since $q_l \mid q_{l'}$, we obtain $q_l \mid b$. Thus, the assumption of Proposition 3.3 for b holds. The definition (3.15) of s' yields that

$$g'(s') = g(s) + (Q_L/Q_0)b = g(s) + (Q_L/Q_0)(q_{L'+1} + q_{L'-1})q_{L'}t, \tag{3.21}$$

and (3.18) implies that

$$h'(s') = h(s) + ((Q_L/Q_0)br_{L'}/q_{L'}).$$

Therefore,

$$\begin{aligned} & f'(s') \\ & = \{(Q_L/Q_0)^2b^2 + 2(Q_L/Q_0)bg(s) + g(s)^2\} + 4h(s) + 4((Q_L/Q_0)br_{L'}/q_{L'}) \\ & = (Q_L/Q_0)^2b^2 + 2(Q_L/Q_0)\frac{b}{q_{L'}}(g(s)q_{L'} + 2r_{L'}) + f(s). \end{aligned}$$

On the other hand, it follows from (3.2), (3.3), $a_l = g(s)$, and (2.20) of Lemma 2.5 that

$$\begin{aligned} & (q_{L'+1} + q_{L'-1})(g(s)q_{L'} + 2r_{L'}) \\ & = g(s)q_{l'} + 2(r_{l'} + (-1)^{L'}) \\ & = 2((G_{l'}/Q_0) - r_{l'}) + 2(r_{l'} + (-1)^{L'}) = 2((G_{l'}/Q_0) + (-1)^{L'}) \\ & = (Q_L/Q_0)(q_{L'+1} + q_{L'-1})^2 \quad \text{(by (3.6))}. \end{aligned}$$

Hence we obtain

$$f'(s') = (Q_L/Q_0)^2b^2 + 2(Q_L/Q_0)^2(q_{L'+1} + q_{L'-1})\frac{b}{q_{L'}} + f(s),$$

which gives (3.20), and our lemma is proved. □

Lemma 3.5. *Under the setting of* Lemma 3.4, *we consider the symmetric string of positive integers* $\overrightarrow{\mathbf{w}}_e$, $\overrightarrow{\mathbf{v}}$, a', $\overleftarrow{\mathbf{v}}$, $\overleftarrow{\mathbf{w}}_e$.
(i) *We assume that* Case (I) *occurs for the symmetric string of positive integers* $a_1, \dots, a_{l-1}$ *and* s *is even. Then, if* t *is even then* Case (I) *occurs for the new symmetric string of positive integers* $\overrightarrow{\mathbf{w}}_e$, $\overrightarrow{\mathbf{v}}$, a', $\overleftarrow{\mathbf{v}}$, $\overleftarrow{\mathbf{w}}_e$, *and* s' *is even. Also, if* t *is odd then* Case (II) *occurs for the new symmetric string.*
(ii) *We assume that* Case (I) *occurs for* $a_1, \dots, a_{l-1}$ *and* s *is odd. If* $e \equiv 1 \bmod 3$ *then* Case (III) *occurs for the new symmetric string. Furthermore, we assume that* $e \equiv 0, 2 \bmod 3$. *Then, if* t *is even then* Case (I) *occurs for the new symmetric string and* s' *is odd. Also, if* t *is odd then* Case (II) *occurs for the new symmetric string.*
(iii) *If* Case (II) *occurs for* $a_1, \dots, a_{l-1}$, *then* Case (II) *occurs for the new symmetric string.*
(iv) *If* Case (III) *occurs for* $a_1, \dots, a_{l-1}$, *then* Case (III) *occurs for the new symmetric string.*

Proof. As we use Lemma 2.2, we note that $a_{L'} = a_{el+L} = a_L$, and that $q'_{L'} = q_{L'}$ by (3.1). Since

$$q_{L'+1} + q_{L'-1} = a_{L'}q_{L'} + 2q_{L'-1} \equiv a_{L'}q_{L'} = a_L q_{L'} \mod 2$$

and $a' = (q_{L'+1} + q_{L'-1})q_{L'}t + a_L$, we have

$$a' \equiv a_L(q_{L'}t + 1) \mod 2. \tag{3.22}$$

Also, as $l' = 2L'$, $(2.9)_{n=L'}$ of Lemma 2.1 yields that

$$q_{(2e+1)l} = q_{l'} = (q_{L'+1} + q_{L'-1})q_{L'} \equiv a_L q_{L'} \mod 2. \tag{3.23}$$

(i) Since Case (I) occurs for $a_1, \dots, a_{l-1}$, both q_l and a_L are odd by Lemma 2.2. Consequently, we have $q_{L'} \equiv q_{(2e+1)l}$ and $a' \equiv q_{L'}t + 1 \bmod 2$ by (3.23) and (3.22).

As s is even and d is a positive integer constructed in Theorem 2.4, now we deal with $\omega = \sqrt{d}$. Therefore, (2.28) of Lemma 2.6 yields that $q_{L'} \equiv q_l \equiv 1 \bmod 2$, so that $a' \equiv t + 1 \bmod 2$. First, we assume that t is even. Then, since both $q'_{L'}$ and a' are odd, we see by Lemma 2.2 that Case (I) occurs for the new symmetric string and $q'_{l'}$ is odd. As $q'_{l'}r'_{l'-1} = {q'_{l'-1}}^2 - (-1)^{l'}$ by (2.8), the parity of $r'_{l'-1}$ does not coincide with that of $q'_{l'-1}$, so that $q'_{l'-1}r'_{l'-1} \equiv 0 \bmod 2$. Furthermore, since $q'_{l'}$ is odd and t is even, (3.19) implies that

$$s' \equiv g(s) = q_l s - (-1)^l q_{l-1} r_{l-1} \mod 2.$$

As q_l is odd, we similarly see that $q_{l-1}r_{l-1} \equiv 0 \bmod 2$. Hence, $s' \equiv s \bmod 2$ and s' is even. Next, we assume that t is odd. Then, since a' is even, Case (II) occurs by Lemma 2.2.

(ii) Similarly, we have $q_{L'} \equiv q_{(2e+1)l}$ and $a' \equiv q_{L'}t+1 \bmod 2$. As s is odd, now we deal with $\omega = (1+\sqrt{d})/2$. First, we assume that $e \equiv 1 \bmod 3$. Then, we see by (2.30) that $q_{L'} \equiv q_{(2e+1)l} \equiv 0 \bmod 2$. Therefore, $q'_{L'}$ is even and a' is odd so that Case (III) occurs by Lemma 2.2. Next, we assume that $e \equiv 0, 2 \bmod 3$. Then, $q_{L'} \equiv q_l \equiv 1 \bmod 2$ by (2.30), hence, $a' \equiv t+1 \bmod 2$. The same argument in (i) implies that $s' \equiv s \bmod 2$, s' is odd, and the same assertion holds.

(iii) Since Case (II) occurs for $a_1, \ldots, a_{l-1}$, Lemma 2.2 yields that a_L is even. By (3.22), a' is also even. We see by Lemma 2.2 that Case (II) occurs again.

(iv) Since Case (III) occurs for $a_1, \ldots, a_{l-1}$, it follows from Lemma 2.2 that q_l is even and a_L is odd. Now we deal with $\omega = (1+\sqrt{d})/2$. We see by (3.23) and (3.22) that $q_{L'} \equiv q_{(2e+1)l}$ and $a' \equiv q_{L'}t + 1 \bmod 2$. As q_l is even, $q_{(2e+1)l}$ is always even by (2.30). Hence, $q'_{L'} = q_{L'}$ is even so that a' is odd. Lemma 2.2 yields that Case (III) occurs. This proves our lemma. □

3.3. Construction of non-square positive integers $d'(t)$. Under the setting of Lemma 3.4, by using Theorem 2.4, we construct a new non-square positive integer d' from a symmetric string of $l'-1$ positive integers $\overrightarrow{\mathbf{w}}_e$, $\overrightarrow{\mathbf{v}}$, a', $\overleftarrow{\mathbf{v}}$, $\overleftarrow{\mathbf{w}}_e$ and the integer s'. When Case (I) occurs for the given symmetric string of positive integers $a_1, \ldots, a_{l-1}$, Case (I) does not always occur for this new symmetric string of positive integers. Indeed, we see by Lemma 3.5 that another Case occurs, depending on e modulo 3 and t modulo 2. Therefore, we consider three cases [A], [B], and [C] separately to prove Theorem 3.6, and construct the following positive integer d':

$$\begin{aligned} &\sqrt{d} \to \sqrt{d'} && \text{in [A]},\\ &(1+\sqrt{d})/2 \to \sqrt{d'} && \text{in [B]},\\ &(1+\sqrt{d})/2 \to (1+\sqrt{d'})/2 && \text{in [C]}. \end{aligned}$$

Theorem 3.6. *We consider a non-square positive integer d constructed in* Theorem 2.4 (i) (*resp*. (ii)), *and assume that the period l of the continued fraction expansion $\omega = \sqrt{d}$ (resp., $= (1+\sqrt{d})/2$) is even: $l = 2L$. Let e be any integer ≥ 0 and put $l' := (2e+1)l$ and $L' := (2e+1)L$ for brevity. For any positive integer t, we put*

$$\begin{aligned} a' &= a'(t) := (q_{L'+1} + q_{L'-1})q_{L'}t + a_L,\\ s' &= s'(t) := \{g(s) + (Q_L/Q_0)(q_{L'+1} + q_{L'-1})q_{L'}t + q'_{l'-1}r'_{l'-1}\}/q'_{l'}, \end{aligned}$$

and define polynomials $g'(x)$, $h'(x)$ of degree 1 and a quadratic polynomial $f'(x)$ in $\mathbb{Z}[x]$ as stated before Lemma 3.4. *Then the following hold.*

[A] *We assume that* "Case (I) *occurs for the symmetric string of positive integers $a_1, \ldots, a_{l-1}$ and s is even*", *or* Case (II) *occurs for it. Put $d' := f'(s')/4$ and $a'_0 :=$*

$g'(s')/2$. *Then,* $Q_L(q_{L'+1}+q_{L'-1})$ *is even and*

$$d' = d'(t) = \frac{Q_L^2(q_{L'+1}+q_{L'-1})^2}{4}q_{L'}^2t^2 + \frac{Q_L^2(q_{L'+1}+q_{L'-1})^2}{2}t + d,$$
$$a_0' = a_0'(t) = \frac{Q_L(q_{L'+1}+q_{L'-1})}{2}q_{L'}t + a_0.$$

Also, d' *is a non-square positive integer and*

$$\omega' := \sqrt{d'} = [a_0', \overline{\overrightarrow{\mathbf{w}}_e, \overrightarrow{\mathbf{v}}, a', \overleftarrow{\mathbf{v}}, \overleftarrow{\mathbf{w}}_e, 2a_0'}]$$

is the continued fraction expansion with the period l' *of* ω'.

