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THE MIYAZAWA POLYNOMIAL OF PERIODIC VIRTUAL LINKS

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Abstract

In this paper, we investigate the behavior of the Miyazawa polynomial of periodic virtual links. As applications, we give some criteria to detect possible periods of a given oriented virtual link.

1. Introduction

A classical link L in S^3 is called a *p-periodic link* ($p \geq 2$ an integer) if there exists an orientation preserving auto-homeomorphism h of S^3 such that $h(L) = L$, h is of order p and the set of fixed points of h is a circle disjoint from L. In this case, $L_* = L/\langle h \rangle$ is called the *factor link* of L. A link diagram D in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ is said to *have period p* if there exists a rotation ϕ of $\mathbb{R}^2$ about the origin $\mathbf{0}$ through $2\pi/p$ such that $\phi(D) = D$. It is well known that every p-periodic link has a diagram of period p.

In 1988, Murasugi [10] found some relationships between the Jones polynomials of a periodic link and its factor link and showed that the knot 10_{105} has no period. In 1990, Traczyk [13] gave a periodicity criterion for links in S^3 by mapping Kauffman's bracket polynomial homomorphically into the group ring over Z_p of a cyclic group C_{p^n} of order p^n (p a prime), and proved that the knots 10_{101} and 10_{105} have no period seven. In addition, several people found criteria to detect possible periods for an oriented link by using polynomial invariants [1, 6, 7, 9, 11, 12, 14, 15, 16].

In 1996, Kauffman introduced the concept of a virtual link [5]. A *virtual link diagram* is a link diagram in $\mathbb{R}^2$ possibly with some encircled crossings without over/under information. Such an encircled crossing is called a *virtual crossing*. Fig. 1 shows an example of a virtual link diagram. If two virtual link diagrams are related by a finite sequence of generalized Reidemeister moves as described in Fig. 2, they are said to be *equivalent*. A *virtual link* is defined to be an equivalence class of virtual link diagrams.

In [5], Kauffman defined a polynomial invariant $f_L \in \mathbb{Z}[A^{\pm 2}]$ for a virtual link L which we call the *Jones-Kauffman polynomial*. For a classical link L, it is equal to the Jones polynomial $V_L(t)$ after substituting $\sqrt{t}$ for A^2. In 2005, Kamada and Miyazawa [4] introduced the concept of virtual magnetic graph diagrams and defined a 2-variable polynomial invariant for a virtual link derived from virtual magnetic graph diagrams. In [8], Miyazawa defined a virtual link invariant, which generalizes the Jones-Kauffman

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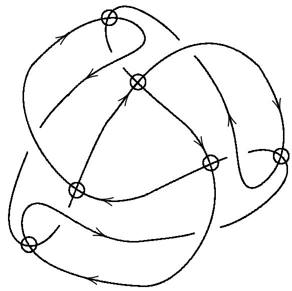
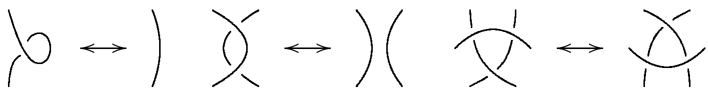
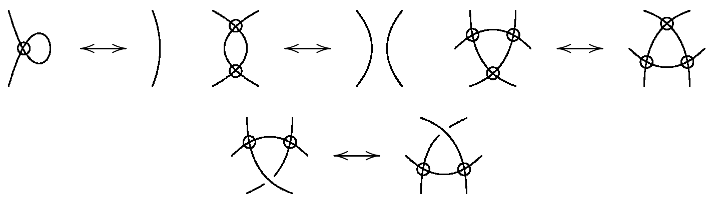


Fig. 1. A virtual link diagram.



Classical Reidemeister moves



Virtual Reidemeister moves

Fig. 2. Generalized Reidemeister moves.

polynomial and the 2-variable polynomial invariant. In [3], Kamada gave some relations of the 2-variable polynomial invariant for a virtual skein triple.

In this paper, we investigate the behavior of the Miyazawa polynomial of periodic virtual links. As applications, we give some criteria to detect possible periods of a given oriented virtual link.

2. The Miyazawa polynomial

In this section, we review the Miyazawa polynomial of a virtual link [3, 4, 8].

