D'Abbicco, M.
Osaka J. Math.
46 (2009), 739–767

SOME RESULTS ON THE WELL-POSEDNESS FOR SECOND ORDER LINEAR EQUATIONS

MARCELLO D'ABBICCO

(Received February 29, 2008, revised May 8, 2008)

Abstract

We investigate the Cauchy problem for second order hyperbolic equations of complete form, and we prove an extension of a classical result of Oleĭnik [10] concerning the well-posedness for equations in which are absent the terms with mixed time-space derivatives. Then, in space dimension $n = 1$, we compare our results with those in [8] for equations with analytic coefficients, and those of [7] and [11] for homogeneous equations with coefficients depending only either on t or on x.

Moreover we exhibit, in space dimension $n \geq 2$, an equation of the form

$$u_{tt} - \sum_{i,j=1}^{n} (a_{ij}(t, x) u_{x_j})_{x_i} = 0, \quad \text{with} \quad \sum a_{ij} \xi_i \xi_j \geq 0,$$

where the coefficients are analytic functions, for which the Cauchy problem is ill-posed.

Finally, we present a sufficient condition for the well-posedness of 2×2 systems.

Introduction

We consider the Cauchy problem

$$\begin{cases} \mathrm{L}\, u = f(t, x), \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x) \end{cases} \tag{1}$$

where L is a differential, second order operator which we write *in variational form*:

$$\mathrm{L}\, u = u_{tt} - \sum_{i,j=1}^{n} (a_{ij}(t, x) u_{x_j})_{x_i} + \sum_{i=1}^{n} [(b_i u_{x_i})_t + (b_i u_t)_{x_i}] + c u_t + \sum_{i=1}^{n} d_i u_{x_i} + e u. \tag{2}$$

With no loss of generality, we can assume that $a_{ij} = a_{ji}$. Note that the polynomial

$$p^s(t, x, \tau, \xi) = c(t, x)\tau + \sum_{i=1}^{n} d_i(t, x) \xi_i \tag{3}$$

2000 Mathematics Subject Classification. 35L15.

is the *sub-principal symbol* of L. We can also write L in the non variational form

$$\mathrm{L}\,u = u_{tt} - \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + 2\sum_{i=1}^{n} b_i u_{tx_i} + (c + c^\sharp)u_t + \sum_{i=1}^{n}(d_i + d_i^\sharp)u_{x_i} + eu, \tag{4}$$

where $c^\sharp = \sum_j \partial_{x_j} b_j$ and $d_i^\sharp = \partial_t b_i - \sum_j \partial_{x_j} a_{ij}$.

Trough this paper, all the coefficients of L are assumed to be real-valued $\mathcal{C}^\infty$ functions, bounded together with all their derivatives in the strip

$$G_T := [0, T] \times \mathbb{R}^n,$$

i.e. which belong to the class $\mathcal{B}^\infty(G_T, \mathbb{R})$.

We say that (1) is *well-posed* (in $\mathcal{C}^\infty$) when, for each $\overline{x} \in \mathbb{R}^n$, there is some nbd. V of $(0, \overline{x})$ in G_T such that there is a unique solution $u \in \mathcal{C}^\infty(V)$ for all $u_0, u_1 \in \mathcal{C}^\infty(\mathbb{R}^n)$ and all $f \in \mathcal{C}^\infty(G_T)$. If, moreover, we can take $V = G_T$, then we say that (1) is *globally well-posed*.

A *necessary condition* for the well-posedness is the *hyperbolicity* of the operator L, which means that, for all $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$,

$$\left(\sum_{i=1}^{n} b_i(t,x)\xi_i\right)^2 + \sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j \geq 0. \tag{5}$$

However, such a condition is far to be *sufficient*. Thus, we look for some additional conditions which can ensure the well-posedness of (1).

A very simple condition was found in 1970 by Oleĭnik, limited to the operators L which do not contain terms with mixed derivatives $\partial_t \partial_{x_i}$, i.e. of the form

$$\mathrm{L}\,u = u_{tt} - \sum_{i,j=1}^{n} (a_{ij}(t,x)u_{x_j})_{x_i} + c(t,x)u_t + \sum_{i=1}^{n} d_i(t,x)u_{x_i} + e(t,x)u. \tag{6}$$

In the following we shall refer to the operators of type (6) as to the *incomplete* operators.

Theorem (O.A. Oleĭnik, [10]). *The Cauchy problem for any hyperbolic operator of type* (6) *is globally well-posed if there exist two positive constants C, A for which*

$$t\left[\sum_{i=1}^{n} d_i(t,x)\xi_i\right]^2 \leq C\left\{A\sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j + \sum_{i,j=1}^{n} \partial_t a_{ij}(t,x)\xi_i\xi_j\right\}. \tag{7}$$

We notice that (7) is fulfilled whenever the d_i's vanish identically and the a_{ij}'s are not depending on t.

In order to extend this theorem to any operator of type (2), a special role is played by the operators such that for each pair $i, j \in \{1, \dots, n\}$, one at least of the following alternatives holds:

$$\text{either} \quad b_i \equiv 0 \quad \text{or} \quad \partial_{x_i} b_j \equiv 0. \tag{8}$$

The condition (8) means that each one of the coefficients $b_j(t, x)$ is depending, besides the variable t, only on those spatial variables x_i for which $b_i \equiv 0$. In particular (8) holds true whenever all the $b_j \equiv b_j(t)$ are depending only on t. If $n = 1$, (8) simply means that $b \equiv b(t)$. If $n = 2$, (8) holds true only in the following cases:

- when $b_1 \equiv b_1(t, x_1)$ and $b_2 \equiv 0$,
- when $b_1 \equiv 0$ and $b_2 \equiv b_2(t, x_1)$,
- when $b_1 \equiv b_1(t)$ and $b_2 \equiv b_2(t)$.

Finally, we put

$$\Delta_{ij} = b_i b_j + a_{ij} \quad (i, j = 1, \dots, n),$$

so that the condition of hyperbolicity (5) reads

$$\sum_{i,j=1}^{n} \Delta_{ij}(t, x)\xi_i \xi_j \geq 0. \tag{9}$$

Theorem 1. *We distinguish two cases*:

- *For any hyperbolic operator of type* (2) *satisfying the condition* (8), *the Cauchy problem is globally well-posed if, for some constants* $C, A > 0$, *one has*

$$t \left[\sum_{i=1}^{n} (d_i - cb_i)\xi_i \right]^2 \leq C\Phi_A(t, x, \xi), \tag{10}$$

where

$$\Phi_A = \sum_{i,j=1}^{n} \left\{ A\Delta_{ij} + \partial_t \Delta_{ij} + \sum_{h=1}^{n} (b_h \partial_{x_h} \Delta_{ij} - 2\Delta_{ih} \partial_{x_h} b_j) \right\} \xi_i \xi_j. \tag{11}$$

- *In absence of the condition* (8), *i.e. for a hyperbolic operator of the general type* (2), *in order to get the well-posedness we must replace* (10) *with the stronger condition*

$$t \left\{ \left[\sum_{i=1}^{n} (d_i - cb_i)\xi_i \right]^2 + \rho(t, x, \xi) \right\} \leq C\Phi_A(t, x, \xi), \tag{12}$$

where

$$\rho = \sum_{i=1}^{n} \Delta_{ii}(t,x)\xi_i^2.$$

Moreover, the well-posedness is no longer global, in general.

In the special case of operators of type (6), the condition (8) is trivially fulfilled and our condition (10) coincides with the Oleĭnik's condition (7), since we have

$$\Delta_{ij} = a_{ij} \quad (i,\ j = 1, \dots, n).$$

REMARK 1. If the *sub-principal symbol* p^s in (3) vanishes identically then our condition (10) (resp. (12)) reduces to:

$$0 \le \Phi_A \quad (\text{resp. } \rho \le C\Phi_A).$$

In this case the operator (2) can be written in the simpler form:

$$\mathrm{L}\, u = e(t,x)u - \sum_{i,j=0}^{n} (a_{ij}(t,x)u_{x_j})_{x_i}, \tag{13}$$

where we used the notation $x_0 = t$ and we put

$$a_{00} = -1, \quad a_{i0} = -b_i = a_{0i} \quad (i = 1, \dots, n).$$

REMARK 2. If the leading coefficients a_{ij}'s and b_i's are not depending on x, then (8) holds true and

$$\Phi_A = A \sum_{i,j=1}^{n} \Delta_{ij}(t)\xi_i\xi_j + \sum_{i,j=1}^{n} \partial_t \Delta_{ij}(t)\xi_i\xi_j.$$

In the one dimensional case, i.e. for the operator

$$\mathrm{L}\, u = u_{tt} - (au_x)_x + [(bu_x)_t + (bu_t)_x] + cu_t + du_x + eu, \tag{14}$$

with

$$\Delta(t,x) \equiv (b(t,x))^2 + a(t,x) \ge 0,$$

our condition (12) takes a much simpler form:

Corollary 1. *Let $n = 1$. The Cauchy problem for a hyperbolic operator of the general type* (14) *is well-posed if, for some constants $C, A > 0$:*

$$t(d - cb)^2 \leq C\{A\Delta + \Delta_t + b\Delta_x\}. \tag{15}$$

Moreover, when $b \equiv b(t)$ the well-posedness is global.