[B] *We assume that* Case (I) *occurs for* $a_1, \dots, a_{l-1}$, *both* s *and* t *are odd, and* $e \equiv 0, 2 \bmod 3$. *Put* $d' := f'(s')/4$ *and* $a_0' := g'(s')/2$. *Then,* $Q_L/2$, $q_{L'+1}+q_{L'-1}$ *and* $q_{L'}$ *are all odd and*

$$\begin{aligned} d' &= d'(t) \\ &= \{(Q_L/2)^2(q_{L'+1}+q_{L'-1})^2q_{L'}^2t^2 + 2(Q_L/2)^2(q_{L'+1}+q_{L'-1})^2t + d\}/4, \\ a_0' &= a_0'(t) = \{(Q_L/2)(q_{L'+1}+q_{L'-1})q_{L'}t + 2a_0 - 1\}/2, \end{aligned}$$

(*so that* d' *and* a_0' *are integers by* $d \equiv 1 \bmod 4$). *Also,* d' *is a non-square positive integer and*

$$\omega' := \sqrt{d'} = [a_0', \overline{\overrightarrow{\mathbf{w}}_e, \overrightarrow{\mathbf{v}}, a', \overleftarrow{\mathbf{v}}, \overleftarrow{\mathbf{w}}_e, 2a_0'}]$$

is the continued fraction expansion with the period l' *of* ω'.

[C] *We assume that* "Case (I) *occurs for* $a_1, \dots, a_{l-1}$ *and* s *is odd*", *or* Case (III) *occurs for it. Here, if* Case (I) *occurs then we also assume that* $e \equiv 1 \bmod 3$, *or* t *is even. Put* $d' := f'(s')$ *and* $a_0' := (g'(s')+1)/2$. *Then,* $q_{L'}t$ *is even and*

$$\begin{aligned} d' &= d'(t) \\ &= (Q_L/2)^2(q_{L'+1}+q_{L'-1})^2q_{L'}^2t^2 + 2(Q_L/2)^2(q_{L'+1}+q_{L'-1})^2t + d, \\ a_0' &= a_0'(t) = (Q_L/2)(q_{L'+1}+q_{L'-1})\frac{q_{L'}t}{2} + a_0. \end{aligned}$$

Also, d' *is a non-square positive integer,* $d' \equiv 1 \bmod 4$, *and*

$$\omega' := (1+\sqrt{d'})/2 = [a_0', \overline{\overrightarrow{\mathbf{w}}_e, \overrightarrow{\mathbf{v}}, a', \overleftarrow{\mathbf{v}}, \overleftarrow{\mathbf{w}}_e, 2a_0'-1}]$$

is the continued fraction expansion with the period l' *of* ω'.

Proof. We see by Proposition 3.3 (i) that s' is an integer and $s' > q'_{l'-1}r'_{l'-1}/q'_{l'}$. It follows from (3.21) and the definition of s that

$$g'(s') > g(s) = a_l > a_1, \dots, a_{l-1}.$$

Also, we have $g'(s') > a'$ from (3.21) and the definition of a'. Hence, the condition (2.13) of Theorem 2.4 for the symmetric string of positive integers $\overrightarrow{\mathbf{w}}_e$, $\overrightarrow{\mathbf{v}}$, a', $\overleftarrow{\mathbf{v}}$, $\overleftarrow{\mathbf{w}}_e$ and s' holds.

[A] Lemma 3.5 (i) and (iii) imply that "Case (I) occurs for this new symmetric string and s' is even", or Case (II) occurs for it. As $\omega = \sqrt{d}$, we have $Q_0 = 1$. By (3.6) of Lemma 3.1, $2(p_{l'} + (-1)^{L'}) = Q_L(q_{L'+1} + q_{L'-1})^2$, so that $Q_L(q_{L'+1} + q_{L'-1})$ is even. Since $d = f(s)/4$ and $a_0 = g(s)/2$ by the definitions, (3.20) of Lemma 3.4 yields that

$$d' = f'(s')/4 = \frac{Q_L^2(q_{L'+1} + q_{L'-1})^2}{4} q_{L'}^2 t^2 + \frac{Q_L^2(q_{L'+1} + q_{L'-1})^2}{2} t + d$$

and by (3.21),

$$a_0' = g'(s')/2 = a_0 + \frac{Q_L(q_{L'+1} + q_{L'-1})}{2} q_{L'} t.$$

Therefore, Theorem 2.4 (i) implies our assertion.

[B] Lemma 3.5 (ii) implies that Case (II) occurs for the new symmetric string, and $q_{L'}$ is odd from its proof. As $Q_0 = 2$, we see by (3.5) of Lemma 3.1 that

$$(Q_L/Q_0)(q_{L'+1} + q_{L'-1}) = G_{L'} = 2p_{L'} - q_{L'} \equiv q_{L'} \equiv 1 \mod 2.$$

Consequently, Q_L/Q_0 and $q_{L'+1} + q_{L'-1}$ are both odd. Since $d = f(s)$ and $a_0 = (g(s) + 1)/2$ by the definitions, (3.20) yields that

$$\begin{aligned} d' &= f'(s')/4 \\ &= \{(Q_L/Q_0)^2(q_{L'+1} + q_{L'-1})^2 q_{L'}^2 t^2 + 2(Q_L/Q_0)^2(q_{L'+1} + q_{L'-1})^2 t + d\}/4 \end{aligned}$$

and by (3.21),

$$a_0' = g'(s')/2 = \{2a_0 - 1 + (Q_L/Q_0)(q_{L'+1} + q_{L'-1})q_{L'} t\}/2.$$

Hence, Theorem 2.4 (i) implies our assertion.

[C] Lemma 3.5 (ii) and (iv) imply that "Case (I) occurs for the new symmetric string and s' is odd", or Case (III) occurs for it. By its proof, $q_{L'}$ or t is even. Since $d = f(s)$ and $a_0 = (g(s) + 1)/2$ by the definitions, (3.20) yields that

$$d' = f'(s') = (Q_L/Q_0)^2(q_{L'+1} + q_{L'-1})^2 q_{L'}^2 t^2 + 2(Q_L/Q_0)^2(q_{L'+1} + q_{L'-1})^2 t + d$$

and by (3.21),

$$a_0' = (g'(s') + 1)/2 = a_0 + (Q_L/Q_0)(q_{L'+1} + q_{L'-1})\frac{q_{L'} t}{2}.$$

As $Q_0 = 2$, Theorem 2.4 (ii) implies our assertion. This proves our theorem. □

REMARK 3.2. We let d' be a non-square positive integer constructed in [A] and [B] (resp. [C]) of Theorem 3.6. We see by Proposition 3.3 (ii) that d' is a positive integer with period l' of minimal type for $\sqrt{d'}$ (resp. $(1+\sqrt{d'})/2$) if and only if

$$s - s_0 \le (q_{L'}^2 - (Q_L/Q_0))(q_{L'+1} + q_{L'-1})q_{L'}t/q_l + (q_{l'}/q_l) - 1.$$

Here, $Q_0 = 1, 2$ (resp., $= 2$). When e is sufficiently large, Remark 3.1 shows that $d' = d'(t)$ becomes of minimal type for $\sqrt{d'}$ (resp. $(1+\sqrt{d'})/2$) for all positive integers t.

Theorem 3.6 [A] implies Theorems 4.1 (c-i) and 4.2 (c-ii) in Mollin [6], and Theorems 2 (ii) and 3 ($e = 0$) in McLaughlin [5].

Proposition 3.7 (Mollin, McLaughlin). *We let d be a non-square positive integer and assume that*

$$\sqrt{d} = [a_0, \overline{a_1, \ldots, a_{l-1}, 2a_0}]$$

is the continued fraction expansion with even period $l = 2L$ of $\sqrt{d}$. Let e be any integer ≥ 0 and put $l' := (2e+1)l$. For any positive integer u, we put

$$d' := (p_{l'} + (-1)^L)^2 q_{l'}^2 u^2 + 2(p_{l'} + (-1)^L)^2 u + d,$$
$$a_0' := (p_{l'} + (-1)^L) q_{l'} u + a_0.$$

Then, d' is a non-square positive integer and

$$\sqrt{d'} = \left[a_0', \overline{\overrightarrow{\mathbf{w}}_e, \overrightarrow{\mathbf{v}}, a', \overleftarrow{\mathbf{v}}, \overleftarrow{\mathbf{w}}_e, 2a_0'}\right]$$

becomes the continued fraction expansion with even period l' of $\sqrt{d'}$. Here,

$$a' := \frac{2(p_{l'} + (-1)^L)}{Q_L} q_{l'} u + a_L.$$

Proof. We see by Theorem 2.4 that d is uniquely of the form $d = f(s)/4$ with some integer $s \ge s_0$. Here, the quadratic polynomial $f(x)$ and the integer s_0 are obtained as in it from the symmetric part of the above continued fraction extension. Furthermore, "Case (I) occurs for $a_1, \ldots, a_{l-1}$ and s is even", or Case (II) occurs for it. We put $t := (q_{L'+1} + q_{L'-1})^2 u$. As $L' \equiv L \bmod 2$, $(-1)^{L'} = (-1)^L$. The equations (3.6) and (3.2) of Lemma 3.1 yield that

$$a' = (q_{L'+1} + q_{L'-1})^2 q_{l'} u + a_L = (q_{L'+1} + q_{L'-1})^3 q_{L'} u + a_L$$
$$= (q_{L'+1} + q_{L'-1}) q_{L'} t + a_L.$$

Also, (3.4) and (3.2) imply that

$$
\begin{aligned}
a_0' &= (q_{L'+1} + q_{L'-1})p_{L'}(q_{L'+1} + q_{L'-1})q_{L'}u + a_0 = p_{L'}q_{L'}t + a_0,\\
d' &= (q_{L'+1} + q_{L'-1})^2 p_{L'}^2 (q_{L'+1} + q_{L'-1})^2 q_{L'}^2 u^2 + 2(q_{L'+1} + q_{L'-1})^2 p_{L'}^2 u + d\\
&= p_{L'}^2 q_{L'}^2 t^2 + 2p_{L'}^2 t + d.
\end{aligned}
$$

By (3.5), we have $p_{L'} = (Q_L/2)(q_{L'+1} + q_{L'-1})$, so that

$$
\begin{aligned}
a_0' &= \frac{Q_L(q_{L'+1} + q_{L'-1})}{2} q_{L'}t + a_0,\\
d' &= \frac{Q_L^2(q_{L'+1} + q_{L'-1})^2}{4} q_{L'}^2 t^2 + \frac{Q_L^2(q_{L'+1} + q_{L'-1})^2}{2} t + d.
\end{aligned}
$$

Hence, Theorem 3.6 [A] implies our proposition. □

We shall use the following lemma in Section 4.3.