Let G be an oriented 2-valent graph in S^3. G is called *magnetic* if the edges of G are oriented alternately as in Fig. 3. We allow G to have components consisting of closed edges without vertices. A *magnetic graph diagram* of a magnetic graph G is a projection image of G on a plane equipped with over/under information on each crossing as in Fig. 4. A *virtual magnetic graph diagram* (or shortly *VMG diagram*) is a magnetic graph diagram possibly with some virtual crossings as in Fig. 5. Two VMG

Fig. 3.

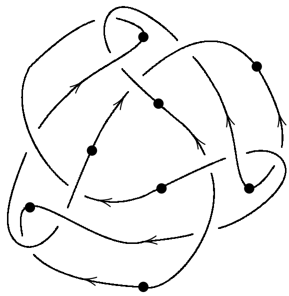


Fig. 4. A magnetic graph diagram.

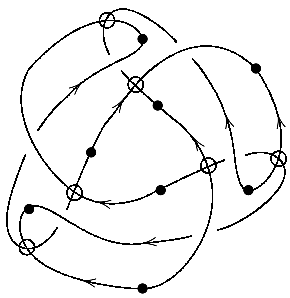


Fig. 5. A virtual magnetic graph diagram.

diagrams are said to be *equivalent* if they are related by a finite sequence of generalized Reidemeister moves. We note that virtual link diagrams are VMG diagrams without vertices. For a VMG diagram D, we denote the sum of the signs of real crossings of D by $w(D)$. It is called the *writhe* of D. A *pure VMG diagram* is a VMG diagram whose crossings are all virtual.

Let D be a pure VMG diagram and $E(D)$ the set of edges of D. A *weight map* of D is a map $f\colon E(D) \to \{+1, -1\}$ such that the product of images of two adjacent edges by f is -1. We denote the set of weight maps of D by $\mathrm{WM}(D)$. For a weight map f of D, we denote D_f a pure VMG diagram of which each edge is labeled its weight as in Fig. 6. It is called a *weighted diagram* corresponding to f. If c is a virtual crossing of a weighted diagram D_f, there exist two types of virtual crossings on D_f. If the product of weights of two edges which intersect at c is +1 (resp. -1), c is called a *regular crossing* (resp. *irregular crossing*).

Let D be a pure VMG diagram and f a weight map of D. Let c be an irregular virtual crossing of D_f. Suppose that c is formed with two edges e_1 and e_{-1} whose

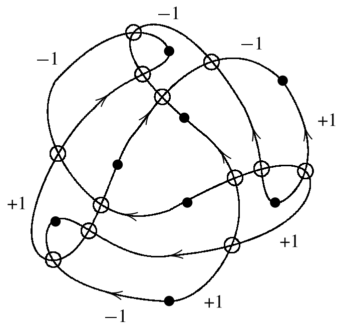


Fig. 6. A weighted pure VMG diagram.

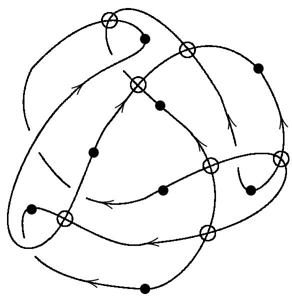


Fig. 7. The raised diagram of the diagram in Fig. 6.

weights are +1 and −1, respectively. Then c can be replaced with a real crossing $\hat{c}$ so that the edges e_1 and e_{-1} are changed into the overpath and the underpath at $\hat{c}$, respectively. Such a replacement is called a *raise* of an irregular crossing. The *raised diagram* of D with respect to f, which is denoted by $\hat{D}_f$, is defined to be the VMG diagram obtained from the weighted diagram D_f by doing raises of all irregular crossings of D_f. For example, the raised diagram derived from the weighted diagram in Fig. 6 is given in Fig. 7.

For a pure VMG diagram D, let F_D be a map from $\mathrm{WM}(D)$ to $\mathbb{Z}$ defined by $F_D(f) = w(\hat{D}_f)$ for all weight map f of D. If we put $\mathrm{WM}_n(D) = \{f \in \mathrm{WM}(D) \mid F_D(f) = n\}$ for any integer n, then we have

Lemma 2.1. *For a pure VMG diagram D and an integer n, there exists a one-to-one correspondence between* $\mathrm{WM}_n(D)$ *and* $\mathrm{WM}_{-n}(D)$.

Proof. For a weight map f of D, we define a map $\tilde{f}$ from $E(D)$ to $\{+1, -1\}$ by

$$\tilde{f}(e) = -f(e), \quad \text{for all} \quad e \in E(D).$$

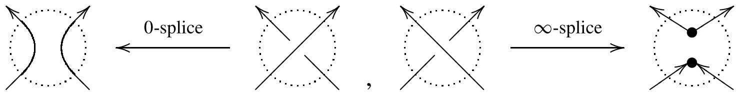


Fig. 8.

