

DECOMPOSITION THEOREM ON G -SPACES

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(Received March 19, 2007, revised November 19, 2007)

Abstract

In this paper, we introduce the weak G -expansivity which is a generalization of both expansivity and G -expansivity. Also, we define G -stable and G -unstable sets of a homeomorphism on a metric G -space X and investigate properties of them. Finally, we consider the decomposition theorem on G -spaces.

1. Introduction

Let X be a topological space, G be a topological group, and $\theta: G \times X \rightarrow X$ be a map. The triple (X, G, θ) is called a *topological G -space* if the following three conditions are satisfied:

- (1) $\theta(e, x) = x$ for all $x \in X$, where e is the identity of G ;
- (2) $\theta(g, \theta(h, x)) = \theta(gh, x)$ for all $x \in X$ and for all $g, h \in G$;
- (3) θ is continuous.

Here, gh is the group operation on G . Simply, we denote $\theta(g, x)$ by gx and X is usually said to be a *topological G -space*.

For any subset A of X , $G(A)$ is denoted by the set $\{ga : g \in G, a \in A\}$. $G(x)$ is called a *G -orbit* of x . A subset A of X is called *G -invariant* if $G(A) = A$. A map $f: X \rightarrow X$ on a G -space X is said to be *pseudoequivariant* provided that $f(G(x)) = G(f(x))$ for all $x \in X$, and f is said to be *equivariant* provided that $f(gx) = gf(x)$ for all $x \in X$ and $g \in G$.

N. Aoki has proved the following topological decomposition theorem in 1983 ([1]), which is an extension of Smale's spectral decomposition theorem and Bowen's decomposition theorem in dynamical systems. All undefined notions can be found in [2].

Theorem 1.1 ([1]). *Let $f: X \rightarrow X$ be a homeomorphism on a compact metric space X and let $CR(f)$ be the chain recurrent set. If $f|_{CR(f)}: CR(f) \rightarrow CR(f)$ is an expansive homeomorphism with the shadowing property, then*

- (1) *$CR(f)$ contains a finite sequence B_i ($1 \leq i \leq k$) of f -invariant closed subsets such that*

2000 Mathematics Subject Classification. Primary 54H20; Secondary 37B05.

This work was supported by the National Institute for Mathematical Sciences¹ and the Korea Research Foundation Grant funded by the Korean Government (MOEHRD) (KRF-2005-037-C00005)².

- (a) $CR(f) = \bigcup_{i=1}^k B_i$ (disjoint union);
- (b) $f|_{B_i} : B_i \rightarrow B_i$ is topologically transitive,
- (2) for each B_i , there exist a subset X_p of B_i and $a > 0$ such that
 - (a) $f^a(X_p) = X_p$;
 - (b) $X_p \cap f^j(X_p) = \emptyset$ ($0 < j < a$);
 - (c) $f^a|_{X_p} : X_p \rightarrow X_p$ is topologically mixing;
 - (d) $B_i = \bigcup_{j=0}^{a-1} f^j(X_p)$.

A point $x \in X$ is called a G -periodic point of f if there exist an integer $n > 0$ and $g \in G$ such that $f^n(x) = gx$. A point $x \in X$ is called a G -nonwandering point of f if for every open neighborhood U of x , there exist $n > 0$ and $g \in G$ such that $gf^n(U) \cap U \neq \emptyset$. $Per_G(f)$ (resp. $\Omega_G(f)$) is denoted by the set of all G -periodic (resp. G -nonwandering) points of f .

For a homeomorphism f on a metric G -space X , a sequence $\{x_i \in X : i \in \mathbb{Z}\}$ is called a (δ, G) -pseudo orbit for f provided that for each i , there exists $g_i \in G$ such that $d(g_i f(x_i), x_{i+1}) < \delta$. A (δ, G) -pseudo orbit $\{x_i\}$ for f is said to be ϵ -traced by a point $x \in X$ provided that for each i , there exists $g_i \in G$ such that $d(f^i(x), g_i x_i) < \epsilon$.

DEFINITION 1.2 ([5]). A homeomorphism $f : X \rightarrow X$ has the G -shadowing property (GSP) provided that for any $\epsilon > 0$, there exists $\delta > 0$ such that every (δ, G) -pseudo orbit $\{x_i\}$ in X for f is ϵ -traced by a point $x \in X$.

REMARK 1.3. It was proved by E. Shah that, when X is a compact metric G -space and the orbit map $\pi : X \rightarrow X/G$ is a covering map, a pseudoequivariant homeomorphism f on X has the GSP if and only if the induced map $\hat{f} : X/G \rightarrow X/G$ has the shadowing property ([5]).

If a pseudoequivariant continuous onto map $f : X \rightarrow X$ has the GSP where X is a compact metric G -space with G compact, then $f|_{\Omega_G(f)}$ has the GSP ([5]).

The main purpose of this paper is to prove the following theorems on compact metric G -spaces.

Theorem A. Let X be a compact metric G -space with G compact. If $f : X \rightarrow X$ is a pseudoequivariant G -expansive homeomorphism with the GSP, then $\Omega_G(f)$ contains a finite sequence B_i ($1 \leq i \leq n$) of f -invariant, G -invariant, and closed subsets such that

- (1) $f|_{\Omega_G(f)}$ is topologically G -transitive;
- (2) $\Omega_G(f) = \bigcup_{i=1}^n B_i$ (disjoint union);
- (3) $f|_{B_i}$ has the GSP.