Lemma 3.8. *Let $d'(t)$ be a non-square positive integer constructed in* Theorem 3.6. *Then, the discriminant of a quadratic polynomial $d'(t)$ in $\mathbb{Z}[t]$ is not equal to* 0.

Proof. We see from the proof of Theorem 3.6 that $d'(t) = f'(s'(t))/4$, or $f'(s'(t))$. By (3.20) of Lemma 3.4, the discriminant of a quadratic polynomial $f'(s'(t))$ is equal to

$$
\begin{aligned}
&4(Q_L/Q_0)^4(q_{L'+1} + q_{L'-1})^4 - 4(Q_L/Q_0)^2(q_{L'+1} + q_{L'-1})^2 q_{L'}^2 f(s)\\
&= 4(Q_L/Q_0)^2(q_{L'+1} + q_{L'-1})^2\{(Q_L/Q_0)^2(q_{L'+1} + q_{L'-1})^2 - q_{L'}^2 f(s)\}.
\end{aligned}
$$

Therefore, if we assume that the discriminant of $d'(t)$ is equal to 0 then $f(s)$ is square. As $d = f(s)/4$ or $f(s)$, d is also square, and this is a contradiction. Our lemma is proved. □

3.4. Yokoi invariant. In this section we let d be a non-square positive integer constructed in Theorem 2.4 (i), or (ii). We assume for a while that d is square-free, and consider a real quadratic field $\mathbb{Q}(\sqrt{d})$. Therefore, if d is a positive integer given in the assertion (i), then we assume that $d \equiv 2, 3 \bmod 4$. We know by the last assertion of Theorem 2.4 that all real quadratic fields are obtained in this way. Let $\varepsilon > 1$ be the fundamental unit of it, and we write uniquely $\varepsilon = (t + u\sqrt{d})/2$ with positive integers t, u. Then, we define the Yokoi invariant m_d of a real quadratic field $\mathbb{Q}(\sqrt{d})$ by putting $m_d := [u^2/t]$. The following hold.

Lemma 3.9 ([4] Lemma 4.1). *Under the above setting, we put $\lambda := A^2/(g(s)A + 2B)$. Then, if $d \equiv 2, 3 \bmod 4$ then $m_d = [4\lambda]$, and if $d \equiv 1 \bmod 4$ then $m_d = [\lambda]$.*

The equations (2.20) of Lemma 2.5 and (2.7) of Lemma 2.1 imply that

$$g(s)A + 2B = a_l q_l + 2q_{l-1} = 2((G_l/Q_0) - r_l) + 2q_{l-1} = 2(G_l/Q_0).$$

When $d \equiv 2, 3 \bmod 4$, as $Q_0 = 1$, we have $4\lambda = (2q_l^2)/G_l = (2q_l^2)/(G_l Q_0)$, and when $d \equiv 1 \bmod 4$, as $Q_0 = 2$, we obtain $\lambda = q_l^2/G_l = (2q_l^2)/(G_l Q_0)$. Thus, $m_d = [(2q_l^2)/(G_l Q_0)]$. Since the right hand side of this equation can be defined also when d has a square factor, we extend the Yokoi invariant in the following way.

DEFINITION 3.1. We let d be any non-square positive integer such that $d \equiv 1, 2, 3 \bmod 4$. First, we assume that $d \equiv 2, 3 \bmod 4$, and consider the continued fraction expansion with period l of $\sqrt{d}$: $\sqrt{d} = [a_0, \overline{a_1, \ldots, a_l}]$. We calculate positive integers p_l, q_l from partial quotients $a_0, a_1, \ldots, a_{l-1}$ by using the recurrence equation (2.1), and put $G_l = p_l$ and $Q_0 = 1$. Then, we define the Yokoi invariant of a non-square positive integer d by putting

$$m_d := \left[\frac{2q_l^2}{G_l Q_0}\right].$$

Next, we assume that $d \equiv 1 \bmod 4$, and consider the continued fraction expansion with period l of $(1 + \sqrt{d})/2$. Similarly, we calculate positive integers p_l, q_l from the partial quotients and put $G_l = 2p_l - q_l$ and $Q_0 = 2$. Then, we define the Yokoi invariant m_d in the same manner. (In fact we can give the similar definition of m_d also when $d \equiv 0 \bmod 4$, and furthermore, we can show that m_d coincides with “the Yokoi invariant” for the fundamental unit of a certain (not necessary maximal) order in “a real quadratic field $\mathbb{Q}(\sqrt{d})$”.)

We show the following proposition which is needed in Section 4. As we have seen in the beginning of Section 3.2, q_l divides q_{kl} for all positive integers k.

Proposition 3.10. *Let e and t be any fixed positive integers, and d, $d' = d'(t)$ and ω, $\omega' = \omega'(t)$, respectively, positive integers and quadratic irrationals constructed in* Theorem 3.6 [A] *or* [C]. *We assume that $d \equiv 2, 3$, $d'(t) \equiv 2, 3 \bmod 4$ in the assertion* [A], *and also assume that $a_0 \geq 2$ in the case where $\omega = \sqrt{d}$* ([A]), *and $a_0 \geq 3$ in the case where $\omega = (1 + \sqrt{d})/2$* ([C]). *We let m_d and $m_{d'(t)}$ be the Yokoi invariants of d and $d'(t)$ defined in* Definition 3.1, *respectively, and put $c_e := q_{(2e+1)l}/q_l$. Then, $m_d c_e - 1 \leq m_{d'(t)} \leq (m_d + 2)c_e$ holds.*

Here, the estimate for $m_{d'(t)}$ is rough and if $m_d = 0$ then the estimate for it from below becomes trivial. For the proof, we first show Lemmas 3.11 and 3.12. Let a' and a'_0 be positive integers as in Theorem 3.6. From a'_0 and the symmetric string of positive integers $\overrightarrow{\mathbf{w}}_e$, $\overrightarrow{\mathbf{v}}$, a', $\overleftarrow{\mathbf{v}}$, $\overleftarrow{\mathbf{w}}_e$, we define positive integers p'_n, $n \geq 0$ by using the recurrence equation (2.1). We write uniquely $\omega' = (P'_0 + \sqrt{d'})/Q'_0$ with positive integers P'_0, Q'_0, and put $G'_{l'} := Q'_0 p'_{l'} - P'_0 q'_{l'}$. Also, we put

$$\lambda := \frac{2q_l^2}{G_l Q_0}, \quad \lambda^* := \frac{2q_{l'}^2}{G_{l'} Q_0}, \quad \lambda' := \frac{2{q'_{l'}}^2}{G'_{l'} Q'_0},$$

and $m^* := [\lambda^*]$ for brevity. Since we deal with the assertions [A] and [C] of Theorem 3.6, note that $Q_0/Q'_0 = 1$ holds. First, we draw a comparison between the value of $m^* = [\lambda^*]$ and that of $m_{d'} = [\lambda']$ in the following lemma. (There we may take $e = 0$.)

Lemma 3.11. *The following hold.*
(i) $G'_{l'}/Q'_0 = ((G_{l'}/Q_0) + (-1)^L)(q_{L'}^2 t + 1)^2 - (-1)^L$.
(ii) *If we put*

$$\varphi(t) := \frac{(q_{L'}^2 t + 1)^2 - 1}{((G_{l'}/Q_0) + (-1)^L)(q_{L'}^2 t + 1)^2 - (-1)^L}$$

for all positive integers t, then

$$\lambda' = \lambda^*(1 - (-1)^L \varphi(t)). \tag{3.24}$$

Also, the function $\varphi(t)$ is strictly, monotonously increasing in the interval $[1, \infty)$, *and* $\varphi(t) \to 1/((G_{l'}/Q_0) + (-1)^L)$ *as* $t \to \infty$. *Furthermore, we have* $0 < \lambda^* \varphi(t) < 1$ *under the assumption of* Proposition 3.10 *for* a_0.

Proof. For brevity, we put

$$g := G_{l'}/Q_0, \quad g' := G'_{l'}/Q'_0, \quad u = u(t) := q_{L'}^2 t + 1,$$

$$\text{so that} \quad \varphi(t) = \frac{u^2 - 1}{(g + (-1)^L)u^2 - (-1)^L}.$$

We show $g \geq 1$ to see that the denominator of it is positive. (In fact, $g > 1$ and $2g$ is an integer.) When $\omega = \sqrt{d}$, as $l' \geq l \geq 2$, we have $g = p_{l'} \geq p_2 \geq 2$. When $\omega = (1 + \sqrt{d})/2$, $(2.19)_{n=l'}$ of Lemma 2.5 implies that $G_{l'} = P_{l'} q_{l'} + Q_{l'} q_{l'-1}$. Since $q_{l'-1} > 0$ from $l' \geq 2$ and $P_{l'}$, $Q_{l'}$ are positive integers, we obtain $G_{l'} \geq 2$, so that $g \geq 1$. This immediately yields that the denominator of $\varphi(t)$ is positive. Let a' be a positive integer defined in Theorem 3.6 and put $b := (q_{L'+1} + q_{L'-1}) q_{L'} t$. Then, $a' = b + a_L$.

(i) By (3.2) of Lemma 3.1, we have $b q_{L'}^2 = q_{l'} q_{L'}^2 t$. Consequently, we see by (3.8) that

$$q'_{l'} = q_{l'}(q_{L'}^2 t + 1) = q_{l'} u. \tag{3.25}$$

Since

$$b q_{L'} r_{L'} = (r_{l'} + (-1)^L) q_{L'}^2 t = (r_{l'} + (-1)^L)(u - 1)$$

by (3.3), the equation (3.9) yields that

$$r'_{l'} = (r_{l'} + (-1)^L)(u-1) + r_{l'} = r_{l'}u + (-1)^L(u-1). \tag{3.26}$$

First, let d' and ω' be a positive integer and a quadratic irrational constructed in Theorem 3.6 [A], respectively. As $2(p_{l'} + (-1)^L) = Q_L(q_{L'+1} + q_{L'-1})^2$ by (3.6), we obtain

$$a'_0 = (p_{l'} + (-1)^L)(q_{L'+1} + q_{L'-1})^{-1} q_{L'} t + a_0.$$

Therefore, (3.2) and (2.4) imply that

$$\begin{aligned} a'_0 q_{l'} &= (p_{l'} + (-1)^L) q_{L'}^2 t + a_0 q_{l'} = (p_{l'} + (-1)^L)(u-1) + p_{l'} - r_{l'} \\ &= gu - r_{l'} + (-1)^L(u-1). \end{aligned} \tag{3.27}$$