Then $\tilde{f}$ is also a weight map of D. Let c be a real crossing of the raised diagram $\hat{D}_f$ and $\tilde{c}$ the real crossing of the raised diagram $\hat{D}_{\tilde{f}}$ corresponding to c. Then $\mathrm{sign}(\tilde{c}) = -\mathrm{sign}(c)$ and hence $w(\hat{D}_{\tilde{f}}) = -w(\hat{D}_f)$. It follows that $\tilde{f} \in \mathrm{WM}_{-n}(D)$ if $f \in \mathrm{WM}_n(D)$. Now we define a map ϕ_n from $\mathrm{WM}_n(D)$ to $\mathrm{WM}_{-n}(D)$ by

$$\phi_n(f) = \tilde{f}, \quad \text{for all} \quad f \in \mathrm{WM}_n(D).$$

Then ϕ_n is well-defined. Since $\phi_{-n} \circ \phi_n$ and $\phi_n \circ \phi_{-n}$ are the identity maps, ϕ_n is a one-to-one correspondence between $\mathrm{WM}_n(D)$ and $\mathrm{WM}_{-n}(D)$. □

Let g be a map from $\mathbb{Z}$ to a Laurent polynomial ring $\mathbb{Z}[h^{\pm 1}]$. The *double bracket polynomial* $\langle\langle D\rangle\rangle_g$ of a pure VMG diagram D associated to g is a Laurent polynomial in $\mathbb{Z}[2^{-1}, h^{\pm 1}]$ defined by

$$\langle\langle D\rangle\rangle_g = 2^{-\mu(D)} \sum_{f \in \mathrm{WM}(D)} (g \circ F_D)(f).$$

If c is a real crossing of D, then there are two kinds of splices at c, which are called 0-*splice* and ∞-*splice* at c as in Fig. 8. A *state* of D is a pure VMG diagram obtained from D by doing 0-splice or ∞-splice at each real crossing of D. We denote the set of states of D by $\mathcal{S}(D)$. For a state s of D, let $C_0(D; s)$ (resp. $C_\infty(D; s)$) be the set of real crossings of D where 0-splices (resp. ∞-splices) are applied to obtain s from D. We put

$$P(D; s) = \sum_{c \in C_0(D;s)} \mathrm{sign}(c) - \sum_{c \in C_\infty(D;s)} \mathrm{sign}(c),$$

where $\mathrm{sign}(c)$ is the crossing sign of c.

Let D be a virtual link diagram of a virtual link L and $g\colon \mathbb{Z} \to \mathbb{Z}[h^{\pm 1}]$. In [8], Miyazawa gave a Laurent polynomial $H_{D,g}(A, h)$ (or briefly, $H(D, g)$) of D associated with g in $\mathbb{Z}[2^{-1}, A^{\pm 1}, h^{\pm 1}]$ defined by

$$H_{D,g}(A, h) = \sum_{s \in \mathcal{S}(D)} A^{P(D;s)} d^{\mu(s)-1} \langle\langle s\rangle\rangle,$$

where $d = -A^2 - A^{-2}$ and $\mu(s)$ is the number of components of s. The *Miyazawa polynomial* $R_{L,g}(A, h)$ (or briefly, $R(L, g)$) of L associated with g is a Laurent polynomial in $\mathbb{Z}[2^{-1}, A^{\pm 1}, h^{\pm 1}]$ defined by

$$R_{L,g}(A, h) = R_{D,g}(A, h) = (-A^3)^{-w(D)} H_{D,g}(A, h).$$

In [8], Miyazawa showed that $R_{L,g}(A, h)$ is a virtual link invariant and gave some properties.