In particular, the homogeneous operator:

$$\mathrm{L} = \mathrm{L}_2 := \partial_t^2 - a(t, x)\partial_x^2 + 2b(t, x)\partial_t\partial_x, \tag{16}$$

can be written in the form (14) with:

$$c = -c^\sharp = -b_x, \tag{17}$$

$$d = -d^\sharp = -b_t + a_x. \tag{18}$$

In such a case our condition (15), i.e.

$$t(-b_t + a_x + bb_x)^2 \leq C\{A\Delta + \Delta_t + b\Delta_x\},$$

reduces to:

$$t(b_t + bb_x)^2 \leq C'\{A\Delta + \Delta_t + b\Delta_x\}. \tag{19}$$

Indeed, we can apply the estimate

$$\Delta_x^2 \leq C''\Delta,$$

for some constant $C'' > 0$, thanks to the following well-known result:

Lemma 1 (Glaeser's inequality [5]). *If $f \in \mathcal{B}^2(\mathbb{R}, \mathbb{R})$, $f(x) \geq 0$, then the following holds true*:

$$(f'(x))^2 \leq \sup_{y\in\mathbb{R}} \| f''(y)\| f(x) \leq Cf(x), \tag{20}$$

for any $x \in \mathbb{R}$ and for some constant $C > 0$ not depending on x.

In particular, the Oleĭnik's condition (7) for the *incomplete* homogeneous operator, that is

$$\mathrm{L} = \partial_t^2 - a(t, x)\partial_x^2, \tag{21}$$

becomes

$$0 \leq Aa + a_t.$$

As a matter of fact, to prove Theorem 1 we use a local change of variables, leaving the lines $t = const$ invariant, that transforms an operator L of the type (2) into an *incomplete* operator of the type (6), to which we apply the Oleĭnik's theorem. The new space variables

$$y_i = g_i(t, x) \quad (i = 1, \dots, n)$$

are implicitly defined as the (unique) solution of the Cauchy problem:

$$\begin{cases} \left(\partial_t + \sum_{j=1}^{n} b_j(t, x)\partial_{x_j}\right) g_i(t, x) = 0, \\ g_i(0, x) = x_i, \end{cases} \tag{22}$$

In particular, if L satisfies (8) this change of variables is global and explicit.

Plan of the work. For the reader's convenience, in §1 we give a direct proof of the Corollary 1 while §2 is devoted to the proof of Theorem 1.

In §3 we assume that the coefficients of L are analytic (at the origin). We recall the result of Nishitani [8] in space dimension $n = 1$, and we compare it with our Corollary 1. In particular, the Nishitani's Theorem states that, if the *sub-principal symbol* is identically zero, then the Cauchy problem (1) for L is well posed (at the origin). It is well-known that in space dimension $n \geq 2$ there exists some operators with analytic coefficients and *sub-principal symbol* zero for which the Cauchy problem is ill-posed. However, one could ask if the Cauchy problem for an *incomplete* operator with analytic coefficients is always well-posed. We prove that the answer to this question is negative by exhibiting a counter-example.

In §4 we consider the *complete operator* in space dimension $n = 1$ with the additional hypothesis:

$$b^2(t, x) \leq C\Delta(t, x). \tag{23}$$

In particular, Spagnolo and Taglialatela [11] proved that (23) is a sufficient condition for the well-posedness of the Cauchy problem for the homogeneous operator with coefficients not depending on t. We show that in this case our condition (12) is more general than (23). Moreover, we present a corollary of Theorem 1 that extends the result in [11] to space dimension $n \geq 2$.

In §5 we present a sufficient condition for the well-posedness of 2×2 systems in space dimension $n = 1$. Again, the proof relies on a change of variables.

1. Proof of Corollary 1

First of all, we notice that Corollary 1 can be easily derived from Theorem 1. Indeed, in space dimension $n = 1$, condition (10) and (12) become:

$$t(d - cb)^2\xi^2 \le C\Phi_A(t, x, \xi), \tag{24}$$

$$t\{(d - cb)^2 + \Delta\}\xi^2 \le C\Phi_A(t, x, \xi), \tag{25}$$

where

$$\Phi_A = \{A\Delta + \Delta_t + b\Delta_x - 2b_x\Delta\}\xi^2. \tag{26}$$

Both conditions (24) and (25) are trivially equivalent to (15). Indeed, we have:

$$2|b_x|\Delta \le 2\|b_x\|_\infty\Delta \le C'\Delta.$$

Now we give a direct proof of Corollary 1.

For the sake of brevity and the ease of reading, we introduce the following:

NOTATION. We write $f \lesssim g$ to mean that there is some constant $C > 0$ such that

$$f(t, x) \le Cg(t, x). \tag{27}$$

We write $f \approx g$ to mean that $f \lesssim g$ and $g \lesssim f$.

The Cauchy problem (22) in space dimension $n = 1$ becomes:

$$\begin{cases} (\partial_t + b(t, x)\partial_x)g(t, x) = 0, \\ g(0, x) = x, \end{cases} \tag{28}$$

that is trivially well-posed. Let $g \in \mathcal{C}^\infty(U, \mathbb{R})$ be the (unique) solution of (28) in some nbd. U of the initial line $\{t = 0\}$ in G_T.

We define the vector-valued function:

$$\mathcal{G} := (\pi_0, g)\colon (t, x) \in U \mapsto (s, y) := (t, g(t, x)) \in G_T,$$

where π_0 is the projection on the time-axis. We remark that $g_x(0, \cdot) \equiv 1$, hence we can take a nbd. $V \subset U$ of $\{t = 0\}$, such that

$$\frac{1}{2} \le g_x(t, x) \le 2. \tag{29}$$

Consequently

$$\mathcal{G}\colon V \to W := \mathcal{G}(V)$$

is a smooth change of variables, since:

$$\det \nabla_{t,x}\mathcal{G} = \det \begin{pmatrix} 1 & 0 \\ g_t & g_x \end{pmatrix} = g_x.$$

Let $\mathcal{H} = (\pi_0, h)\colon W \to V$ be the inverse of $\mathcal{G}$.

For the sake of simplicity, we introduce the following notation:

NOTATION. For any function $u \equiv u(t, x)$ or $v \equiv v(s, y)$ we define:

$$\tilde{u}(s, y) := u \circ \mathcal{H}(s, y),$$
$$\hat{v}(t, x) := v \circ \mathcal{G}(t, x).$$

We can write explicitly some relations between g and h. Indeed $h \circ \mathcal{G}(t, x) = x$, hence we have:

$$0 \equiv (h \circ \mathcal{G})_t = \hat{h}_s(\pi_0)_t + \hat{h}_y g_t = \hat{h}_s + \hat{h}_y g_t, \tag{30}$$

$$1 \equiv (h \circ \mathcal{G})_x = \hat{h}_s(\pi_0)_x + \hat{h}_y g_x = \hat{h}_y g_x. \tag{31}$$

From (31) we get:

$$h_y = \frac{1}{g_x} \circ \mathcal{H}, \tag{32}$$

hence

$$\frac{1}{2} \leq h_y(s, y) \leq 2$$

in W thanks to (29). On the other hand, thanks to (30) and using (28) and (32) we obtain

$$h_s = -\tilde{g}_t h_y = -\frac{g_t}{g_x} \circ \mathcal{H} = b \circ \mathcal{H}. \tag{33}$$

Therefore the function h solves the following Cauchy problem in W:

$$\begin{cases} \partial_s h(s, y) - b(s, h(s, y)) = 0, \\ h(0, y) = y. \end{cases} \tag{34}$$

REMARK 3. If (8) holds true, that is if $b \equiv b(t)$, then we can write explicitly the solution $g(t, x)$ to (28):

$$y = g(t, x) = x - \int_0^t b(\tau)\, d\tau. \tag{35}$$

We notice that $g_x \equiv 1$, hence (29) is trivially satisfied in G_T.

Therefore we can take $V = G_T$ and $W = \mathcal{G}(G_T) = G_T$. For any $(s, y) \in G_T$ we can write explicitly:

$$h(s, y) := x = y + \int_0^s b(\sigma)\, d\sigma. \tag{36}$$

Lemma 2. *Let* L *be a complete operator of type* (2)*. Therefore*

$$\mathrm{L}\, u = [\mathrm{L}^{\mathcal{H}}(u \circ \mathcal{H})] \circ \mathcal{G} \tag{37}$$

in V*, where*

$$\mathrm{L}^{\mathcal{H}}(s, y, \partial_s, \partial_y)v = v_{ss} - (\Delta^{\mathcal{H}} v_y)_y + c^{\mathcal{H}} v_s + d^{\mathcal{H}} v_y + e^{\mathcal{H}} v,$$

is an incomplete operator with:

$$\begin{aligned}
\Delta^{\mathcal{H}}(s, y) &:= [g_x^2 \Delta] \circ \mathcal{H}(s, y),\\
c^{\mathcal{H}}(s, y) &:= [c + b_x] \circ \mathcal{H}(s, y),\\
d^{\mathcal{H}}(s, y) &:= [(d - cb + g_{xx}(h_y \circ \mathcal{G})\Delta) g_x] \circ \mathcal{H}(s, y),\\
e^{\mathcal{H}}(s, y) &:= e \circ \mathcal{H}(s, y).
\end{aligned}$$

We remark that the coefficients of $\mathrm{L}^{\mathcal{H}}$ belong to $\mathcal{C}^\infty(W, \mathbb{R})$.