Hence,

$$\begin{aligned} g' &= p'_{l'} = a'_0 q'_{l'} + r'_{l'} = a'_0 q_{l'} u + r'_{l'} \quad \text{(by (3.25))} \\ &= gu^2 - r_{l'}u + (-1)^L u(u-1) + r_{l'}u + (-1)^L(u-1) \quad \text{(by (3.27), (3.26))} \\ &= gu^2 + (-1)^L(u^2-1) = (g + (-1)^L)u^2 - (-1)^L. \end{aligned}$$

Thus, the assertion (i) holds. Next, let d' and ω' be a positive integer and a quadratic irrational constructed in Theorem 3.6 [C], respectively. As $2(g+(-1)^L) = (Q_L/2)(q_{L'+1} + q_{L'-1})^2$ by (3.6), we obtain

$$a'_0 = (g + (-1)^L)(q_{L'+1} + q_{L'-1})^{-1} q_{L'} t + a_0,$$

so that

$$\begin{aligned} a'_0 q_{l'} &= (g + (-1)^L) q_{L'}^2 t + a_0 q_{l'} = (g + (-1)^L)(u-1) + p_{l'} - r_{l'} \\ &= gu + \frac{q_{l'}}{2} - r_{l'} + (-1)^L(u-1). \end{aligned} \tag{3.28}$$

Therefore,
(3.29)

$$\begin{aligned} p'_{l'} &= a'_0 q'_{l'} + r'_{l'} = a'_0 q_{l'} u + r'_{l'} \quad \text{(by (3.25))} \\ &= gu^2 + \frac{q_{l'}}{2}u - r_{l'}u + (-1)^L u(u-1) + r_{l'}u + (-1)^L(u-1) \quad \text{(by (3.28), (3.26))} \\ &= gu^2 + \frac{q_{l'}}{2}u + (-1)^L(u^2-1). \end{aligned}$$

Hence,

$$\begin{aligned} g' &= p'_{l'} - \frac{q'_{l'}}{2} = gu^2 + \frac{q_{l'}}{2}u + (-1)^L(u^2-1) - \frac{q_{l'}}{2}u \quad \text{(by (3.29), (3.25))} \\ &= gu^2 + (-1)^L(u^2-1) = (g + (-1)^L)u^2 - (-1)^L. \end{aligned}$$

Thus, the assertion (i) holds.

(ii) As $Q_0/Q_0' = 1$, it follows from (3.25) and the assertion (i) that

$$\lambda'\lambda^{*-1} = \frac{gu^2}{g'} = \frac{gu^2}{(g+(-1)^L)u^2 - (-1)^L} = 1 - (-1)^L\varphi(t),$$

which gives (3.24). Since the derivative of $\varphi(t)$ satisfies

$$\frac{d\varphi}{dt}(t) = \frac{2uq_{L'}^2 g}{\{(g+(-1)^L)u^2 - (-1)^L\}^2} > 0,$$

the function $\varphi(t)$ is strictly, monotonously increasing in the interval $[1, \infty)$. Also, we see from the definition of $\varphi(t)$ that $\varphi(t) \to 1/(g+(-1)^L)$ as $t \to \infty$. This implies that $\varphi(t) < 1/(g+(-1)^L)$. Therefore,

$$\lambda^*\varphi(t) < \frac{1}{Q_0{}^2} \cdot \frac{2q_{l'}^2}{g} \cdot \frac{1}{g+(-1)^L} \leq \frac{2}{g/q_{l'}} \cdot \left(\frac{g}{q_{l'}} + \frac{(-1)^L}{q_{l'}}\right)^{-1}.$$

First, we assume that $\omega = \sqrt{d}$ to show the last assertion. As $g = p_{l'}$, (2.4) yields that $g/q_{l'} = a_0 + (r_{l'}/q_{l'}) \geq a_0$. Hence we see by $a_0 \geq 2$ that

$$\lambda^*\varphi(t) < \frac{2}{a_0} \cdot \left(a_0 + \frac{(-1)^L}{q_{l'}}\right)^{-1} \leq \frac{2}{a_0(a_0-1)} \leq 1.$$

Next, we assume that $\omega = (1+\sqrt{d})/2$. As $g = p_{l'} - (q_{l'}/2)$, (2.4) yields that

$$g/q_{l'} = a_0 + \frac{r_{l'}}{q_{l'}} - \frac{1}{2} \geq a_0 - \frac{1}{2}.$$

Hence we see by $a_0 \geq 3$ that

$$\lambda^*\varphi(t) < \frac{2}{a_0 - (1/2)} \cdot \left(a_0 - \frac{1}{2} + \frac{(-1)^L}{q_{l'}}\right)^{-1} \leq \frac{2}{(a_0-1)(a_0-2)} \leq 1.$$

Thus, the assertion (ii) holds and our lemma is proved. □

Next, we draw a comparison between the value of $m_d = [\lambda]$ and that of $m^* = [\lambda^*]$ in the following lemma. There it is not necessary for the period l to be even.

Lemma 3.12. *We let k be a positive integer ≥ 2, and put $\lambda = 2q_l^2/(G_l Q_0)$, $\lambda^* := 2q_{kl}^2/(G_{kl} Q_0)$ and*

$$\psi := \frac{q_{(k-1)l}}{q_l}\left(\frac{a_l q_{kl}}{2} + r_{kl}\right)^{-1} > 0.$$

Then the following hold.

(i) $\lambda^* = (q_{kl}/q_l)\lambda(1 + (-1)^l \psi)$.
(ii) $0 < \lambda\psi < (1/Q_0^2 q_l) \cdot (2/a_l)^3$.

Proof. As $k \geq 2$, note that $l \leq kl - 1$. It follows from $(2.9)_{n=l}$ and $(2.10)_{n=l}$ of Lemma 2.1 that

$$q_{kl} = q_{l+1}q_{(k-1)l} + q_l q_{(k-1)l-1}, \tag{3.30}$$

$$r_{kl} = q_{(k-1)l}r_{l+1} + q_{(k-1)l-1}r_l. \tag{3.31}$$

(i) By adding (3.31) times $-q_l$ to (3.30) times r_l, we obtain

$$q_{kl}r_l - q_l r_{kl} = q_{(k-1)l}(q_{l+1}r_l - q_l r_{l+1}) = (-1)^l q_{(k-1)l} \quad \text{(by (2.3))},$$

so that

$$(-1)^l \psi = \left(q_{kl}\frac{r_l}{q_l} - r_{kl}\right)\left(\frac{a_l q_{kl}}{2} + r_{kl}\right)^{-1} = \left(\frac{r_l}{q_l} - \frac{r_{kl}}{q_{kl}}\right)\left(\frac{a_l}{2} + \frac{r_{kl}}{q_{kl}}\right)^{-1}.$$

Hence,

$$1 + (-1)^l \psi = \left(\frac{a_l}{2} + \frac{r_l}{q_l}\right)\left(\frac{a_l}{2} + \frac{r_{kl}}{q_{kl}}\right)^{-1}. \tag{3.32}$$

If $\omega = \sqrt{d}$ (resp., $= (1 + \sqrt{d})/2$) then, since $G_l/q_l = a_0 + (r_l/q_l)$ (resp., $= 2a_0 - 1 + (2r_l/q_l)$) by (2.4), we have

$$G_l/q_l = \frac{Q_0}{2}a_l + Q_0\frac{r_l}{q_l} = Q_0\left(\frac{a_l}{2} + \frac{r_l}{q_l}\right).$$

Similarly, we see that $G_{kl}/q_{kl} = Q_0(a_l/2 + r_{kl}/q_{kl})$. Therefore, (3.32) yields that

$$\lambda^*\lambda^{-1} = \frac{q_{kl}}{q_l} \cdot \frac{G_l/q_l}{G_{kl}/q_{kl}} = \frac{q_{kl}}{q_l} \cdot \left(\frac{a_l}{2} + \frac{r_l}{q_l}\right)\left(\frac{a_l}{2} + \frac{r_{kl}}{q_{kl}}\right)^{-1} = \frac{q_{kl}}{q_l}(1 + (-1)^l \psi),$$

which gives the assertion (i).

(ii) Dividing both sides of (3.30) by $q_{kl}q_l q_{l+1}$ implies that

$$\frac{1}{q_l q_{l+1}} = \frac{q_{(k-1)l}}{q_{kl}q_l} + \frac{q_{(k-1)l-1}}{q_{kl}q_{l+1}}.$$

Consequently, we have

$$\frac{q_{(k-1)l}}{q_{kl}q_l} \leq \frac{1}{q_l q_{l+1}} = \frac{1}{q_l(a_l q_l + q_{l-1})} \leq \frac{1}{a_l q_l^2}.$$

As $kl \geq 2$, $r_{kl} > 0$. Hence,

$$\psi = \frac{q_{(k-1)l}}{q_{kl}q_l}\left(\frac{a_l}{2} + \frac{r_{kl}}{q_{kl}}\right)^{-1} < \frac{1}{a_l q_l^2} \cdot \frac{2}{a_l} = \frac{2}{a_l^2 q_l^2}.$$

On the other hand, as $G_l/q_l = Q_0(a_l/2 + r_l/q_l) \geq Q_0 a_l/2$, we have

$$\lambda = \frac{2}{Q_0} \cdot \frac{q_l}{G_l/q_l} \leq (2/Q_0)^2 \frac{q_l}{a_l}.$$

Therefore we obtain

$$\lambda\psi < (2/Q_0)^2 \frac{q_l}{a_l} \cdot \frac{2}{a_l^2 q_l^2} = \frac{1}{Q_0^2 q_l}(2/a_l)^3.$$

This proves our lemma. □

Proof of Proposition 3.10. Let $k := 2e + 1 \geq 3$. As l is even, we see by Lemma 3.12 (i) that $\lambda^* = c_e\lambda + c_e\lambda\psi$. If $\omega = \sqrt{d}$ (resp., $= (1 + \sqrt{d})/2$) then, as $a_0 \geq 2$ (resp., ≥ 3) by our assumption, we have $a_l \geq 4$. Lemma 3.12 (ii) yields that

$$0 < c_e\lambda\psi < c_e/(8Q_0^2 q_l) \leq c_e.$$

Hence we obtain $c_e\lambda < \lambda^* < c_e(\lambda+1)$. Consequently, $c_e m_d \leq m^* < c_e(\lambda+1) < c_e(m_d+2)$, so that

$$c_e m_d \leq m^* \leq c_e(m_d + 2) - 1. \tag{3.33}$$

First, we assume that L is even. The equation (3.24) of Lemma 3.11 implies that $\lambda' = \lambda^* - \lambda^*\varphi(t)$. Since $0 < \lambda^*\varphi(t) < 1$ by the assertion (ii) of it, we have $\lambda^* - 1 < \lambda' < \lambda^*$. Therefore, $m^* - 1 \leq m_{d'} < \lambda^* < m^* + 1$, so that $m^* - 1 \leq m_{d'} \leq m^*$. Hence, by (3.33), we obtain $c_e m_d - 1 \leq m_{d'} \leq c_e(m_d + 2) - 1$. Next, we assume that L is odd. We see by (3.24) that $\lambda' = \lambda^* + \lambda^*\varphi(t)$. By Lemma 3.11 (ii), we have $\lambda^* < \lambda' < \lambda^* + 1$. Therefore, $m^* \leq m_{d'} < \lambda^* + 1 < m^* + 2$, so that $m^* \leq m_{d'} \leq m^* + 1$. Hence, by (3.33), we obtain $c_e m_d \leq m_{d'} \leq c_e(m_d + 2)$. This proves our proposition. □

REMARK 3.3. We see by the above proof that $m^* - 1 \leq m_{d'(t)} \leq m^* + 1$ for all positive integers t. Since the integer m^* depends on an integer $e \geq 0$, if e is fixed then the values of $m_{d'(t)}$ do not change very much when t is various. Indeed, they are constant in the tables of Section 5. Also, we can similarly show that $4m_d c_e - 4 \leq m_{d'(t)} \leq 4(m_d + 2)c_e + 3$ holds under the assumption that $d'(t) \equiv 2, 3 \bmod 4$ in the assertion [B] of Theorem 3.6.