Proposition 2.2 ([8]). (1) *If $g\colon \mathbb{Z} \to \mathbb{Z}[h^{\pm 1}]$ is defined by $g(n) = 1$, then $R(L, g)$ is identical with the Jones-Kauffman polynomial of L.*
(2) *If $g\colon \mathbb{Z} \to \mathbb{Z}[h^{\pm 1}]$ is defined by $g(n) = |n|$ and L is a classical link, then $R(L, g)$ is equal to zero.*
(3) *If $g\colon \mathbb{Z} \to \mathbb{Z}[h^{\pm 1}]$ is defined by $g(n) = h^{(1-(-1)^n)/2}$, then $R(L, g)$ coincides with the 2-variable polynomial defined by Kamada and Miyazawa.*
(4) *If $g\colon \mathbb{Z} \to \mathbb{Z}[h^{\pm 1}]$ is defined by $g(n) = h^n$ and $v(L)$ is the virtual crossing number of L, then $v(L) \geq \max \deg_h R(L, g)$.*

REMARK 2.3. In [8], Miyazawa used an arbitrary Laurent polynomial ring Γ over $\mathbb{Q}$ as the range of g. If $\Gamma = \mathbb{Q}[h^{\pm 1}]$, then $\langle\langle D \rangle\rangle_g \in \mathbb{Q}[h^{\pm 1}]$ and $R(L, g) \in \mathbb{Q}[h^{\pm 1}, A^{\pm 1}]$. Since the ideal of $\mathbb{Q}[h^{\pm 1}, A^{\pm 1}]$ generated by a non-zero integer is itself, our theorems in Section 3 are meaningless for $g\colon \mathbb{Z} \to \mathbb{Q}[h^{\pm 1}]$. On the other hand, the rage of g in propositions of [8] can be restricted in $\mathbb{Z}[h^{\pm 1}]$. Thus we can use the Laurent polynomial ring $\mathbb{Z}[h^{\pm 1}]$ as the range of g. Since the ideals in Section 3 are proper, our theorems are meaningful.

3. Periodic virtual links

An oriented virtual link L is said to *have period* $p \geq 2$ if it admits an oriented virtual link diagram D in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ that invariant under the rotation ζ of $\mathbb{R}^2$ about the origin $\mathbf{0}$ through $2\pi/p$. The virtual link L_* represented by the quotient $D/\langle\zeta\rangle$ is called the *factor link* of L. The diagram described in Fig. 1 is a virtual link diagram of a virtual link having period 3.

Theorem 3.1 (Fermat's little theorem, [2]). *If p is a prime and a an integer relatively prime to p, then*

$$a^{p-1} \equiv 1 \bmod p.$$

Theorem 3.2. *Let p be an odd prime and L a virtual link that has period p^r $(r \geq 1)$. Let g be a map from $\mathbb{Z}$ to $\mathbb{Z}[h^{\pm 1}]$.*
(1) *If $g\colon (\mathbb{Z}, +) \to (\mathbb{Z}[h^{\pm 1}], \cdot\,)$ is a homomorphism, then*

$$R(L, g) \equiv [R(L_*, g)]^{p^r} \bmod (p, (-A^2 - A^{-2})^{p-1} - 1).$$

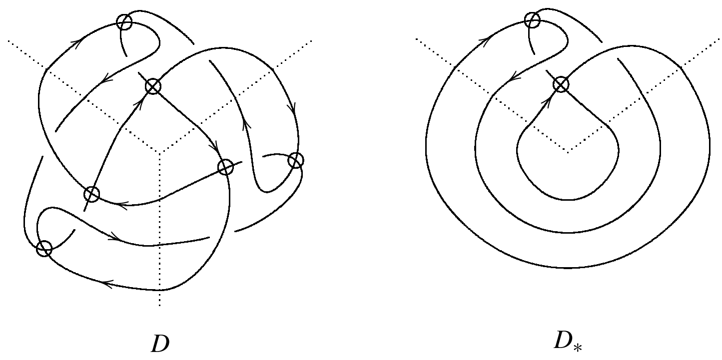


Fig. 9.

(2) *If* $g\colon \mathbb{Z} \to \mathbb{Z}[h^{\pm 1}]$ *is defined by* $g(n) = h^{(1-(-1)^n)/2}$*, then*

$$R(L, g) \equiv [R(L_*, g)]^{p^r} \bmod (p, (-A^2 - A^{-2})^{p-1} - 1, h^{p-1} - 1).$$