Proof. We immediately obtain, for any function $v \in \mathcal{C}^\infty(W, \mathbb{R})$, that:

$$\begin{aligned}
\partial_t \hat{v} &= \hat{v}_s + g_t \hat{v}_y = \hat{v}_s - b g_x \hat{v}_y,\\
\partial_x \hat{v} &= g_x \hat{v}_y,
\end{aligned}$$

hence

$$\begin{aligned}
\partial_t^2 \hat{v} + \partial_t(b \partial_x \hat{v}) &= \partial_t \hat{v}_s - \partial_t(b g_x \hat{v}_y) + \partial_t(b g_x \hat{v}_y)\\
&= \partial_t \hat{v}_s = \hat{v}_{ss} - b g_x \hat{v}_{sy},
\end{aligned}$$

whereas

$$\begin{aligned}
\partial_x(b \partial_t \hat{v}) - \partial_x(a \partial_x \hat{v}) &= \partial_x(b \hat{v}_s) - \partial_x(b^2 g_x \hat{v}_y) - \partial_x(a g_x \hat{v}_y)\\
&= \partial_x(b \hat{v}_s) - \partial_x(\Delta g_x \hat{v}_y)\\
&= b_x \hat{v}_s + b g_x \hat{v}_{sy} - \partial_x(\Delta g_x \hat{v}_y).
\end{aligned}$$

We have that:

$$\begin{aligned}
\partial_x(\Delta g_x \hat{v}_y) &= \partial_x[(\tilde{\Delta} \tilde{g}_x v_y) \circ \mathcal{G}] = g_x[(\tilde{\Delta} \tilde{g}_x v_y)_y \circ \mathcal{G}]\\
&= [(\tilde{g}_x \tilde{\Delta} \tilde{g}_x v_y)_y] \circ \mathcal{G} - [(\partial_y \tilde{g}_x) \tilde{\Delta} \tilde{g}_x v_y] \circ \mathcal{G}\\
&= [(\tilde{g}_x^2 \tilde{\Delta} v_y)_y] \circ \mathcal{G} - [h_y \tilde{g}_{xx} \tilde{\Delta} \tilde{g}_x v_y] \circ \mathcal{G},
\end{aligned}$$

thus we get:

$$\partial_t^2 \hat{v} + \partial_t(b\partial_x \hat{v}) + \partial_x(b\partial_t \hat{v}) - \partial_x(a\partial_x \hat{v}) = \hat{v}_{ss} + b_x \hat{v}_s - \partial_x(\Delta g_x \hat{v}_y).$$

Analogously we have

$$c\partial_t \hat{v} + d\partial_x \hat{v} = c\hat{v}_s + (d - cb)g_x \hat{v}_y,$$

hence we get

$$\mathrm{L}(v \circ \mathcal{G}) = (\mathrm{L}^{\mathcal{H}} v) \circ \mathcal{G},$$

that is (37) with $u := v \circ \mathcal{G}$. □

REMARK 4. If the condition (8) holds true then we can write explicitly:

$$\begin{aligned} \Delta^{\mathcal{H}}(s, y) &= \Delta(s, h(s, y)), \\ c^{\mathcal{H}}(s, y) &= c(s, h(s, y)), \\ d^{\mathcal{H}}(s, y) &= d(s, h(s, y)) - c(s, h(s, y))b(s), \\ e^{\mathcal{H}}(s, y) &= e(s, h(s, y)), \end{aligned}$$

since $g_x \equiv 1$ and $g_{xx} \equiv 0$ (see Remark 3). We remark that $\mathrm{L}^{\mathcal{H}}$ has coefficients in $\mathcal{B}^\infty(G_T, \mathbb{R})$.

Direct Proof of Corollary 1. In order to apply the Oleĭnik's theorem to the transformed operator $\mathrm{L}^{\mathcal{H}}$, we prove the Oleĭnik's condition (7), that is:

$$s(d^{\mathcal{H}})^2 \lesssim A\Delta^{\mathcal{H}} + \partial_s \Delta^{\mathcal{H}}, \tag{38}$$

in $W = \mathcal{G}(V)$. Thanks to (34) we can develop the last term in the right-hand side of (38):

$$\begin{aligned} (\partial_s \Delta^{\mathcal{H}}) \circ \mathcal{G} &= \partial_t(g_x^2 \Delta) + b\partial_x(g_x^2 \Delta) \\ &= g_x^2(\Delta_t + b\Delta_x) + 2g_x \Delta(g_{tx} + bg_{xx}) \\ &= g_x^2(\Delta_t + b\Delta_x) + 2g_x \Delta[\partial_x(g_t + bg_x) - b_x g_x] \\ &= g_x^2(\Delta_t + b\Delta_x - 2\Delta b_x). \end{aligned} \tag{39}$$

Now we compose (38) on the right-hand with $\mathcal{G}$ and we replace the coefficients of $\mathrm{L}^{\mathcal{H}}$ with their explicit expressions, thus (38) becomes:

$$t[(d - cb + g_{xx}(h_y \circ \mathcal{G})\Delta)g_x]^2 \lesssim (A\Delta + \Delta_t + b\Delta_x - 2\Delta b_x)g_x^2. \tag{40}$$

Thanks to (29), we can divide both the left-hand term and the right-hand term of (40) by g_x^2. We can estimate

$$(g_{xx}(h_y \circ \mathcal{G})\Delta)^2 \lesssim \Delta \quad \text{and} \quad 2|b_x|\Delta \lesssim \Delta,$$

hence (40) is equivalent to our condition (15).

Hence we have proved the Oleĭnik's condition (7) for the operator $\mathrm{L}^{\mathcal{H}}$ in W. Now we distinguish two cases.

If the condition (8) holds true then we take $V = G_T = W$ and (7) holds globally for $\mathrm{L}^{\mathcal{H}}$. We apply the Oleĭnik's theorem: there exists a unique global solution v of the Cauchy problem for $\mathrm{L}^{\mathcal{H}}$. Therefore $u := v \circ \mathcal{G}$ is the unique global solution of the Cauchy problem (1) for the operator L and this concludes the proof.

Now we prove the local well-posedness in absence of the condition (8). We fix arbitrarily $\overline{x} \in \mathbb{R}$ and we prove the well-posedness of (1) at $(0, \overline{x})$ for L. We take $T_1 \in (0, T]$ and $\varepsilon > 0$ in such a way that we have

$$W_1 := [0, T_1] \times B_{\varepsilon/2} \subset K := [0, T_1] \times \overline{B_\varepsilon} \subset W,$$

where $B_r := \{|x - \overline{x}| < r\}$.

By compactness arguments, the transformed operator $\mathrm{L}^{\mathcal{H}}$ has coefficients in $\mathcal{B}^\infty(K, \mathbb{R})$ hence we can extend the coefficients of $\mathrm{L}^{\mathcal{H}}$ from $\mathcal{B}^\infty(K, \mathbb{R})$ to $\mathcal{B}^\infty(G_{T_1}, \mathbb{R})$. We take $\phi \in \mathcal{B}^\infty(\mathbb{R}, \mathbb{R}_+)$ in such a way that:

$$\phi|_{B_{\varepsilon/2}} \equiv 0, \quad \phi|_{G_{T_1} \setminus \overline{B_\varepsilon}} \equiv C + 1,$$

where C is a suitable positive constant such that $\Delta^{\mathcal{H}} \geq -C$ in G_{T_1} (we notice that the extended operator $\mathrm{L}^{\mathcal{H}}$ may be no longer hyperbolic in G_{T_1}). Now the operator

$$\mathrm{M}\, v := \mathrm{L}^{\mathcal{H}}\, v - (\phi(y) v_y)_y$$

satisfies (7) in G_{T_1}, hence we can apply the Oleĭnik's theorem to M.

We have a (unique) solution $v \in \mathcal{C}^\infty(G_{T_1}, \mathbb{R})$ of the Cauchy problem for M in G_{T_1}; in particular, v is the (unique) solution of the Cauchy problem for $\mathrm{L}^{\mathcal{H}}$ in W_1. Therefore $u := v \circ \mathcal{G}$ is the unique solution of the Cauchy problem (1) for L in $V_1 := \mathcal{H}(W_1)$, nbd. of $(0, \overline{x})$. This concludes the proof. □

2. Proof of Theorem 1

Through this section we assume space dimension $n \geq 2$. For the sake of brevity and the ease of reading, we introduce the following notation.