4. Main results

We begin with quadratic irrationals ω with period 2, 4 given in [4], and by using results of Sections 3.3 and 3.4, construct real quadratic fields $\mathbb{Q}(\sqrt{d'})$ with even period of minimal type whose Yokoi invariant is relatively large. For brevity we put $c_e := q_{(2e+1)l}/q_l$.

4.1. The case where $l = 2$.

Proposition 4.1. *Let e and m be any positive integers, and a any positive integer such that $a \geq 2$ and $4a^4 + 8a^2 + 2 > m$. We define positive integers q_n, $n \geq 1$ by using partial quotients $a, 2a$, appeared in the continued fraction expansion $\sqrt{a^2+2} = [a, \overline{a, 2a}]$, and the recurrence equation (2.1). For any positive integer t, we put*

$$d'(t) := (q_{2e+2} + q_{2e})^2 q_{2e+1}^2 t^2 + 2(q_{2e+2} + q_{2e})^2 t + (a^2 + 2).$$

Then the following hold.
(i) *Each $d'(t)$ is a positive integer with period $2(2e+1)$ of minimal type for $\sqrt{d'(t)}$.*
(ii) *When a is even, we have $d'(t) \equiv 2 \bmod 4$. When a is odd, if t is even then $d'(t) \equiv 3 \bmod 4$, and if t is odd then $d'(t) \equiv 2 \bmod 4$.*
(iii) *For all positive integers t, we have $c_e - 1 \leq m_{d'(t)} \leq 3c_e$. Also, $m_{d'(t)} > m$.*

Proof. We put $d := a^2 + 2$, $l' := 2(2e+1)$ and $L' := 2e+1$ for brevity. We know from [4, Example 4.2] that when a is odd (resp. even), Case (I) (resp. Case (II)) occurs for "the symmetric string of a positive integer a", and $\sqrt{d} = [a, \overline{a, 2a}]$ is the continued fraction expansion of $\sqrt{d}$. Also, $d \equiv 2, 3 \bmod 4$, $s_0 = 1$, $s = 2$ and $m_d = 1$. In [4] we calculated the Yokoi invariant m_d under the assumption that d is square-free. However, as we have explained in the beginning of Section 3.4, this value is obtained from the continued fraction expansion of $\sqrt{d}$. Hence, $m_d = 1$ holds without this assumption. Since $P_1 = aQ_0 - P_0 = a$ from (2.16), we see by (2.17) that $Q_1 = d - a^2 = 2$.

(i) The definition of $d'(t)$ and Theorem 3.6 [A] imply that the period of $\sqrt{d'(t)}$ is equal to l'. As $l' \geq 2$, $L' \geq 3$ and $q_3 = 2a^2 + 1 \geq 3$, we obtain

$$\begin{aligned}&(q_{L'}^2 - (Q_1/Q_0))(q_{L'+1} + q_{L'-1})q_{L'}t/q_2 + (q_{l'}/q_2) - 1\\ &\geq (q_3^2 - 2)(q_4 + q_2)q_3/q_2 \geq 1 = s - s_0.\end{aligned}$$

Therefore, we see by Remark 3.2 that $d'(t)$ is of minimal type for $\sqrt{d'(t)}$.

(ii) For brevity, we put $A_0 := (q_{L'+1} + q_{L'-1})^2 q_{L'}^2$ and $A_1 := 2(q_{L'+1} + q_{L'-1})^2$, and write $d'(t) = A_0t^2 + A_1t + d$. First, we assume that a is even. Then, we easily see by the definition of q_n that the parity of n coincides with that of q_n. Consequently, as $q_{L'+1} + q_{L'-1}$ is even, we have $A_0 \equiv A_1 \equiv 0$, so that $d'(t) \equiv d \equiv 2 \bmod 4$. Next, we assume that a is odd. Then, we easily see that

$$q_{4k} \equiv 0, \quad q_{4k+1} \equiv q_{4k+2} \equiv q_{4k+3} \equiv 1 \mod 2$$

for any integer $k \geq 0$. As L' is odd, this yields that both $q_{L'}$ and $q_{L'+1} + q_{L'-1}$ are odd. Consequently, $A_0 \equiv 1$, $A_1 \equiv 2$, so that $d'(t) \equiv t^2 + 2t + 3 \bmod 4$, which gives the assertion (ii).

(iii) As $d'(t) \equiv 2, 3 \bmod 4$, $a \geq 2$ and $m_d = 1$, Proposition 3.10 implies that $c_e - 1 \leq m_{d'(t)} \leq 3c_e$. By the definition of c_e, we obtain

$$m_{d'(t)} \geq c_e - 1 \geq c_1 - 1 = 4a^4 + 8a^2 + 2 > m.$$

This proves our proposition. □

4.2. The case where $l = 4$. For brevity we put $l' := 4(2e+1)$ and $L' := 2(2e+1)$.

Proposition 4.2. *Let e and m be any positive integers, u any integer ≥ 0, and a any positive integer such that $16a - 1 > m$. We put*

$$d := \{(8a^2 + 6a + 1)u + 8a^2 + 4a + 1\}^2 + (4a + 2)u + 4a + 1,$$

and define positive integers q_n, $n \geq 1$ by using partial quotients appeared in the periodic part of the continued fraction expansion

$$\sqrt{d} = [(8a^2 + 6a + 1)u + 8a^2 + 4a + 1, \overline{4a + 1, (4a + 1)u + 4a, 4a + 1, (16a^2 + 12a + 2)u + 16a^2 + 8a + 2}]$$

and the recurrence equation (2.1). *For any positive integer t, we put*

$$d'(t) := (2a + 1)^2(q_{4e+3} + q_{4e+1})^2 q_{4e+2}^2 t^2 + 2(2a + 1)^2(q_{4e+3} + q_{4e+1})^2 t + d.$$

Then the following hold.

(i) *Each $d'(t)$ is a positive integer with period $4(2e + 1)$ of minimal type for $\sqrt{d'(t)}$.*

(ii) *When u is even, we have $d'(t) \equiv 2 \bmod 4$. When u is odd, if t is even then $d'(t) \equiv 3 \bmod 4$, and if t is odd then $d'(t) \equiv 2 \bmod 4$.*

(iii) *For all positive integers t, we have $16ac_e - 1 \leq m_{d'(t)} \leq (16a+2)c_e$. Also, $m_{d'(t)} > m$.*

Proof. We know from [4, Proposition 5.2 (i)] that when u is odd (resp. even), Case (I) (resp. Case (II)) occurs for the symmetric string of positive integers $4a + 1$, $(4a+1)u+4a$, $4a+1$, and the continued fraction expansion of $\sqrt{d}$ has the above form. Also, when u is even (resp. odd), we have $d \equiv 2$ (resp., $\equiv 3$) mod 4, $s_0 = u + 1$, and $m_d = 16a$ (without the assumption that d is square-free). Furthermore, d is of minimal type for $\sqrt{d}$. For brevity, we put $a_0 := (8a^2+6a+1)u+8a^2+4a+1$. By (2.16) and (2.17), we have $P_1 = a_0$ and $Q_1 = d - a_0^2 = (4a+2)u+4a+1$, so that $P_2 = (4a+1)Q_1 - P_1$ and

$$\begin{aligned} Q_2 &= 1 + (4a+1)(P_1 - P_2) = 1 + 2(4a+1)P_1 - (4a+1)^2 Q_1 \\ &= 1 + 2(4a+1)\{(8a^2+6a+1)u + 8a^2 + 4a + 1\} \\ &\quad - (4a+1)^2\{(4a+2)u + 4a + 1\} \\ &= 1 + (4a+1) = 4a+2. \end{aligned}$$

Thus, $Q_2/2 = 2a+1$.

(i) The definition of $d'(t)$ and Theorem 3.6 [A] imply that the period of $\sqrt{d'(t)}$ is equal to l'. As $l' \geq 4$, $L' \geq 2$ and $q_2 = 4a+1$, we obtain

$$\begin{aligned} &(q_{L'}^2 - (Q_2/Q_0))(q_{L'+1} + q_{L'-1})q_{L'}t/q_4 + (q_{l'}/q_4) - 1 \\ &\geq (q_2^2 - 4a - 2)(q_3 + q_1)q_2/q_4 > 0 = s - s_0. \end{aligned}$$

Therefore, we see by Remark 3.2 that $d'(t)$ is of minimal type for $\sqrt{d'(t)}$.

(ii) For brevity, we put $A_0 := (2a+1)^2(q_{L'+1}+q_{L'-1})^2 q_{L'}^2$ and $A_1 := 2(2a+1)^2(q_{L'+1}+q_{L'-1})^2$, and write $d'(t) = A_0t^2 + A_1t + d$. By the definition of q_n, we easily see that

$$\begin{aligned} &q_{8k} \equiv 0,\ q_{8k+1} \equiv q_{8k+2} \equiv 1,\ q_{8k+3} \equiv u+1, \\ &q_{8k+4} \equiv u,\ q_{8k+5} \equiv u+1,\ q_{8k+6} \equiv q_{8k+7} \equiv 1 \bmod 2 \end{aligned}$$

for any integer $k \geq 0$. Consequently, as $L' \equiv 2 \bmod 4$, $q_{L'}$ is odd and $q_{L'+1} + q_{L'-1} \equiv u \bmod 2$. First, we assume that u is even. Then, as $q_{L'+1} + q_{L'-1}$ is even, we have $A_0 \equiv A_1 \equiv 0$, so that $d'(t) \equiv d \equiv 2 \bmod 4$. Next, we assume that u is odd. Then, since both $q_{L'}$ and $q_{L'+1} + q_{L'-1}$ are odd, we have $A_0 \equiv 1$, $A_1 \equiv 2$, so that $d'(t) \equiv t^2 + 2t + 3 \bmod 4$, which gives the assertion (ii).