Proof. It suffices to prove the theorem for $r = 1$ (the theorem for $r > 1$ is proved by applying the argument for $r = 1$ repeatedly). Let D be a virtual link diagram of L in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ that invariant under the rotation ζ of $\mathbb{R}^2$ about the origin $\mathbf{0}$ through $2\pi/p$. Then D can be divided into p pieces $D_0, D_1, \ldots, D_{p-1}$ such that $\zeta(D_i) = D_{i+1}$ $(i = 0, 1, \ldots, p-1)$ and $D_p = D_0$. Let $I(0, 2\pi/p)$ be the closed domain bounded by two half lines $\theta = 0$ and $\theta = 2\pi/p$ in the polar coordinate system. We may assume that $D_0 = D \cap I(0, 2\pi/p)$. Let $A_1, A_2, \ldots, A_l$ be the points of intersection of D_0 and the line $\theta = 0$ and let $\zeta(A_i) = B_i$ $(i = 1, 2, \ldots, l)$. By joining A_i and B_i on $\mathbb{R}^2 \setminus I(0, 2\pi/p)$ by circle C_i centered 0, we obtain a diagram D_* of the factor link L_*. For example, see Fig. 9. For simplicity, we write $D_* = D/\zeta$. We note that the rotation $\zeta\colon \mathbb{R}^2 \to \mathbb{R}^2$ maps D onto itself preserving the sign of each crossing. If s is a state in $\mathcal{S}(D)$, then either $\zeta(s) \neq s$ or $\zeta(s) = s$.

If $\zeta(s) \neq s$, then $s, \zeta(s), \zeta^2(s), \ldots, \zeta^{p-1}(s)$ are all distinct. Since any two of these are isomorphic, we have p identical terms in $H(D, g)$, and they vanish by reducing modulo p.

If $\zeta(s) = s$, then s defines a unique quotient state s_* $(= s/\zeta)$. Let α and α_* be the terms in $H(D, g)$ and $H(D_*, g)$ which are associated with s and s_*, respectively. Since $\sum_{C_0(D;s)} \operatorname{sign}(c) = p \cdot \sum_{C_0(D_*;s_*)} \operatorname{sign}(c)$ and $\sum_{C_\infty(D;s)} \operatorname{sign}(c) = p \cdot \sum_{C_\infty(D_*;s_*)} \operatorname{sign}(c)$, we have

$$P(D; s) = p \cdot P(D_*; s_*).$$

Then we have that

$$\alpha = A^{p \cdot P(D_*;s_*)} d^{\mu(s)-1} \langle\langle s \rangle\rangle, \quad \alpha_* = A^{P(D_*;s_*)} d^{\mu(s_*)-1} \langle\langle s_* \rangle\rangle. \tag{3.1}$$

We will compare $\mu(s)-1$ and $\mu(s_*)-1$. Let $G=\{id,\zeta,\ldots,\zeta^{p-1}\}$ and $\mathcal{C}=\{C \mid C$ is a component of $s\}$, where id is the identity of $\mathbb{R}^2$. Then G acts on $\mathcal{C}$ by $\zeta^i \cdot C = \zeta^i(C)$. We put $\mathcal{C}_G=\{C\in\mathcal{C} \mid gC=C,\ \forall g\in G\}$ and $\mathcal{C}/G=\{G(C) \mid G(C)$ is the orbit of $C\in\mathcal{C}\}$. For a set S, we denote by $|S|$ the number of elements in S. If $\zeta^i(C)=C$ for some i $(1\le i\le p-1)$, then $\zeta^j(C)=C$ for all j because p is prime. Thus $|G(C)|=p$ or 1. We note that $|G(C)|=1$ if and only if $C\in\mathcal{C}_G$. Since $\mu(s_*)=|\mathcal{C}/G|$, we calculate that

$$\mu(s)=|\mathcal{C}|=|\mathcal{C}_G|+p(|\mathcal{C}/G|-|\mathcal{C}_G|)=p\cdot\mu(s_*)-(p-1)|\mathcal{C}_G|. \tag{3.2}$$

Since $\mu(s)-1=p(\mu(s_*)-1)-(p-1)(|\mathcal{C}_G|-1)$, we have that

$$d^{\mu(s)-1}\equiv d^{p\cdot(\mu(s_*)-1)} \bmod (d^{p-1}-1). \tag{3.3}$$

By Theorem 3.1 and (3.2), it follows that

$$2^{-\mu(s)}\equiv 2^{p\cdot(-\mu(s_*))} \bmod p. \tag{3.4}$$

Let f be a wight map of s. We define a weight map $\zeta(f)$ of s by, for each edge e of s,

$$\zeta(f)(e)=f(e') \quad \text{whenever} \quad \zeta(e')=e.$$

If $\zeta(f)\neq f$, then $f,\zeta(f),\ldots,\zeta^{p-1}(f)$ are all distinct but $\widehat{s_f},\widehat{s_{\zeta(f)}},\ldots,\widehat{s_{\zeta^{p-1}(f)}}$ are equivalent. Thus $w(\widehat{s_f})=w(\widehat{s_{\zeta(f)}})=\cdots=w(\widehat{s_{\zeta^{p-1}(f)}})$. If $\zeta(f)=f$, then f defines a unique weight map f_* $(=f/\zeta)$ of s_*. Let $\mathrm{WD}(s)$ denote the set of weighted diagram of s, that is, $\mathrm{WD}(s)=\{s_f \mid f\in\mathrm{WM}(s)\}$. Then G acts on $\mathrm{WD}(s)$ by