NOTATION. For any $(t, x, \xi) \in G_T \times \mathbb{R}^n$, we put:

$$a(t, x, \xi) := \sum_{i,j=1}^{n} a_{ij}(t, x)\xi_i\xi_j, \tag{41}$$

$$\Delta(t, x, \xi) := \sum_{i,j=1}^{n} \Delta_{ij}(t, x)\xi_i\xi_j, \tag{42}$$

$$b(t, x, \xi) := \sum_{i=1}^{n} b_i(t, x)\xi_i, \tag{43}$$

$$d(t, x, \xi) := \sum_{i=1}^{n} d_i(t, x)\xi_i. \tag{44}$$

Moreover, we define:

$$\gamma(t, x, \xi) := \sum_{i,j=1}^{n} \gamma_{ij}(t, x)\xi_i\xi_j, \quad \text{with} \quad \gamma_{ij} := -2\sum_{k=1}^{n} \Delta_{ik}\partial_{x_k} b_j.$$

NOTATION. We write $f \lesssim g$ to mean that there is some constant $C > 0$ such that we have:

$$f(t, x, \xi) \leq Cg(t, x, \xi). \tag{45}$$

We write $f \approx g$ to mean that $f \lesssim g$ and $g \lesssim f$.

With this notation, our conditions (10) and (12) become:

$$t(d - cb)^2 \lesssim \Phi_A, \tag{46}$$

$$t((d - cb)^2 + \rho) \lesssim \Phi_A. \tag{47}$$

NOTATION. We write $\mathbf{b}(t, x)$ to mean the vector-valued function

$$\mathbf{b} = (b_i)_{i=1,\dots,n} \colon G_T \to \mathbb{R}^n, \tag{48}$$

and similarly we define $\mathbf{d}(t, x)$.

With this notation we have:

$$b(t, x, \xi) = \mathbf{b}(t, x) \cdot \xi, \quad \text{and} \quad d(t, x, \xi) = \mathbf{d}(t, x) \cdot \xi,$$

where $\cdot$ denotes the scalar product in $\mathbb{R}^n$. Moreover, (11) reads:

$$\Phi_A = A\Delta + \Delta_t + \mathbf{b} \cdot \nabla_x \Delta + \gamma. \tag{49}$$

As in §1, let

$$g_i : (t, x) \in V \mapsto y \in \mathbb{R} \quad (i = 1, \ldots, n),$$

be the the unique $\mathcal{C}^\infty$ solution of the Cauchy problem (22) in V, nbd. of $\{t = 0\}$. We put

$$\mathcal{G} = (\pi_0, \mathbf{g}), \quad \text{where} \quad \mathbf{g} = (g_i)_{i=1,\ldots,n}.$$

We can take V in a such way that:

$$\inf_V \det \nabla_x \mathbf{g}(t, x) > 0, \tag{50}$$

since $\nabla_x \mathbf{g}(0, \,\cdot\,) \equiv \mathrm{I}$. Consequently

$$\mathcal{G} \colon V \to W := \mathcal{G}(V),$$

is a smooth change of variables. Let

$$\mathcal{H} = (\pi_0, \mathbf{h}) \colon W \to V, \quad \text{where} \quad \mathbf{h} = (h_i)_{i=1,\ldots,n},$$

be the inverse of $\mathcal{G}$.

Proceeding as in Lemma 2 we get (37) where now

$$\mathrm{L}^{\mathcal{H}}(s, y, \partial_s, \nabla_y)v = v_{ss} - \sum_{i,j=1}^{n} (\Delta_{ij}^{\mathcal{H}} v_{y_j})_{y_i} + c^{\mathcal{H}} v_s + \sum_{i=1}^{n} d_i^{\mathcal{H}} v_{y_i} + e^{\mathcal{H}} v,$$

is an *incomplete* operator with

$$\begin{aligned}
\Delta_{ij}^{\mathcal{H}}(s, y) &:= \left[\sum_{k,l=1}^{n} (\partial_{x_k} g_i) \Delta_{kl} (\partial_{x_l} g_j) \right] \circ \mathcal{H}(s, y),\\
c^{\mathcal{H}}(s, y) &:= \left[c + \sum_{k=1}^{n} \partial_{x_k} b_k \right] \circ \mathcal{H}(s, y),\\
d_i^{\mathcal{H}}(s, y) &:= \left[\sum_{k=1}^{n} ((d_k - c b_k) + r_k))(\partial_{x_k} g_i) \right] \circ \mathcal{H}(s, y),\\
e^{\mathcal{H}}(s, y) &:= e \circ \mathcal{H}(s, y),
\end{aligned}$$

and

$$r_k(t, x) := \sum_{l,p,q=1}^{n} \big((\partial_{x_l} \partial_{x_p} g_q)(\widehat{\partial_{y_q} h_p}) \big) \Delta_{lk}(t, x).$$

Moreover the vector-valued function $\mathbf{h}$ solves in W the Cauchy problem:

$$\begin{cases} \partial_s \mathbf{h}(s, y) = \mathbf{b}(s, \mathbf{h}(s, y)), \\ \mathbf{h}(0, y) = y. \end{cases} \tag{51}$$

Proposition 1. *If* L *satisfies the condition* (8) *then we can write explicitly*

$$\mathbf{g}(t, x) = x - \int_0^t \mathbf{b}(\tau, x)\, d\tau, \quad (t, x) \in G_T, \tag{52}$$

$$\mathbf{h}(s, y) = y + \int_0^s \mathbf{b}(\sigma, y)\, d\sigma, \quad (s, y) \in G_T. \tag{53}$$

In particular, (50) *holds trivially true since we have*:

$$\det \nabla_x \mathbf{g} \equiv 1. \tag{54}$$

Moreover we can write:

$$\Delta_{ij}^{\mathcal{H}}(s, y) = \left[b_i b_j + \sum_{k,l=1}^{n} (\partial_{x_k} g_i) a_{kl} (\partial_{x_l} g_j) \right] \circ \mathcal{H}(s, y), \tag{55}$$

$$c^{\mathcal{H}}(s, y) = c \circ \mathcal{H}(s, y), \tag{56}$$

$$d_i^{\mathcal{H}}(s, y) = \left[\sum_{k=1}^{n} d_k (\partial_{x_k} g_i) - c b_i \right] \circ \mathcal{H}(s, y). \tag{57}$$

We remark that the coefficients of $\mathrm{L}^{\mathcal{H}}$ are in $\mathcal{B}^\infty(G_T, \mathbb{R})$, since $\nabla_x \mathbf{g} \in \mathcal{B}^\infty(G_T, M_n(\mathbb{R}))$.

Proof. Condition (8) implies immediately (56).

It is easy to check that each function g_i solves the Cauchy problem (22). Indeed, by (8) we have:

$$\partial_t g_i(t, x) + \sum_{j=1}^{n} b_j(t, x) \partial_{x_j} g_i(t, x) = - \int_0^t \sum_{j=1}^{n} b_j(t, x) \partial_{x_j} b_i(\tau, x)\, d\tau = 0.$$

We can assume with no loss of generality that

$$b_i \equiv 0 \quad (i = 1, \dots, m),$$
$$\partial_{x_i} \mathbf{b} \equiv 0 \quad (i = m+1, \dots, n),$$

for some $m \in \{0, \dots, n\}$. We put:

$$x \equiv (x', x''), \quad x' = (x_i)_{i=1,\dots,m}, \quad x'' = (x_i)_{i=m+1,\dots,n},$$
$$y \equiv (y', y''), \quad y' = (y_i)_{i=1,\dots,m}, \quad y'' = (y_i)_{i=m+1,\dots,n},$$

$$\mathbf{b} \equiv (\mathbf{b}', \mathbf{b}''), \quad \mathbf{b}' = (b_i)_{i=1,\dots,m}, \quad \mathbf{b}'' = (b_i)_{i=m+1,\dots,n},$$
$$\mathbf{g} \equiv (\mathbf{g}', \mathbf{g}''), \quad \mathbf{g}' = (g_i)_{i=1,\dots,m}, \quad \mathbf{g}'' = (g_i)_{i=m+1,\dots,n},$$
$$\mathbf{h} \equiv (\mathbf{h}', \mathbf{h}''), \quad \mathbf{h}' = (h_i)_{i=1,\dots,m}, \quad \mathbf{h}'' = (h_i)_{i=m+1,\dots,n}.$$

We remark that $\mathbf{b}' \equiv 0$. We get immediately:

$$\nabla_x \mathbf{g} = \begin{pmatrix} \nabla_{x'}\mathbf{g}' & \nabla_{x''}\mathbf{g}' \\ \nabla_{x'}\mathbf{g}'' & \nabla_{x''}\mathbf{g}'' \end{pmatrix} = \begin{pmatrix} \mathrm{I}_m & 0 \\ -\int_0^t \nabla_{x'}\mathbf{b}''(\tau, x')\, d\tau & \mathrm{I}_{n-m} \end{pmatrix}, \tag{58}$$

that implies trivially (54).