(iii) As $d'(t) \equiv 2, 3 \bmod 4$ and $m_d = 16a$, Proposition 3.10 implies that $16ac_e - 1 \leq m_{d'(t)} \leq (16a+2)c_e$. Hence we obtain $m_{d'(t)} \geq 16ac_e - 1 \geq 16a - 1 > m$, and our proposition is proved. □

Proposition 4.3. *Let e and m be any positive integers, u any integer ≥ 0, and a any odd integer such that $a > m+1$. We put*

$$d := \{(a^2+3a+2)u + a^2 + 2a + 2\}^2 + 4\{(a+2)u + a + 1\}$$

and define positive integers q_n, $n \geq 1$ by using partial quotients appeared in the periodic part of the continued fraction expansion

$$(1+\sqrt{d})/2 = \left[\frac{(a^2+3a+2)u + a^2 + 2a + 3}{2}, \overline{a+1, (a+1)u + a, a+1, (a^2+3a+2)u + a^2 + 2a + 2}\right]$$

and the recurrence equation (2.1). *For any positive integer t, we put*

$$d'(t) := (a+2)^2(q_{4e+3}+q_{4e+1})^2 q_{4e+2}^2 t^2 + 2(a+2)^2(q_{4e+3}+q_{4e+1})^2 t + d.$$

Then the following hold.

(i) *Each $d'(t)$ is a positive integer with period $4(2e+1)$ of minimal type for $(1+\sqrt{d'(t)})/2$, and $d'(t) \equiv 1 \bmod 4$ holds.*

(ii) *For all positive integers t, we have $ac_e - 1 \le m_{d'(t)} \le (a+2)c_e$. Also, $m_{d'(t)} > m$.*

Proof. We know from [4, Proposition 5.2 (ii)] that Case (III) occurs for the symmetric string of positive integers $a+1$, $(a+1)u+a$, $a+1$, and the continued fraction expansion of $(1+\sqrt{d})/2$ has the above form. Also, we have $s_0 = u+1$, and $m_d = a$ (without the assumption that d is square-free). Furthermore, d is of minimal type for $(1+\sqrt{d})/2$. If we put $a_0 := \{(a^2+3a+2)u + a^2 + 2a + 3\}/2$, then $d - (2a_0-1)^2 = 4\{(a+2)u+a+1\}$. Since $P_1 = 2a_0 - 1$ and

$$Q_1 = (d-1)/2 + a_0(1-P_1) = \{d-(2a_0-1)^2\}/2 = 2\{(a+2)u+a+1\}$$

from (2.16) and (2.17), we see that $P_2 = (a+1)Q_1 - P_1$ and

$$\begin{aligned} Q_2 &= 2 + (a+1)(P_1 - P_2) = 2 + 2(a+1)P_1 - (a+1)^2 Q_1 \\ &= 2 + 2(a+1)\{(a^2+3a+2)u + a^2 + 2a + 2\} \\ &\quad - 2(a+1)^2\{(a+2)u + a + 1\} \\ &= 2 + 2(a+1) = 2(a+2). \end{aligned}$$

Thus, $Q_2/2 = a+2$.

(i) The definition of $d'(t)$ and Theorem 3.6 [C] imply that the period of $(1+\sqrt{d'(t)})/2$ is equal to l' and $d'(t) \equiv 1 \bmod 4$. As $l' \ge 4$, $L' \ge 2$ and $q_2 = a+1$, we obtain

$$\begin{aligned} &(q_{L'}^2 - (Q_2/Q_0))(q_{L'+1}+q_{L'-1})q_{L'}t/q_4 + (q_{l'}/q_4) - 1 \\ &\ge (q_2^2 - a - 2)(q_3+q_1)q_2/q_4 > 0 = s - s_0. \end{aligned}$$

Therefore, we see by Remark 3.2 that $d'(t)$ is of minimal type for $(1+\sqrt{d'(t)})/2$.

(ii) As $m_d = a$, Proposition 3.10 implies that $ac_e - 1 \le m_{d'(t)} \le (a+2)c_e$. Hence we obtain $m_{d'(t)} \ge ac_e - 1 \ge a - 1 > m$, and our proposition is proved. □

4.3. Proof of Theorem 1.1. We denote by m_d and h_d the Yokoi invariant and the class number (in the wide sense) of a real quadratic field $\mathbb{Q}(\sqrt{d})$, respectively. We shall show the following by using Propositions 4.1 and 4.2.

Proposition 4.4. *Let l' be an even integer ≥ 4 which is not divisible by* 8, *and h and m any positive integers. Also, let $\delta = 2$ or* 3. *Then, there exist infinitely many real quadratic fields $\mathbb{Q}(\sqrt{d})$, $d \equiv \delta \bmod 4$ with period l' of minimal type such that $h_d > h$ and $m_d > m$.*

Also, we shall see by Proposition 4.3:

Proposition 4.5. *Let e, h and m be any positive integers. Then, there exist infinitely many real quadratic fields $\mathbb{Q}(\sqrt{d})$, $d \equiv 1 \bmod 4$ with period $4(2e+1)$ of minimal type such that $h_d > h$ and $m_d > m$.*

Hence, by Proposition 4.4, we find the existence of an infinite family of real quadratic fields $\mathbb{Q}(\sqrt{d})$ satisfying $d \equiv \delta \bmod 4$ which is asserted in Theorem 1.1. Furthermore, if we assume that the period is congruent to 4 modulo 8 then, by Proposition 4.5, we also find the existence of an infinite family of real quadratic fields $\mathbb{Q}(\sqrt{d})$ satisfying $d \equiv 1 \bmod 4$. To prove Propositions 4.4 and 4.5, we use the same argument in [4]. A theorem of Nagell [7, Section 2] yields:

Lemma 4.6 ([4] Proposition 6.1). *Let $f(x) = ax^2 + bx + c$ be a quadratic polynomial in $\mathbb{Z}[x]$ with $a > 0$. As $a > 0$, there is some integer t_1 for all integers $t \geq t_1$ such that $f(t) > 0$. We suppose that the discriminant $d(f) = b^2 - 4ac$ of $f(x)$ is not equal to* 0, *the greatest common divisor (a, b, c) is square-free, and there is some integer t for which $f(t) \not\equiv 0 \bmod 4$. Then, the set $\{f(t) \mid t \in \mathbb{Z},\ t \geq t_1\}$ contains infinite square-free elements.*

Yokoi [12, Theorem 1.1] and a theorem of Siegel (Narkiewicz [8, Theorem 8.14]) imply:

Lemma 4.7 ([4] Lemma 4.3). *We suppose that a sequence $\{d_n\}_{n\geq 1}$ of square-free positive integers is strictly monotonously increasing. Let m_{d_n} and h_{d_n} denote the Yokoi invariant and the class number of a real quadratic field $\mathbb{Q}(\sqrt{d_n})$, respectively. We assume that $m_{d_n} \geq 1$ for all $n \geq 1$ and the sequence $\{m_{d_n}\}_{n\geq 1}$ of positive integers is bounded. Then, the sequence $\{h_{d_n}\}_{n\geq 1}$ of positive integers is not bounded. Namely, for any positive integer h, there exist infinitely many numbers $n \geq 1$ such that $h_{d_n} > h$.*

We remark in Propositions 4.1, 4.2 and 4.3 that a sequence $\{m_{d'(t)}\}_{t\geq 1}$ of positive integers is bounded if an integer e is fixed and the continued fraction expansion of $\sqrt{d}$ or $(1+\sqrt{d})/2$ is given.

Proof of Proposition 4.4. When $2 \parallel l'$, as $l' > 2$, there is some positive integer e such that $l' = 2(2e+1)$, and when $2^2 \mid l'$, as $2^3 \nmid l'$, there is some integer $e \geq 0$ such that $l' = 4(2e+1)$. Then, since our proposition follows from [4, Proposition 5.2 (i)] if $e = 0$ ($l' = 4$), we may assume that $e > 0$.

(i) The case where $l' = 2(2e+1)$. We suppose that a is a positive integer such that $a \geq 2$ and $4a^4 + 8a^2 + 2 > m$. For any positive integer t, we let $d'(t)$ be a non-square positive integer as in Proposition 4.1. For brevity, we put $A_0 := (q_{2e+2} + q_{2e})^2 q_{2e+1}^2$, $A_1 := 2(q_{2e+2} + q_{2e})^2$ and $d := a^2 + 2$, and write $d'(t) = A_0 t^2 + A_1 t + d$. As $a^2 \equiv -2 \bmod d$, we easily see by induction in e that

$$q_{2e} \equiv (-1)^{e-1} ea, \quad q_{2e+1} \equiv (-1)^e (2e+1) \mod d.$$

Consequently, since $q_{2e+2} + q_{2e} \equiv (-1)^e a \bmod d$, we obtain $A_0 \equiv a^2(2e+1)^2$ and $A_1 \equiv 2a^2 \bmod d$, so that

$$g := (A_0, A_1, d) = (a^2(2e+1)^2, 2a^2, d).$$

If we assume that g has an odd prime divisor p, then $p \mid a$ from $p \mid 2a^2$. As $p \mid d$, we have $0 \equiv d \equiv 2 \bmod p$, and this is a contradiction. Hence, g is a power of 2. On the other hand, as $d \equiv 2, 3 \bmod 4$, $\mathrm{ord}_2(g) \leq \mathrm{ord}_2(d) \leq 1$. Here, $\mathrm{ord}_p(*)$ denotes the additive valuation on the rationals $\mathbb{Q}$ with $\mathrm{ord}_p(p) = 1$ for a prime number p. Therefore, $g = 1$ or 2. In particular, g is square-free. Also, Lemma 3.8 yields that the discriminant of a quadratic polynomial $d'(t)$ is not equal to 0, and we see by Proposition 4.1 (i) that $d'(t)$ is a positive integer with period l' of minimal type for $\sqrt{d'(t)}$.

First, we take an even integer a. By Proposition 4.1 (ii) and (iii), we have $d'(t) \equiv 2 \bmod 4$ and $m_{d'(t)} > m$. In particular, there is some integer t for which $d'(t) \not\equiv 0 \bmod 4$. Hence, Lemma 4.6 implies that the set $\{d'(t) \mid t \in \mathbb{N}\}$ contains infinite square-free elements. Consequently, as $A_0 > 0$, we can choose a sequence $\{d_n\}_{n\geq 1}$ of square-free positive integers which is strictly monotonously increasing such that $d_n \equiv 2 \bmod 4$ and $m_{d_n} > m$. Since the sequence $\{m_{d_n}\}_{n\geq 1}$ of positive integers is bounded by Proposition 4.1 (iii), we see by Lemma 4.7 that $\{h_{d_n}\}_{n\geq 1}$ is not bounded. Therefore we obtain the assertion for $\delta = 2$.