$$\zeta(s_f)=s_{\zeta(f)}.$$

We can put that $\mathrm{WD}(s)=\{s_{f_1},s_{f_2},\ldots,s_{f_m}\}\cup\{s_{f_{1,0}},s_{f_{1,1}},\ldots,s_{f_{1,p-1}}\}\cup\cdots\cup\{s_{f_{n,0}},s_{f_{n,1}},\ldots,s_{f_{n,p-1}}\}$ where $\zeta(s_{f_i})=s_{f_i}$ for all i $(1\le i\le m)$ and $f_{j,k}=\zeta^k(f_{j,0})$ for each $k=1,2,\ldots,p-1$, $j=1,2,\ldots,n$. We set that $w_i=w(\widehat{s_{f_i}})$ and $w_{j,k}=w(\widehat{s_{f_{j,k}}})$ for each $i=1,2,\ldots,m$, $k=1,2,\ldots,p-1$, $j=1,2,\ldots,n$ and set $w_i^*=w(\widehat{(s_*)_{(f_i)_*}})$ for each $i=1,2,\ldots,m$. For each $i=1,2,\ldots,m$, we have

$$w_i=p\cdot w_i^*. \tag{3.5}$$

For any map $g\colon\mathbb{Z}\to\mathbb{Z}[h^{\pm1}]$, we have that

$$\langle\langle s_*\rangle\rangle=2^{-\mu(s_*)}[g(w_1^*)+\cdots+g(w_m^*)]$$

and

$$\langle\langle s\rangle\rangle=2^{-\mu(s)}[g(w_1)+\cdots+g(w_m)+p\cdot g(w_{1,0})+\cdots+p\cdot g(w_{m,0})].$$

By (3.4) and (3.5), it follows that

$$\langle\langle s\rangle\rangle \equiv 2^{p\cdot(-\mu(s_*))}[g(p\cdot w_1^*)+\cdots+g(p\cdot w_m^*)] \bmod p.$$

(1) If $g\colon (Z,+)\to(\mathbb{Z}[h^{\pm1}],\ \cdot\ )$ is a homomorphism, then

$$\begin{aligned}\langle\langle s_*\rangle\rangle^p &= 2^{p\cdot(-\mu(s_*))}[g(w_1^*)+\cdots+g(w_m^*)]^p\\ &\equiv 2^{p\cdot(-\mu(s_*))}[g(w_1^*)^p+\cdots+g(w_m^*)^p] \bmod p\\ &\equiv 2^{p\cdot(-\mu(s_*))}[g(p\cdot w_1^*)+\cdots+g(p\cdot w_m^*)] \bmod p\\ &\equiv \langle\langle s\rangle\rangle \bmod p.\end{aligned} \tag{3.6}$$

By (3.1), (3.3) and (3.6), it follows that

$$\alpha_*^p \equiv \alpha \bmod (p, (-A^2-A^{-2})^{p-1}-1).$$

Hence we have

$$H(D,g)\equiv[H(D_*,g)]^p \bmod (p,(-A^2-A^{-2})^{p-1}-1).$$

Since $w(D)=p\cdot w(D_*)$, $(-A^3)^{-w(D)}=[(-A^3)^{-w(D_*)}]^p$. Therefore we have

$$R(L,g)\equiv[R(L_*,g)]^p \bmod (p,(-A^2-A^{-2})^{p-1}-1).$$

(2) Suppose that $g\colon\mathbb{Z}\to\mathbb{Z}[h^{\pm1}]$ is defined by $g(n)=h^{(1-(-1)^n)/2}$. Since $w_i = p\cdot w_i^*$ and p is an odd prime, w_i and w_i^* have the same parity and hence $g(w_i)=g(w_i^*)$. Since $g(w_i^*)$ is either h or 1, we have

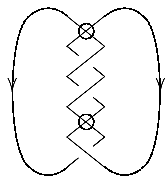
$$g(w_i^*)^p\equiv g(w_i^*) \bmod (h^{p-1}-1).$$