For any $(t, y) \in G_T$ there exists $x \in \mathbb{R}^n$ such that $\mathbf{g}(t, x) = y$. Hence, using (52), we get:

$$\mathbf{h}'(t, y) = x' = y'. \tag{59}$$

Now we can use (52) and (59) to have:

$$\mathbf{h}''(t, y) = x'' = y'' + \int_0^t \mathbf{b}''(\tau, x')\, d\tau = y'' + \int_0^t \mathbf{b}''(\tau, y')\, d\tau. \tag{60}$$

This proves (53).

On the other hand, using (52) and (53), we get

$$\begin{aligned} &\sum_{q=1}^n \partial_{x_l}\partial_{x_p} g_q(t, x)\, \partial_{y_q} h_p(t, y) \\ &= \sum_{q=1}^n \partial_{x_l}\partial_{x_p} g_q(t, x)\, \partial_{y_q} y_p \\ &= \partial_{x_l}\partial_{x_p} g_p(t, x) = \partial_{x_l} 1 = 0 \quad (p = 1, \dots, m,\ l = 1, \dots, n), \end{aligned}$$

and

$$\begin{aligned} &\sum_{q=1}^n \partial_{x_l}\partial_{x_p} g_q(t, x)\, \partial_{y_q} h_p(t, y) \\ &= \sum_{q=1}^n \partial_{x_l}(\partial_{x_p} x_q - 0)\, \partial_{y_q} h_p(t, y) \\ &= \partial_{x_l} 1\ \ \partial_{y_p} h_p(t, y) = 0 \quad (p = m+1, \dots, n,\ l = 1, \dots, n). \end{aligned}$$

This proves that:

$$r_k \equiv 0\ (k = 1, \dots, n), \quad \text{hence} \quad d_i^{\mathcal{H}} = \left[\sum_{k=1}^n (d_k - cb_k)(\partial_{x_k} g_i)\right] \circ \mathcal{H}.$$

We notice that:

$$\sum_{k=1}^{n} b_k(\partial_{x_k} g_i) = \sum_{k=1}^{n} \left(b_k \partial_{x_k} x_i - b_k \int_0^t \partial_{x_k} b_i \right) = b_i,$$

thanks to (8). Hence we have proved (57). By the same way we prove (55). □

Proof of Theorem 1. By the same arguments used in §1, we get:

$$(\partial_s \Delta_{ij}^{\mathcal{H}}) \circ \mathcal{G} = \sum_{k,l=1}^{n} (\partial_{x_k} g_i) \left(\partial_t \Delta_{kl} + \sum_{m=1}^{n} b_m \partial_{x_m} \Delta_{kl} + \gamma_{kl} \right) (\partial_{x_l} g_j).$$

We put:

$$\eta_k := \sum_{i=1}^{n} (\partial_{x_k} g_i) \xi_i \quad (k = 1, \ldots, n),$$

for any $\xi \in \mathbb{R}^n$. We remark that for any $(t, x) \in V$ the linear map

$$\xi \in \mathbb{R}^n \mapsto \eta := (\nabla_x \mathbf{g}(t, x)) \cdot \xi \in \mathbb{R}^n,$$

is bijective, thanks to (50). Therefore the Oleĭnik's condition (7) for $L^{\mathcal{H}}$ is verified if:

$$t(d - cb + r)^2 \lesssim A\Delta + \Delta_t + \mathbf{b} \cdot \nabla_x \Delta + \gamma, \tag{61}$$

where $r(t, x, \xi) := \sum_{i=1}^{n} r_i(t, x) \xi_i$.

If the condition (8) holds true, then $r \equiv 0$ hence (61) reduces to our condition (10). On the other hand, in absence of the condition (8) we can prove that $r^2 \lesssim \rho$.

By the positivity of Δ it follows that:

$$0 \leq \Delta(t, x, \lambda e_l + e_k) = \lambda^2 \Delta_{ll}(t, x) + 2\lambda \Delta_{lk}(t, x) + \Delta_{kk}(t, x) \quad (l, k = 1, \ldots, n),$$

for any $\lambda \in \mathbb{R}$, hence:

$$(\Delta_{lk}(t, x))^2 \leq \Delta_{ll}(t, x) \Delta_{kk}(t, x) \quad (l, k = 1, \ldots, n). \tag{62}$$

Thanks to (62), we can estimate:

$$r_k^2(t, x) \lesssim \sum_{l=1}^{n} \Delta_{lk}^2(t, x) \leq \left(\sum_{l=1}^{n} \Delta_{ll}(t, x) \right) \Delta_{kk}(t, x) \lesssim \Delta_{kk}(t, x) \quad (k = 1, \ldots, n). \tag{63}$$

Using (63) we have:

$$(r(t, x, \xi))^2 \lesssim \sum_{k=1}^{n} r_k^2(t, x) \xi_k^2 \lesssim \sum_{k=1}^{n} \Delta_{kk}(t, x) \xi_k^2 = \rho(t, x, \xi).$$

Therefore (61) reduces to our condition (12).

The conclusion of the proof is the same as in Corollary 1. □

Our condition (12) is far to be equivalent to (61), but the last one does contain the term $r(t, x, \xi)$ that depends explicitly on the solution of (22). On the other hand, under assumption (8), (10) and (61) are equivalent. We can obtain $r \equiv 0$ for some operator also in absence of the condition (8). In these cases, condition (61) reduces to our condition (10).

EXAMPLE 1. Let L be some operator of type (2) with $\mathbf{b} = x$. Therefore:

$$g_i(t, x) = x_i e^{-t} \quad (i = 1, \ldots, n),$$

solves the Cauchy problem (22). Moreover, we have that:

$$h_i(s, y) = y_i e^s \quad (i = 1, \ldots, n).$$

The condition (8) is not satisfied. Nevertheless, it is easy to notice that $r \equiv 0$ and we can write explicitly the coefficients of $\mathrm{L}^{\mathcal{H}}$:

$$\begin{aligned}
\Delta^{\mathcal{H}}_{ij}(s, y) &= \Delta_{ij}(s, ye^s)e^{-2s} = y_i y_j + a_{ij}(s, ye^s)e^{-2s},\\
c^{\mathcal{H}}(s, y) &= c(s, ye^s) + n,\\
d^{\mathcal{H}}_i(s, y) &= d_i(s, ye^s)e^{-s} - y_i c(s, ye^s),\\
e^{\mathcal{H}}(s, y) &= e(s, ye^s).
\end{aligned}$$

3. The case of analytic coefficients

Through this section, we assume that the coefficients of L are analytic (at the origin). In space dimension $n = 1$ we recall the following well-known result.

Lemma (T. Nishitani [8]). *Let* L *be a hyperbolic operator with coefficients in* $\mathcal{C}^\omega(U, \mathbb{C})$ *where* U *is a nbd. of the origin in* $\mathbb{R}^2$.

Then we can find another nbd. V *of the origin, in a such way that the characteristic roots* τ_1 *and* τ_2 *of* L *can be taken continuous in* V *and analytic in* $V \setminus \{(0, 0)\}$.

Theorem (T. Nishitani [8]). *Let* L, V *and* τ_1, τ_2 *be as described in the previous lemma. Assume that there are two constants* $A, B > 0$ *such that, for any* $(t, x) \in W$, *where* $W \subset V$ *is a nbd. of the origin, and for any* $\xi \in \mathbb{R}$, *we have*:

$$|p^s(t, x, \tau_1(t, x)\xi, \xi)| \le A|\{\tau - \tau_1\xi, \tau - \tau_2\xi\}| + B|\xi(\tau_1 - \tau_2)|. \tag{64}$$

Here $\{f, g\}$ *is the Poisson bracket, i.e.*

$$\{f, g\} := (\partial_\tau f \partial_t g + \partial_\xi f \partial_x g) - (\partial_t f \partial_\tau g + \partial_x f \partial_\xi g).$$

Then the Cauchy problem (1) *for* L *is well-posed in* $\mathcal{C}^\infty$ (*at the origin*).

We remark that the assumption of analyticity in t of the coefficients of L is essential. Indeed there exists an operator

$$\mathrm{L} = \partial_t^2 - a(t)\partial_x^2, \quad \text{with} \quad a(t) \in \mathcal{C}^\infty([0, T]) \quad a(t) \geq 0, \tag{65}$$

for which the Cauchy problem is ill-posed [3].

From the results in [6] it follows that, if $\Delta(t, x)$ can be written in the form

$$\Delta(t, x) = (\Phi(t, x))^N \Psi(t, x), \tag{66}$$

for some integer $N \in \mathbb{N}$ and some smooth functions Φ and Ψ satisfying

$$\Phi(0, 0) = 0, \quad (\Phi_t(0, 0), \Phi_x(0, 0)) \neq (0, 0), \quad \Psi(0, 0) \neq 0,$$

then (64) is also a necessary condition for the well-posedness in $\mathcal{C}^\infty$. In particular, if either $\Delta \equiv \Delta(t)$ or $\Delta \equiv \Delta(x)$ then (66) holds true. On the other hand, in the general case $\Delta \equiv \Delta(t, x)$, Nishitani extended (64) to a necessary and sufficient condition in [9].

In order to compare the Nishitani's theorem with our Corollary 1, we re-state (64) in the following form.