Next, we take an odd integer a. Furthermore, we take an even integer t, and write $t = 2u$ with some $u \in \mathbb{N}$. Since the discriminant of a quadratic polynomial $d'(2u)$ in $\mathbb{Z}[u]$ is equal to the product of 2^2 and that of $d'(t)$, it is not equal to 0. By Proposition 4.1 (ii) and (iii), we have $d'(2u) \equiv 3 \bmod 4$ and $m_{d'(2u)} > m$. (In particular, there is some integer u for which $d'(2u) \not\equiv 0 \bmod 4$.) As $d = a^2 + 2$ is odd, the greatest common divisor of coefficients of a quadratic polynomial $d'(2u) = 4A_0 u^2 + 2A_1 u + d$ is equal to $(4A_0, 2A_1, d) = g = 1$. Hence, Lemma 4.6 implies that the set $\{d'(2u) \mid u \in \mathbb{N}\}$ contains infinite square-free elements. Consequently, we can choose a sequence $\{d_n\}_{n\geq 1}$ of square-free positive integers which is strictly monotonously increasing such that $d_n \equiv 3 \bmod 4$ and $m_{d_n} > m$. Similarly, Proposition 4.1 (iii) and Lemma 4.7 yield the assertion for $\delta = 3$.

(ii) The case where $l' = 4(2e+1)$. We suppose that u is an integer ≥ 0 and a is a positive integer such that $16a - 1 > m$ and $2a + 1$ is square-free. For any positive integer

t, we let $d'(t)$ be a non-square positive integer as in Proposition 4.2. For brevity, we put $L' := 2(2e+1)$, $A_0 := (2a+1)^2(q_{L'+1}+q_{L'-1})^2 q_{L'}^2$ and $A_1 := 2(2a+1)^2(q_{L'+1}+q_{L'-1})^2$, and write $d'(t) = A_0 t^2 + A_1 t + d$. Also, we put $a_2 := (4a+1)u+4a$. We see by the proof of Proposition 4.2 that $Q_2/2 = 2a+1$, and the proof of (3.5) in Lemma 3.1 yields that $P_3 = P_2 = a_2 Q_2/2$. Therefore, by (2.18),

$$2d = 2P_3^2 + 2Q_2Q_3 = Q_2\left(a_2^2\frac{Q_2}{2} + 2Q_3\right),$$

so that we obtain a factorization of d: $d = (Q_2/2)\Delta$. Here, we put

$$\Delta := a_2^2(Q_2/2) + 2Q_3. \tag{4.1}$$

If we put $g := (A_0, A_1, d)$ and

$$g' := ((Q_2/2)(q_{L'+1}+q_{L'-1})^2 q_{L'}^2,\ 2(Q_2/2)(q_{L'+1}+q_{L'-1})^2,\ \ \Delta)$$

then, as $d = (Q_2/2)\Delta$, we have

$$g = (Q_2/2)g'. \tag{4.2}$$

We look for g. The proof of Lemma 3.1 implies that $Q_{L'} = Q_2$. As L' is even, it follows from Lemma 2.7 that

$$G_{L'}^2 - dq_{L'}^2 = Q_2Q_0.$$

Since $G_{L'} = (Q_2/2)(q_{L'+1}+q_{L'-1})$ from (3.5) and $d = (Q_2/2)\Delta$, we obtain $(Q_2/2)(q_{L'+1}+q_{L'-1})^2 - \Delta q_{L'}^2 = 2Q_0$, so that

$$(Q_2/2)(q_{L'+1}+q_{L'-1})^2 \equiv 2Q_0 \mod \Delta.$$

If we assume that g' has an odd prime divisor p, then the definition of g' yields that $p \mid \Delta$, so that

$$(Q_2/2)(q_{L'+1}+q_{L'-1})^2 \equiv 2Q_0 \mod p.$$

Also, since $p \mid 2(Q_2/2)(q_{L'+1}+q_{L'-1})^2$ and p is odd, we have $0 \equiv 2Q_0 \bmod p$. As $Q_0 = 1$, we obtain $p = 2$ and this is a contradiction. Thus, g' is a power of 2. Also, the proof of Proposition 4.2 implies that $P_1 = a_0 \equiv (2a+1)u+1$, $Q_1 \equiv 2u+1$, $P_2 \equiv Q_1 - P_1 \equiv (2a+1)u$ and $Q_2 \equiv 2 \bmod 4$. Consequently, we see by (2.16) and (2.17) that $P_3 = a_2Q_2 - P_2 \equiv (2a+1)u$ and $Q_3 \equiv Q_1 + u(P_2 - P_3) \equiv 2u+1 \bmod 4$. As $Q_2/2 = 2a+1$, (4.1) yields that

$$\Delta \equiv u^2(2a+1) + 2 \mod 4.$$

When u is even, $\Delta \equiv 2 \bmod 4$. As we have seen in the proof of Proposition 4.2, $q_{L'+1} + q_{L'-1}$ is even. Therefore, since g' is a power of 2, we have $g' = 2$ by the definition of g'. We see by (4.2) that $g = 2(2a+1)$. When u is odd, $\Delta \equiv 2a+3 \bmod 4$, so that Δ is odd. Therefore, g' is also odd from the definition of g'. Since g' is a power of 2, we have $g' = 1$. We see by (4.2) that $g = 2a + 1$. Thus, g is square-free by our assumption. Also, Lemma 3.8 yields that the discriminant of a quadratic polynomial $d'(t)$ is not equal to 0, and we see by Proposition 4.2 (i) that $d'(t)$ is a positive integer with period l' of minimal type for $\sqrt{d'(t)}$.

First, we take an even integer u. By Proposition 4.2 (ii) and (iii), we have $d'(t) \equiv 2 \bmod 4$ and $m_{d'(t)} > m$. In particular, there is some integer t for which $d'(t) \not\equiv 0 \bmod 4$. Hence, Lemma 4.6 implies that the set $\{d'(t) \mid t \in \mathbb{N}\}$ contains infinite square-free elements. Consequently, we can choose a sequence $\{d_n\}_{n\geq 1}$ of square-free positive integers which is strictly monotonously increasing such that $d_n \equiv 2 \bmod 4$ and $m_{d_n} > m$. Since the sequence $\{m_{d_n}\}_{n\geq 1}$ of positive integers is bounded by Proposition 4.2 (iii), we see by Lemma 4.7 that $\{h_{d_n}\}_{n\geq 1}$ is not bounded. Therefore we obtain the assertion for $\delta = 2$.

Next, we take an odd integer u. Furthermore, we take an even integer t, and write $t = 2v$ with some $v \in \mathbb{N}$. Since the discriminant of a quadratic polynomial $d'(2v)$ in $\mathbb{Z}[v]$ is equal to the product of 2^2 and that of $d'(t)$, it is not equal to 0. By Proposition 4.2 (ii) and (iii), we have $d'(2v) \equiv 3 \bmod 4$ and $m_{d'(2v)} > m$. (In particular, there is some integer v for which $d'(2v) \not\equiv 0 \bmod 4$.) As $g = 2a + 1$ is odd, the greatest common divisor of coefficients of a quadratic polynomial $d'(2v) = 4A_0v^2 + 2A_1v + d$ is equal to $(4A_0, 2A_1, d) = g = 2a + 1$. Hence, Lemma 4.6 implies that the set $\{d'(2u) \mid u \in \mathbb{N}\}$ contains infinite square-free elements. Consequently, we can choose a sequence $\{d_n\}_{n\geq 1}$ of square-free positive integers which is strictly monotonously increasing such that $d_n \equiv 3 \bmod 4$ and $m_{d_n} > m$. Similarly, Proposition 4.2 (iii) and Lemma 4.7 yield the assertion for $\delta = 3$. Our proposition is proved. □

Proof of Proposition 4.5. We suppose that u is an integer ≥ 0 and a is a positive odd integer such that $a > m + 1$ and $a + 2$ is square-free. For any positive integer t, we let $d'(t)$ be a non-square positive integer as in Proposition 4.3. For brevity, we put $L' := 2(2e + 1)$, $A_0 := (a + 2)^2(q_{L'+1} + q_{L'-1})^2 q_{L'}^2$ and $A_1 := 2(a + 2)^2(q_{L'+1} + q_{L'-1})^2$, and write $d'(t) = A_0t^2 + A_1t + d$. Also, we put $g := (A_0, A_1, d)$ and $a_2 := (a+1)u + a$. We see by the proof of Proposition 4.3 that $Q_2/2 = a + 2$. If we put $\Delta := a_2^2(Q_2/2) + 2Q_3$ and

$$g' := ((Q_2/2)(q_{L'+1} + q_{L'-1})^2 q_{L'}^2,\ 2(Q_2/2)(q_{L'+1} + q_{L'-1})^2,\ \Delta)$$

then the argument in the proof of Proposition 4.4 (ii) implies that $d = (Q_2/2)\Delta$, $g = (Q_2/2)g'$, and (as $Q_0 = 2$) g' is a power of 2. Since $d \equiv 1 \bmod 4$, Δ is odd. By the definition of g', g' is also odd so that $g' = 1$. Consequently, we have $g = a + 2$ so that g is square-free by our assumption. Also, Lemma 3.8 yields that the discriminant of a quadratic polynomial $d'(t)$ is not equal to 0, and we see by Proposition 4.3 (i) and

(ii) that $d'(t)$ is a positive integer with period l' of minimal type for $(1+\sqrt{d'(t)})/2$, $d'(t) \equiv 1 \bmod 4$, and $m_{d'(t)} > m$. (In particular, there is some integer t for which $d'(t) \not\equiv 0 \bmod 4$.) Hence, Lemma 4.6 implies that the set $\{d'(t) \mid t \in \mathbb{N}\}$ contains infinite square-free elements. Consequently, we can choose a sequence $\{d_n\}_{n\geq 1}$ of square-free positive integers which is strictly monotonously increasing such that $d_n \equiv 1 \bmod 4$ and $m_{d_n} > m$. Since the sequence $\{m_{d_n}\}_{n\geq 1}$ of positive integers is bounded by Proposition 4.3 (ii), we see by Lemma 4.7 that $\{h_{d_n}\}_{n\geq 1}$ is not bounded. This proves our proposition. □

REMARK 4.1. We begin with quadratic irrationals ω with period 8, and then by using the above argument, it may be possible to find the existence of an infinite family of real quadratic fields with even period ≥ 4 which is not divisible by 16 satisfying the same property as in Theorem 1.1. However, that is a open problem.