Then we know that

$$\begin{aligned}\langle\langle s_*\rangle\rangle^p &= 2^{p\cdot(-\mu(s_*))}[g(w_1^*)+\cdots+g(w_m^*)]^p\\ &\equiv 2^{p\cdot(-\mu(s_*))}[g(w_1^*)^p+\cdots+g(w_m^*)^p] \bmod p\\ &\equiv 2^{p\cdot(-\mu(s_*))}[g(w_1^*)+\cdots+g(w_m^*)] \bmod (p,h^{p-1}-1)\\ &\equiv 2^{p\cdot(-\mu(s_*))}[g(w_1)+\cdots+g(w_m)] \bmod (p,h^{p-1}-1)\\ &\equiv \langle\langle s\rangle\rangle \bmod (p,h^{p-1}-1).\end{aligned} \tag{3.7}$$

By (3.1), (3.3) and (3.7), it follows that

$$\alpha_*^p\equiv\alpha \bmod (p,(-A^2-A^{-2})^{p-1}-1,h^{p-1}-1).$$



L_1

Fig. 10.

Thus we have

$$H(D, g) \equiv [H(D_*, g)]^p \bmod (p, (-A^2 - A^{-2})^{p-1} - 1, h^{p-1} - 1).$$

Hence we get

$$R(L, g) \equiv [R(L_*, g)]^p \bmod (p, (-A^2 - A^{-2})^{p-1} - 1, h^{p-1} - 1).$$

This completes the proof. □

EXAMPLE 3.3. Let L_1 be a virtual knot as in Fig. 10. Then the Jones-Kauffman polynomial of L_1 is equal to 1 [8]. Let $g\colon \mathbb{Z} \to \mathbb{Z}[h^{\pm 1}]$ be a map given by $g(n) = h^n$. It is known [8] that

$$R^g_{L_1} = \frac{1}{2}(1 + A^{-8}) + \frac{1}{4}(1 - A^{-8})(h^2 + h^{-2}).$$

Suppose that L_1 has period 3. From Theorem 3.2, it follows that

$$\frac{1}{4}(1 - A^{-8}) \equiv 0 \bmod (3, (-A^2 - A^{-2})^2 - 1).$$

Since $(-A^2 - A^{-2})^2 - 1 = A^4 + 1 + A^{-4} = A^{-4}(A^8 + A^4 + 1)$ and $1 - A^{-8} = (A^{-4} - A^{-8})(A^8 + A^4 + 1) + (1 - A^4)$,

$$\frac{1}{4}(1 - A^{-8}) \equiv 1 + 2A^4 \bmod (3, A^8 + A^4 + 1).$$

Let $\mathcal{I}$ be the ideal of $\mathbb{Z}[2^{-1}, A^{\pm 1}]$ generated by 3. We note that the quotient ring of $\mathbb{Z}[2^{-1}, A^{\pm 1}]$ by $\mathcal{I}$ is isomorphic to the ring $\mathbb{Z}_3[A^{\pm 1}]$. So it is not true that $1 + 2A^4 \equiv 0 \bmod (3, A^8 + A^4 + 1)$. Hence L_1 does not have period 3.

Theorem 3.4. *Let p be a prime and L a virtual link that has period p^r $(r \geq 1)$. Let g be a map from $\mathbb{Z}$ to $\mathbb{Z}[h^{\pm 1}]$. Then*

$$R_{L,g}(A, h) \equiv R_{L,g}(A^{-1}, h) \bmod (p, A^{p^r} - 1).$$

Proof. Let D be a virtual link diagram of L in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ that invariant under the rotation ζ of $\mathbb{R}^2$ about the origin $\mathbf{0}$ through $2\pi/p^r$ and $D_* = D/\zeta$. Let s be a state of D.

If $\zeta(s) \neq s$, then there exist p^n distinct but equivalent states $s, \zeta(s), \dots, \zeta^{p^n-1}(s)$ for some n $(1 \leq n \leq r)$. Contribution of these states to the polynomial vanishes by reducing modulo p.