Proposition 2. *We define*

$$\delta(t, x) := \frac{1}{2}(\tau_1(t, x) - \tau_2(t, x)),$$

that is continuous in V *and analytic in* $V \setminus \{(0, 0)\}$, *thanks to the Nishitani's lemma* [8]. *Now the Nishitani's condition* (64) *is equivalent to*:

$$|d - cb| \lesssim |\delta_t + b\delta_x| + |\delta|. \tag{67}$$

We remark that $\delta^2(t, x) = \Delta(t, x)$.

Proof. We notice that:

$$b(t, x) = -\frac{1}{2}(\tau_1(t, x) + \tau_2(t, x)),$$

hence we can compute:

$$\begin{aligned}
\{\tau - \tau_1, \tau - \tau_2\} &= \{\tau - (\delta - b)\xi, \tau + (\delta + b)\xi\} \\
&= ((\delta + b)_t \xi - (\delta - b)(\delta + b)_x \xi) \\
&\quad - (-(\delta - b)_t \xi - (\delta + b)(\delta - b)_x \xi) \\
&= 2\xi(\delta_t + b\delta_x - b_x \delta),
\end{aligned}$$

By estimating $|b_x\delta| \lesssim |\delta|$, we obtain:

$$A|\{\tau - \tau_1\xi,\, \tau - \tau_2\xi\}| + B|\xi(\tau_1 - \tau_2)| \approx |\xi|(A'|\delta| + |\delta_t + b\delta_x|).$$

In order to conclude our proof, it is sufficient to notice that:

$$|p^s(t, x, \tau_1\xi, \xi)| = |\xi|\,|d + c(\delta - b)| \leq |\xi|(|d - cb| + |c\delta|) \lesssim |\xi|(|d - cb| + |\delta|). \qquad \square$$

Proposition 3. *If $\Delta \equiv \Delta(t)$ is depending only on t and belongs to $\mathcal{C}^\omega((-T, T), \mathbb{R})$ for some $T > 0$, then our condition* (15) *is locally equivalent to* (67).

In particular, Proposition 3 proves that (15) is a necessary and sufficient condition for the well-posedness, provided that the coefficients are depending only on the time variable and are analytic at the origin.

Proof. If L is strictly hyperbolic, that is $\Delta(0) > 0$, then both (15) and (67) hold locally true, hence we can assume $\Delta(0) = 0$ in the following.

Thanks to the positive analyticity of $\Delta(t)$ we can write for some integer $\kappa \geq 1$:

$$\Delta(t) = \alpha_{2\kappa} t^{2\kappa} + \mathcal{O}(t^{2\kappa+1}), \quad \text{with} \quad \alpha_{2\kappa} > 0,$$

near to the origin. Hence (15) is locally equivalent to

$$t(d - cb)^2 \lesssim t^{2\kappa-1}, \tag{68}$$

whereas (67) is locally equivalent to

$$|d - cb| \lesssim |t^{\kappa-1}|. \tag{69}$$

We notice that (68) and (69) are both equivalent to

$$(d - cb)^2 \lesssim t^{2(\kappa-1)}.$$

This concludes the proof. $\square$

Proposition 4. *If the leading coefficients $a \equiv a(x)$ and $b \equiv b(x)$ are depending only on x and belong to $\mathcal{C}^\omega((-\varepsilon, \varepsilon), \mathbb{R})$ for some $\varepsilon > 0$, then our condition* (15) *is locally equivalent to* (67), *provided that $a(0) = b(0) = 0$.*

In particular, Proposition 4 proves that (15) is a necessary and sufficient condition for the well-posedness of L, provided that the coefficients are depending only on the space variable and are analytic at the origin, and that $a(0) = b(0) = 0$.

Proof of Proposition 4. Thanks to the analyticity of the coefficients, we have $b(x) = \mathcal{O}(x)$ and

$$\Delta(x) = \alpha_{2\kappa} x^{2\kappa} + \mathcal{O}(x^{2\kappa+1}), \quad \text{with} \quad \alpha_{2\kappa} > 0,$$

for some integer $\kappa \in \mathbb{N}^*$. Therefore:

$$A\Delta + b\Delta_x \approx x^{2k},$$

in (15) whereas

$$|b\delta_x| + |\delta| \approx x^{\kappa},$$

in (67). Consequently both the conditions (15) and (67) are locally equivalent to:

$$(d - cb)^2 \lesssim x^{2k}. \qquad \square$$

We remark that the Nishitani's condition (64) holds trivially true whenever the subprincipal symbol vanishes identically. On the other hand, condition (67) for an homogeneous operator (16) become:

$$|b_t + bb_x| \lesssim |\delta_t + b\delta_x| + |\delta|, \tag{70}$$

since we can estimate

$$|2bb_x + a_x| = |\Delta_x| = 2|\delta\delta_x| \lesssim |\delta|.$$

We notice that (70) does not necessarily hold true: the homogeneous *complete* operator

$$\mathrm{L} = \partial_t^2 + 2t\partial_t\partial_x + t^2\partial_x^2 \tag{71}$$

does not satisfy (70). Indeed $b_t \equiv 1$ whereas the right-hand term vanishes identically, since $\Delta \equiv 0$. In facts it is well known that the Cauchy problem (1) for the operator (71) is ill-posed. Indeed, via the change of variables

$$\begin{cases} s = t, \\ y = x - \dfrac{1}{2}t^2, \end{cases}$$

the operator (71) is transformed into:

$$\mathrm{L}^{\mathcal{H}} = \partial_s^2 - \partial_y,$$

for which the Cauchy problem is ill-posed.

On the other hand, (70) trivially holds true for an *incomplete* homogeneous operator (21) since the left-hand term vanishes.

In space dimension $n \geq 2$, the Cauchy problem for operators with identically vanishing sub-principal symbol is not necessarily well-posed. In particular, the Cauchy problem (1) for the *complete* operator

$$\mathrm{L} = \partial_t^2 + 2x_2\partial_t\partial_{x_1} + x_2^3\partial_{x_1}^2 - \partial_{x_2}^2, \tag{72}$$

is ill-posed at the origin [1]. We notice that such an operator has analytic (in facts polynomial) coefficients and that its sub-principal symbol is identically zero. Moreover, (72) is homogeneous.

However, one could ask if the Cauchy problem for an *incomplete* operator with analytic coefficients is always well-posed. We prove that the answer to this question is negative by exhibiting a counter-example obtained by applying the change of variables in §2 to (72):

Theorem 2. *The Cauchy problem* (1) *for the incomplete operator*

$$\mathrm{L}^{\mathcal{H}} = \partial_s^2 - (y_2^2(1-y_2) + s^2)\partial_{y_1}^2 + 2s\partial_{y_1}\partial_{y_2} - \partial_{y_2}^2, \tag{73}$$

is ill-posed at the origin. We notice that such an operator has analytic (*in facts polynomial*) *coefficients and that its sub-principal symbol is identically zero. Moreover,* (73) *is homogeneous.*

Proof. By applying the change of variables

$$\begin{cases} s = t, \\ y_1 = x_1 - tx_2, \\ y_2 = x_2, \end{cases}$$

to (72), we get the transformed operator (73). Indeed:

$$\begin{aligned}
\Delta_{11}^{\mathcal{H}}(s, y) &= [(\partial_{x_1}g_1)^2\Delta_{11} + (\partial_{x_2}g_1)^2\Delta_{22}] \circ \mathcal{H}(s, y) \\
&= [x_2^2(1-x_2) + t^2] \circ \mathcal{H}(s, y) \\
&= y_2^2(1-y_2) + s^2, \\
\Delta_{12}^{\mathcal{H}}(s, y) &= [(\partial_{x_2}g_1)(\partial_{x_2}g_2)\Delta_{22}] \circ \mathcal{H}(s, y) = -s, \\
\Delta_{22}^{\mathcal{H}}(s, y) &= [(\partial_{x_2}g_2)^2\Delta_{22}] \circ \mathcal{H}(s, y) = 1.
\end{aligned}$$

Therefore the Cauchy problem for the *complete* operator (73) and the Cauchy problem for the *incomplete* operator (72) are equivalent: both of them are ill-posed (at the origin). This concludes the proof. $\square$

4. On the Mizohata's condition

Let space dimension be $n = 1$. We compare our condition (15) with the Mizohata's condition (23):

$$b^2(t, x) \lesssim \Delta(t, x).$$

We introduce the following:

DEFINITION 1. We say that $f(t)$, smooth, has finite degeneracy (at the origin) when there is some integer $\kappa \in \mathbb{N}$ such that we have

$$\partial_t^\kappa f(0) \neq 0.$$

Theorem (S. Mizohata [7]). *We assume that the homogeneous operator* (16) *has coefficients depending only on* t, *i.e.*

$$\mathrm{L} = \partial_t^2 - a(t)\partial_x^2 + 2b(t)\partial_t\partial_x.$$

We also assume that $a(0) = b(0) = 0$ *and that* $a(t)$ *or* $b(t)$ *has finite degeneracy. Therefore the Cauchy problem* (1) *is well-posed if and only if* L *satisfies* (23).