5. Numerical examples

In this section, we give numerical examples of Propositions 4.1, 4.2 and 4.3 in Tables 1, 2 and 3, respectively. In the beginning of each table below, the symbol $*$ in the values of t means that $d'(t)$ has a square factor. Then the class number is not given. (In fact, as we have mentioned in Definition 3.1, for any non-square positive integer d, we can give the definition as the class number of a certain (not necessary maximal) order in "a real quadratic field $\mathbb{Q}(\sqrt{d})$".) Also, a factorization of $d'(t)$ into prime numbers is symbolically written in the last term of each table. There the notations p, q and $p_1, p_2, \ldots$ denote distinct prime numbers, and they satisfy $p < q$ and $p_1 < p_2 < \cdots$. Since these values and the values of a'_0 and a' in the footnote are relatively large, we do not give them explicitly. In particular, we note that the values of $m_{d'(t)}$ are constant as we have stated in Remark 3.3.

Table 1. $e = 4$, $m = 397$, $a = 3$ $d = 11$, $l = 2$, $\sqrt{11} = [3, \overline{3, 6}]$, $s_0 = 1$, $s = 2$, Case (I), $h_d = 1$, $m_d = 1$.

t	$d'(t)$	$h_{d'}$	$m_{d'}$	s' $(= s'_0)$	factorization of $d'(t)$
1	5694692744076689288198	586731780	45506014561	54501530706758363042	$p_1 p_2 p_3 p_4$
2	22778770975305624832403	1500801728	45506014561	109003061411121371042	$p_1 p_2 p_3$
3	51252234693686806632626	3796614660	45506014561	163504592115484379042	$p_1 p_2 p_3 p_4$
4	91115083899220234688867	2241483780	45506014561	218006122819847387042	$p_1 p_2 p_3$
5	142367318591905909001126	5939930848	45506014561	272507653524210395042	$p_1 p_2 p_3 p_4$
6	205008938771743829569403	4039479852	45506014561	327009184228573403042	pq
7	279039944438733996393698	4437850032	45506014561	381510714932936411042	$p_1 p_2 p_3$
8	364460335592876409474011	6691740720	45506014561	436012245637299419042	$p_1 p_2 p_3 p_4$
9	461270112234171068810342	8133745152	45506014561	490513776341662427042	$p_1 p_2 p_3$
10	569469274362617974402691	11140664040	45506014561	545015307046025435042	$p_1 p_2 p_3 p_4$
11	689057821978217126251058	9049583040	45506014561	599516837750388443042	$p_1 p_2 p_3 p_4 p_5 p_6 p_7$
12	820035755080968524355443	8329322828	45506014561	654018368454751451042	pq
13	962403073670872168715846	13437783832	45506014561	708519899159114459042	$p_1 p_2 p_3$
14	1116159777747928059332267	8932263352	45506014561	763021429863477467042	$p_1 p_2 p_3$
15	1281305867312136196204706	14029074272	45506014561	817522960567840475042	$p_1 p_2 p_3 p_4$
16	1457841342363496579333163	12262575704	45506014561	872024491272203483042	$p_1 p_2 p_3 p_4$
17	1645766202902009208717638	8326036656	45506014561	926526021976566491042	$p_1 p_2 p_3 p_4$
18	1845080448927674084358131	24836590641	45506014561	981027552680929499042	p
19	2055784080440491206254642	12565341686	45506014561	1035529083385292507042	$p_1 p_2 p_3$
20	2277877097440460574407171	30966590388	45506014561	1090030614089655515042	$p_1 p_2 p_3$

$l = 18$, Case (I), $\sqrt{d'(t)} = [a_0{}', \overline{3, 6, 3, 6, 3, 6, 3, 6, a', 6, 3, 6, 3, 6, 3, 6, 3, 2a_0{}'}]$.
Distinct prime numbers p, q and p_i satisfy $p < q$ and $p_1 < p_2 < \cdots$.

Table 2. $e = 1$, $m = 14$, $u = 1$, $a = 1$ $d = 795$, $l = 4$, $\sqrt{795} = [28, \overline{5, 9, 5, 56}]$, $s_0 = 2$, $s = 2$, Case (I), $h_d = 4$, $m_d = 16$.

t	$d'(t)$	$h_{d'}$	$m_{d'}$	s' $(= s_0')$	factorization of $d'(t)$
1	15328651059393793906782	1941840102	2927366816	1359148436947933929	$p_1 p_2 p_3$
2	61314604223611635909219	5178887184	2927366816	2718296873586340896	$p_1 p_2 p_3 p_4 p_5$
3	137957859492653526008106	6180051072	2927366816	4077445310224747863	$p_1 p_2 p_3 p_4 p_5$
4	245258416866519464203443	5438690864	2927366816	5436593746863154830	pq
5	383216276345209450495230	7979517984	2927366816	6795742183501561797	$p_1 p_2 p_3 p_4 p_5$
6	551831437928723484883467	8923863728	2927366816	8154890620139968764	$p_1 p_2 p_3 p_4$
7	751103901617061567368154	21121124856	2927366816	9514039056778375731	$p_1 p_2 p_3 p_4 p_5$
8	981033667410223697949291	13950612192	2927366816	10873187493416782698	$p_1 p_2 p_3 p_4 p_5$
9	1241620735308209876626878	16576692168	2927366816	12232335930055189665	$p_1 p_2 p_3 p_4 p_5$
10	1532865105311020103400915	24818176448	2927366816	13591484366693596632	$p_1 p_2 p_3 p_4 p_5$
11	1854766777418654378271402	16450950752	2927366816	14950632803332003599	$p_1 p_2 p_3 p_4 p_5 p_6$
12	2207325751631112701238339	25745388768	2927366816	16309781239970410566	$p_1 p_2 p_3 p_4 p_5$
13	2590542027948395072301726	25143543850	2927366816	17668929676608817533	$p_1 p_2 p_3$
14	3004415606370501491461563	22814342688	2927366816	19028078113247224500	pq
15*	3448946486897431958717850		2927366816	20387226549885631467	$p_1 p_2 p_3^2 p_4 p_5 p_6$
16	3924134669529186474070587	21586636896	2927366816	21746374986524038434	$p_1 p_2 p_3$
17	4429980154265765037519774	32048761984	2927366816	23105523423162445401	$p_1 p_2 p_3 p_4 p_5 p_6$
18	4966482941107167649065411	44724503880	2927366816	24464671859800852368	$p_1 p_2 p_3 p_4$
19	5533643030053394308707498	31120884336	2927366816	25823820296439259335	$p_1 p_2 p_3 p_4 p_5 p_6$
20	6131460421104445016446035	52617867776	2927366816	27182968733077666302	$p_1 p_2 p_3 p_4 p_5$

$l = 12$, Case (I), $\sqrt{d'(t)} = [a_0', \overline{5, 9, 5, 56, 5, a', 5, 56, 5, 9, 5, 2a_0'}]$. The symbol * in the values of t means that $d'(t)$ has a square factor, and distinct prime numbers p, q and p_i satisfy $p < q$ and $p_1 < p_2 < \cdots$.

Table 3. $e = 2$, $m = 1$, $u = 1$, $a = 3$, $d = 1405$, $l = 4$, $(1 + \sqrt{1405})/2 = [19, \overline{4, 7, 4, 37}]$, $s_0 = 2$, $s = 2$, Case (III), $h_d = 2$, $m_d = 3$.

t	$d'(t)$	$h_{d'}$	$m_{d'}$	s' $(= s'_0)$	factorization of $d'(t)$
1	60319534423282785183184709635126405	1006920891909546	1310451112713603	4542449226015418508544687950672	pq
2	241278137693131103909062571288251405	2140128310867456	1310451112713603	9084898452030836323824410585822	$p_1 p_2 p_3 p_4 p_5 p_6 p_7$
3	542875809809544956177633584959376405	4087239709328032	1310451112713603	13627347678046254139104133220972	$p_1 p_2 p_3 p_4 p_5 p_6$
4	965112550772524341988897750648501405	3395111490225808	1310451112713603	18169796904061671954383855856122	$p_1 p_2 p_3 p_4 p_5$
5	1507988360582069261342855068355626405	5646542135303784	1310451112713603	22712246130077089769663578491272	$p_1 p_2 p_3 p_4$
6	2171503239238179714239505538080751405	6262132059357472	1310451112713603	27254695356092507584943301126422	$p_1 p_2 p_3 p_4 p_5 p_6$
7	2955657186740855700678849159823876405	5932897382333814	1310451112713603	31797144582107925400223023761572	pq
8	3860450203090097220660885933585001405	7120095324529294	1310451112713603	36339593808123343215502746396722	pq
9	4885882288285904274185615859364126405	10666555286013088	1310451112713603	40882043034138761030782469031872	$p_1 p_2 p_3 p_4$
10*	6031953442328276861253038937161251405		1310451112713603	45424492260154178846062191667022	$p_1 p_2^2 p_3 p_4 p_5 p_6$
11	7298663665217214981863155166976376405	8742568815707016	1310451112713603	49966941486169596661341914302172	$p_1 p_2 p_3$
12	8686012956952718636015964548809501405	11975429794558924	1310451112713603	54509390712185014476621636937322	$p_1 p_2 p_3$
13	10194001317534787823711467082660626405	19734644954653392	1310451112713603	59051839938200432291901359572472	$p_1 p_2 p_3 p_4$
14	11822628746963422544949662768529751405	8684679689795424	1310451112713603	63594289164215850107181082207622	$p_1 p_2 p_3 p_4$
15	13571895245238622799730551606416876405	14110791881487128	1310451112713603	68136738390231267922460804842772	$p_1 p_2 p_3$
16	15441800812360388588054133596322001405	15184541144140632	1310451112713603	72679187616246685737740527477922	$p_1 p_2 p_3 p_4$
17	17432345448328719909920408738245126405	16096720191517056	1310451112713603	77221636842262103553020250113072	$p_1 p_2 p_3 p_4$
18	19543529153143616765329377032186251405	16362586409444832	1310451112713603	81764086068277521368299972748222	$p_1 p_2 p_3 p_4 p_5$
19	21775351926805079154281038478145376405	14816081371825224	1310451112713603	86306535294292939183579695383372	$p_1 p_2 p_3$
20	24127813769313107076775393076122501405	25636583247086656	1310451112713603	90848984520308356998859418018522	$p_1 p_2 p_3 p_4$

$l = 20$, Case (III), $(1 + \sqrt{d'(t)})/2 = [a_0{}', \overline{4, 7, 4, 37, 4, 7, 4, 37, 4, a', 4, 37, 4, 7, 4, 37, 4, 7, 4, 2a_0{}' - 1}]$.
The symbol * in the values of t means that $d'(t)$ has a square factor, and distinct prime numbers p, q and p_i satisfy $p < q$ and $p_1 < p_2 < \cdots$.

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