If $\zeta(s) = s$, then s defines a unique quotient states s_* $(= s/\zeta)$. Since $P(D; s) = p^r \cdot P(D_*; s_*)$, we get

$$A^{P(D;s)} = A^{p^r \cdot P(D_*;s_*)} \equiv 1 \bmod (A^{p^r} - 1).$$

Since $d = -A^2 - A^{-2}$ is symmetric and $\langle\langle s \rangle\rangle \in \mathbb{Z}[2^{-1}, h^{\pm 1}]$, we obtain

$$H_{D,g}(A, h) \equiv H_{D,g}(A^{-1}, h) \bmod (p, A^{p^r} - 1).$$

Since $w(D) = p^r \cdot w(D_*)$,

$$(-A^3)^{-w(D)} \equiv (-A^{-3})^{-w(D)} \bmod (A^{p^r} - 1).$$

Hence we have

$$R_{L,g}(A, h) \equiv R_{L,g}(A^{-1}, h) \bmod (p, A^{p^r} - 1).$$

This completes the proof. □

Corollary 3.5. *Let p be a prime and L a virtual link that has period p^r. Let $g\colon (\mathbb{Z}, +) \to (\mathbb{Z}[h^{\pm 1}], \cdot)$ be a homomorphism. Then*

$$R_{L,g}(A, h) \equiv R_{L,g}(A^{-1}, h^{-1}) \bmod (p, A^{p^r} - 1).$$

Proof. Let D a virtual link diagram of L in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ that invariant under the rotation ζ of $\mathbb{R}^2$ about the origin $\mathbf{0}$ through $2\pi/p^r$ and $D_* = D/\zeta$.

Let s be a state of D. Since $g\colon (\mathbb{Z}, +) \to (\mathbb{Z}[h^{\pm 1}], \cdot)$ is a homomorphism, we have $\langle\langle s \rangle\rangle(h) = \langle\langle s \rangle\rangle(h^{-1})$ by Lemma 2.1. By the similar argument to Theorem 3.4, we have

$$R_{L,g}(A, h) \equiv R_{L,g}(A^{-1}, h^{-1}) \bmod (p, A^{p^r} - 1).$$

This completes the proof. □

EXAMPLE 3.6. Let L_1 be a virtual knot as in Fig. 10. Then

$$R^g_{L_1}(A, h) - R^g_{L_1}(A^{-1}, h) = \frac{1}{2}(A^{-8} - A^8) + \frac{1}{4}(A^8 - A^{-8})(h^2 + h^{-2}).$$

We observe that

$$\frac{1}{2}(A^{-8} - A^{8}) \equiv 2A + A^{2} \not\equiv 0 \bmod (3,\ A^{3} - 1).$$

Hence this is an another proof to show that L_1 does not have period 3.

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References

[1] J.F. Davis and C. Livingston: *Alexander polynomials of periodic knots*, Topology **30** (1991), 551–564.

[2] U. Dudley: Elementary Number Theory, second edition, Freeman, San Francisco, Calif., 1978.

[3] N. Kamada: *Some relations on Miyazawa's virtual knot invariant*, Topology Appl. **154** (2007), 1417–1429.

[4] N. Kamada and Y. Miyazawa: *A* 2*-variable polynomial invariant for a virtual link derived from magnetic graphs*, Hiroshima Math. J. **35** (2005), 309–326.

[5] L.H. Kauffman: *Virtual knot theory*, European J. Combin. **20** (1999), 663–690.

[6] S.Y. Lee: *$\mathbb{Z}_n$-equivariant Goeritz matrices for periodic links*, Osaka J. Math. **40** (2003), 393–408.

[7] Y. Miyazawa: *Conway polynomials of periodic links*, Osaka J. Math. **31** (1994), 147–163.

[8] Y. Miyazawa: *Magnetic graphs and an invariant for virtual links*, J. Knot Theory Ramifications **15** (2006), 1319–1334.

[9] K. Murasugi: *On periodic knots*, Comment. Math. Helv. **46** (1971), 162–174.

[10] K. Murasugi: *Jones polynomials of periodic links*, Pacific J. Math. **131** (1988), 319–329.

[11] J.H. Przytycki: *On Murasugi's and Traczyk's criteria for periodic links*, Math. Ann. **283** (1989), 465–478.

[12] M. Sakuma: *On the polynomials of periodic links*, Math. Ann. **257** (1981), 487–494.

[13] P. Traczyk: 10_{101} *has no period* 7*: a criterion for periodic links*, Proc. Amer. Math. Soc. **108** (1990), 845–846.

[14] Y. Yokota: *The skein polynomial of periodic knots*, Math. Ann. **291** (1991), 281–291.

[15] Y. Yokota: *The Kauffman polynomial of periodic knots*, Topology **32** (1993), 309–324.

[16] Y. Yokota: *Polynomial invariants of periodic knots*, J. Knot Theory Ramifications **5** (1996), 553–567.

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