The Mizohata's condition (23) trivially holds true whenever L is *incomplete*.

REMARK 5. If we assume the Mizohata's condition (23), then our condition (15) for *complete* operators (14) in space dimension $n = 1$ reduces to the Oleĭnik's condition (7). Indeed, in (15) the left-hand term $(cb)^2$ can be easily estimated by $A\Delta$, whereas by Glaeser's inequality (20) it follows that:

$$|b\Delta_x| \lesssim b^2 + (\Delta_x)^2 \lesssim \Delta.$$

In particular, if the homogeneous *complete* operator (16) verifies (23), then our condition (19) reduces to:

$$t(b_t(t, x))^2 \lesssim A\Delta(t, x) + \Delta_t(t, x). \tag{74}$$

Consequently, the Cauchy problem for an homogeneous *complete* operator with coefficients depending only on x, is well-posed, provided Mizohata's condition (23) be fulfilled.

REMARK 6. If L is homogeneous and has analytic coefficients depending only on t, then our condition (19) and the Nishitani's condition (70) are locally equivalent to the Mizohata's condition (23), provided that $a(0) = b(0) = 0$. Indeed we have:

$$b(t) = \beta_l t^l + \mathcal{O}(t^{l+1}), \quad \text{with} \quad \beta_l \neq 0,$$
$$\Delta(t) = \alpha_{2\kappa} t^{2\kappa} + \mathcal{O}(t^{2\kappa+1}), \quad \text{with} \quad \alpha_{2\kappa} > 0,$$

for some integer $\kappa, l \in \mathbb{N}^*$. Hence (19), (23) and (70) are locally equivalent to ask that $l \geq \kappa$.

Colombini and Orrú [2] proved, under a finite degeneracy assumption, that:

$$\mu^2 + \lambda^2 \lesssim (\mu - \lambda)^2, \quad \text{for any pair of } \textit{characteristic roots} \quad \mu, \lambda, \tag{75}$$

is a necessary and sufficient condition for the well-posedness of the Cauchy problem for higher order homogeneous operators with $\mathcal{C}^\infty([0, T], \mathbb{R})$ coefficients depending only on t. If the operator has order $m = 2$, then the Colombini-Orrú's condition (75) is equivalent to the Mizohata's condition (23).

On the other hand, Spagnolo and Taglialatela [11] proved that (75) is a sufficient condition for the well-posedness of the Cauchy problem for homogeneous operators of any order, with coefficients depending only on x.

Corollary 1 is more general than Theorem 1.1 in [11] for second-order homogeneous operators. Indeed, in this case the Mizohata's condition (23) implies our condition (19), since (74) holds trivially true. However (19) does not imply (23), as the following counter-example shows.

EXAMPLE 2. We consider the homogeneous operator

$$\mathrm{L} = \partial_t^2 + 2x^2 \partial_t \partial_x + x^4(1 - x^2)\partial_x^2. \tag{76}$$

We have $(b(x))^2 = x^4$ whereas $\Delta(x) = x^6$: in $[-\varepsilon, \varepsilon]$ the Mizohata's condition (23) does not hold true. Nevertheless we have the following estimates:

$$|b(x)\Delta_x(x)| = 6x^7 \leq 6\varepsilon x^6 = 6\varepsilon\Delta(x),$$
$$(b(x)b_x(x))^2 = 4x^6 = 4\Delta(x),$$

hence our condition (19) holds true.

In space dimension $n \geq 2$, we could consider the following generalization of the Mizohata's condition (23):

$$(b(t, x, \xi))^2 \lesssim \Delta(t, x, \xi), \tag{77}$$

but such a condition does not ensure the well-posedness. Indeed, the operator (72) fulfills (77) but the corresponding Cauchy problem is ill-posed [1]. The following corollary of Theorem 1 extends Theorem 1.1 in [11] for second order operators in space dimension $n \geq 2$.

Corollary 2. *Assume that* L *has coefficients depending only on* x *and assume that*

$$\Delta(x,\xi)=\varphi(x)\Delta^{(0)}(x,\xi), \quad \textit{with} \quad \Delta^{(0)}(x,\xi)\approx|\xi|^2. \tag{78}$$

If L *satisfies* (77) *together with the Oleĭnik's condition* (7) (*that reduces to* $d^2 \lesssim \Delta$), *then the Cauchy problem* (1) *for* L *is well-posed. Moreover, if* L *satisfies the condition* (8) *then* (1) *is globally well-posed.*

REMARK 7. In space dimension $n=1$, the condition (78) is trivially satisfied. The operator (72) does not satisfy (78) since $\Delta_{11}(x)=x_2^3$ whereas $\Delta_{22}\equiv 1$.

Proof. First we prove that

$$|\Delta_{ij}(x)|\lesssim\varphi(x) \quad (i,\ j=1,\dots,n). \tag{79}$$

Indeed we have:

$$0\le\Delta_{ii}(x)=\Delta(x,e_i)=\varphi(x)\Delta^{(0)}(x,e_i)\approx\varphi(x) \quad (i=1,\dots,n),$$

thanks to (78). On the other hand (see (62)):

$$|\Delta_{ij}|\le\sqrt{\Delta_{ii}\Delta_{jj}}\lesssim\varphi \quad (i,\ j=1,\dots,n).$$

Now we can prove that $|\gamma|\lesssim\Delta$. Indeed, thanks to (79) we have that:

$$\sum_{k=1}^n|\Delta_{ik}(x)|\,|\partial_{x_k}b_j(x)|\,|\xi_i\xi_j|\lesssim\varphi(x)|\xi|^2\approx\Delta(x,\xi) \quad (i,\ j=1,\dots,n).$$

Analogously $|\mathbf{b}\cdot\nabla_x\Delta|\lesssim\Delta$. Indeed, thanks to (77) and (78) we get:

$$\begin{aligned}|b_k(x)\Delta_{x_k}(x,\xi)| &\le |\xi|^2b_k^2(x)+|\xi|^{-2}(\Delta_{x_k}(x,\xi))^2\\ &\lesssim |\xi|^2(b(x,e_k))^2+\Delta(x,\xi)\\ &\lesssim |\xi|^2\Delta(x,e_k)+\Delta(x,\xi)\\ &\approx \Delta(x,\xi) \quad (k=1,\dots,n).\end{aligned}$$

Here we applied Glaeser's inequality (20) to

$$x_k\mapsto\Delta(t,x,\xi),$$

which is a positive function in $\mathcal{B}^\infty(\mathbb{R},\mathbb{R})$ depending on the space variable x_k and on the $(2n-1)$-dimensional parameter (x',ξ) where $x':=(x_j)_{j\neq k}$. In facts:

$$(\Delta_{x_k}(x,\xi))^2\le\Delta(x,\xi)\sup_{x_k\in\mathbb{R}}|\partial_{x_k}^2\Delta(x,\xi)|\lesssim|\xi|^2\Delta(x,\xi).$$

Analogously, $|\rho| \lesssim \Delta$ thanks to (79).

Thus our conditions (10) and (12) reduce to

$$(d - cb)^2 \lesssim \Delta,$$

which holds true thanks to the Oleĭnik's condition (7) and to the generalized Mizohata's condition (77). To conclude the proof, we apply Theorem 1. □

5. The 2 x 2 first-order systems

Through this section we study the Cauchy problem for the 2×2 first-order systems:

$$\begin{cases} \mathrm{L}(t, x, \partial_t, \partial_x)U(t, x) = F(t, x), \\ U(0, x) = U_0(x), \end{cases} \tag{80}$$

in space dimension $n = 1$ with

$$\mathrm{L} = \mathrm{I}\partial_t + A(t, x)\partial_x + B(t, x). \tag{81}$$

We assume that $A, B \in \mathcal{B}^\infty(G_T, M_2(\mathbb{R}))$.

NOTATION. We define:

$$A := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad A^+ := \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix},$$

and we notice that

$$A + A^+ = (\mathrm{tr}\, A)\mathrm{I}, \quad \text{and} \quad AA^+ = (\det A)\mathrm{I}.$$

Moreover, we put:

$$\mathcal{A} := \frac{1}{2}(A - A^+) = \begin{pmatrix} \dfrac{A_{11} - A_{22}}{2} & A_{12} \\ A_{21} & \dfrac{A_{22} - A_{11}}{2} \end{pmatrix}.$$

We assume that L is hyperbolic, that is:

$$\Delta := \left(\frac{\mathrm{tr}\, A}{2}\right)^2 - \det A \equiv -\det \mathcal{A} \geq 0.$$

Theorem 3. *Assume that there is some constant $A > 0$ for which*:

$$t\{(\mathrm{tr}(A^+B) - (\mathrm{tr}\, B)b)^2 + \|\mathcal{A}_t + b\mathcal{A}_x\|^2\} \lesssim \Phi_A, \tag{82}$$

where $b := (1/2)\,\mathrm{tr}\,A$, $\|M\| := \max_{i,j=1,2}|M_{ij}|$ *and*

$$\Phi_A := A\Delta + \Delta_t + b\Delta_x. \tag{83}$$

Hence the Cauchy problem (80) *is well-posed in* $\mathcal{C}^\infty$.

Moreover, if $\mathrm{tr}\,A \equiv \mathrm{tr}\,A(t)$ *is not depending on* x, *then* (80) *is globally well-posed.*

REMARK 8. If we assume that

$$(\mathrm{tr}\,A)^2 \lesssim \Delta, \tag{84}$$

then our condition (82) reduces to:

$$t\{(\mathrm{tr}(A^+B))^2 + \|\mathcal{A}_t\|^2\} \lesssim A\Delta + \Delta_t. \tag{85}$$

Moreover, if $\mathcal{A} = \mathcal{A}(x)$ then (85) reduces to:

$$(\mathrm{tr}(A^+B))^2 \lesssim \Delta. \tag{86}$$

We remark that (84) and (86) involve only matrices invariants.

In order to prove Theorem 3 we recall a result of Ebert [4] that extends the Oleĭnik's theorem [10] to 2×2 second-order systems with a scalar principal part.

Theorem (M. Ebert [4]). *The Cauchy problem*

$$\begin{cases} \mathcal{L}U(t,x) = F(t,x), \\ U(0,x) = U_0(x), \\ U_t(0,x) = U_1(x), \end{cases} \tag{87}$$

for the second-order system

$$\mathcal{L}U = U_{tt} - (\Delta(t,x)U_x)_x + C(t,x)U_t + D(t,x)U_x + E(t,x)U,$$

with coefficients in $\mathcal{B}^\infty(G_T, M_2(\mathbb{R}))$, *is globally well-posed in* $\mathcal{C}^\infty$ *if there is some constant* $A > 0$ *such that*

$$t\|D\|^2 \lesssim A\Delta + \Delta_t. \tag{88}$$

Proof of Theorem 3. We put:

$$a := \det A, \quad b := \frac{1}{2}\,\mathrm{tr}\,A, \quad c := \mathrm{tr}\,B, \quad d := \mathrm{tr}(A^+B),$$

and we define

$$\mathcal{L}(t, x, \partial_t, \partial_x)U := U_{tt} + (bU_x)_t + (bU_t)_x + (aU_x)_x + cU_t + dU_x.$$

We compose L on the left-hand with the operator

$$N^{(l)}(t, x, \partial_t, \partial_x) := \mathrm{L}^+(t, x, \partial_t, \partial_x) + A_x^+(t, x),$$

where

$$\mathrm{L}^+ = \mathrm{I}\partial_t + A^+\partial_x + B^+,$$

hence we have

$$\begin{aligned} N^{(l)}\,\mathrm{L}\,U &= U_{tt} + 2bU_{tx} + aU_{xx} + cU_t + dU_x \\ &\qquad + (A_t + A^+A_x)U_x + A_x^+U_t + A_x^+AU_x + E^{(l)}U \\ &= \mathcal{L}U + (A_x^+ - b_x)U_t + (A_t + A^+A_x + A_x^+A - b_t - a_x)U_x + E^{(l)}U. \end{aligned}$$

By the identities $A_xA^+ + AA_x^+ = a_x\mathrm{I}$ and $A - b\mathrm{I} = \mathcal{A} = -(A^+ - b\mathrm{I})$, we get:

$$N^{(l)}\,\mathrm{L}\,U = \mathcal{L}U - \mathcal{A}_xU_t + \mathcal{A}_tU_x + E^{(l)}U.$$

As in §1 we take V, nbd. of the initial line, and $\mathcal{G} = (\pi_0, g)$, smooth change of variables on V, such that $N^{(l)}\,\mathrm{L}$ is equivalent to:

$$(N^{(l)}\,\mathrm{L})^{\mathcal{H}}V = V_{ss} - (\Delta^{\mathcal{H}}(s, y)V_y)_y + C^{\mathcal{H}}(s, y)V_s + D^{\mathcal{H}}(s, y)V_y + E^{\mathcal{H}}(s, y)V, \tag{89}$$

where:

$$\begin{aligned} \Delta^{\mathcal{H}} &= [g_x^2\Delta] \circ \mathcal{H}, \\ C^{\mathcal{H}} &= [(c - b_x)\mathrm{I} - \mathcal{A}_x] \circ \mathcal{H}, \\ D^{\mathcal{H}} &= [(\hat{h}_y g_{xx}\Delta + d - cb)\mathrm{I} + \mathcal{A}_t + b\mathcal{A}_x] \circ \mathcal{H}, \\ E^{\mathcal{H}} &= E^{(l)} \circ \mathcal{H}. \end{aligned}$$

The system

$$(N^{(l)}\,\mathrm{L}\,a)^{\mathcal{H}}V = [N^{(r)}\,\mathrm{L}(V \circ \mathcal{G})] \circ \mathcal{H},$$

verifies (88) in $W := \mathcal{G}(V)$, since

$$t\|(d - cb)\mathrm{I} + (\partial_t + b\partial_x)\mathcal{A}\|^2 \lesssim A\Delta + \Delta_t + b\Delta_x, \tag{90}$$

holds true thanks to condition (82).

We compose L on the right-hand with the operator

$$N^{(r)}(t, x, \partial_t, \partial_x) := \mathrm{L}^{+}(t, x, \partial_t, \partial_x) - A_x^{+}(t, x),$$

thus (here we use again $\mathcal{A} = -(A^{+} - b\mathrm{I})$):

$$\begin{aligned} \mathrm{L}\, N^{(r)} U &= U_{tt} + 2bU_{tx} + aU_{xx} + cU_t + dU_x - A_x^{+} U_t + A_t^{+} U_x + E^{(r)} U \\ &= \mathcal{L} U + (\mathcal{A}_x - 2b_x) U_t - (\mathcal{A}_t + a_x) U_x + E^{(l)} U. \end{aligned}$$

The system $(\mathrm{L}\, N^{(r)})^{\mathcal{H}} V$ verifies (88) in W. Indeed by applying the Glaeser's inequality (20) to

$$(2bb_x - a_x)^2 = \Delta_x^2 \lesssim \Delta,$$

it follows that

$$t \|(d - a_x - cb + 2bb_x)\mathrm{I} - (\partial_t + b\partial_x)\mathcal{A}\|^2 \lesssim A\Delta + \Delta_t + b\Delta_x,$$

holds true thanks to condition (82).

Following the proof of Corollary 1 we can prove that the Cauchy problem (87) is well-posed for both the operators $N^{(l)}\,\mathrm{L}$ and $\mathrm{L}\, N^{(r)}$. Consequently, the Cauchy problem (80) for L is well-posed. Moreover, if $\operatorname{tr} A \equiv \operatorname{tr} A(t)$ then we can take $V = G_T = W$ and (87) is globally well-posed. $\square$

ACKNOWLEDGMENT. This work was carried out mainly during the stay of the author at the Department of Mathematics, University of Pisa, in April-September 2006. The author wishes to warmly express his sincere gratitude to professor S. Spagnolo, who proposed and supervised the work, for his constant encouragement and many useful advices. The author wishes to thank G. Taglialatela for his help in reviewing the work.

References

[1] E. Bernardi and A. Bove: *On the Cauchy problem for some hyperbolic operator with double characteristics*; in Phase Space Analysis of Partial Differential Equations, Birkhäuser, Boston, MA, 2006, 29–44.

[2] F. Colombini and N. Orrú: *Well-posedness in C^∞ for some weakly hyperbolic equations*, J. Math. Kyoto Univ. **39** (1999), 399–420.

[3] F. Colombini and S. Spagnolo: *An example of a weakly hyperbolic Cauchy problem not well posed in C^∞*, Acta Math. **148** (1982), 243–253.

[4] M.R. Ebert: *On the Cauchy problem for 2×2 weakly hyperbolic systems*, Osaka J. Math. **42** (2005), 831–860.

[5] G. Glaeser: *Racine carrée d'une fonction différentiable*, Ann. Inst. Fourier (Grenoble) **13** (1963), 203–210.

[6] V.Ja. Ivrii and V.M. Petkov: *Necessary conditions for the Cauchy problem for non-strictly hyperbolic equations to be well-posed*, Russ. Math. Surveys. **29** (1974), 1–70.

[7] S. Mizohata: On the Cauchy Problem, Academic Press, Orlando, FL, 1985.

[8] T. Nishitani: *The Cauchy problem for weakly hyperbolic equations of second order*, Comm. Partial Differential Equations **5** (1980), 1273–1296.

[9] T. Nishitani: *A necessary and sufficient condition for the hyperbolicity of second order equations with two independent variables*, J. Math. Kyoto Univ. **24** (1984), 91–104.

[10] O.A. Oleĭnik: *On the Cauchy problem for weakly hyperbolic equations*, Comm. Pure Appl. Math. **23** (1970), 569–586.

[11] S. Spagnolo and G. Taglialatela: *Homogeneous hyperbolic equations with coefficients depending on one space variable*, J. Hyperbolic Differ. Equ. **4** (2007), 533–553.

Dipartimento di Matematica
Università di Bari
via E. Orabona 4
70125 Bari
Italy