A homeomorphism $f : X \rightarrow X$ is said to be topologically G -mixing provided that for every nonempty open subsets U and V of X , there exists an integer N such that

for each $n \geq N$, there is $g_n \in G$ satisfying $g_n f^n(U) \cap V \neq \emptyset$.

Theorem B. *Let $f|_{\Omega_G(f)}: \Omega_G(f) \rightarrow \Omega_G(f)$ be a G -expansive homeomorphism with the GSP. Then, for any f -invariant, G -invariant, open and closed subset $B \subset \Omega_G(f)$ such that $f|_B: B \rightarrow B$ is topologically G -transitive, there are $X_p \subset B$ and $a > 0$ such that*

- (1) $f^a(X_p) = X_p$;
- (2) $X_p \cap f^j(X_p) = \emptyset$ ($0 < j < a$);
- (3) $f^a|_{X_p}: X_p \rightarrow X_p$ is topologically G -mixing;
- (4) $B = \bigcup_{j=0}^{a-1} f^j(X_p)$.

DEFINITION 1.4. A homeomorphism $f: X \rightarrow X$ on a metric G -space X is said to be *weak G -expansive* provided that there exists $\delta > 0$ such that for every $x, y \in X$ with $G(x) \neq G(y)$ if $u \in G(x)$ and $v \in G(y)$, there exists $n = n(u, v) \in \mathbb{Z}$ such that

$$d(f^n(u), f^n(v)) > \delta.$$

The constant δ is called a *weak G -expansive constant* for f .

The weak G -expansivity is a generalization of both expansivity and G -expansivity. Here, G -expansivity has been defined by R. Das ([4]). A homeomorphism $f: X \rightarrow X$ is said to be *G -expansive* provided that there exists $\delta > 0$ such that for every $x, y \in X$ with $G(x) \neq G(y)$, there exists $n \in \mathbb{Z}$ such that

$$d(f^n(u), f^n(v)) > \delta \quad \text{for all } u \in G(x), v \in G(y).$$

The constant δ is called a *G -expansive constant* for f .

REMARK 1.5. R. Das proved that there is no implication between G -expansivity and expansivity by giving counterexamples ([4]).

EXAMPLE 1.6 ([4]). Consider the compact space $X = \{1/n, 1 - 1/n: n \in \mathbb{N}\}$ with the usual metric and let the topological group $G = \{-1, 1\}$ act on X with the action θ defined by $\theta(1, x) = x$ and $\theta(-1, x) = 1 - x$. Define a homeomorphism $f: X \rightarrow X$ by

$$f(x) = \begin{cases} x & \text{if } x = 0, 1; \\ \text{next to the right of } x & \text{if } x \in X \setminus \{0, 1\}. \end{cases}$$

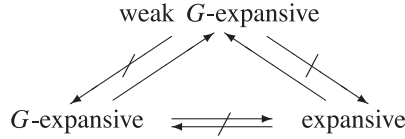
Then f is an expansive map with expansive constant δ ($0 < \delta < 1/6$). But, it is easy to see that for $x, y \in X \setminus \{1/2\}$ with $G(x) \neq G(y)$, there is no $n \in \mathbb{Z}$ such that

$$|f^n(u) - f^n(v)| > \delta \quad \text{for all } u \in G(x), v \in G(y),$$

whatever $\delta > 0$ may be. This means that f is not G -expansive.

EXAMPLE 1.7 ([4]). Consider the compact space $X = \bigcup_{i=1}^n C_i$ with the usual metric, where each C_i is the circle in \mathbb{R}^2 with center the origin and radius i . Denote $G = SO(2)$ by the set of all 2×2 matrices whose determinants are ± 1 and define an action $\theta: G \times X \rightarrow X$ by the usual rotations on X . Then the identity map on X is G -expansive with G -expansive constant δ ($0 < \delta < 1$).

Therefore, all properties of the following diagram are distinguished as we see in Examples 1.6 and 1.7:



DEFINITION 1.8. Let $f: X \rightarrow X$ be a homeomorphism of a metric G -space X . We define a *local G -stable set* $W_\epsilon^s(x)$ and a *local G -unstable set* $W_\epsilon^u(x)$ by

$$\begin{aligned}
 W_\epsilon^s(x) &= \{y \in X : \text{for each } n \geq 0, \\
 &\quad \text{there is } g_n \in G \text{ such that } d(f^n(g_n x), f^n(y)) \leq \epsilon\}, \\
 W_\epsilon^u(x) &= \{y \in X : \text{for each } n \geq 0 \\
 &\quad \text{there is } g_n \in G \text{ such that } d(f^{-n}(g_n x), f^{-n}(y)) \leq \epsilon\}.
 \end{aligned}$$

We modify results of [3] into the following results by weakening the condition “equivariant” into “pseudoequivariant” and deleting the condition “invariant metric”. A metric d on a G -space X is called an *invariant metric* provided that $d(x, y) = d(gx, gy)$ for all $x, y \in X$ and $g \in G$.

REMARK 1.9. Let X be a compact metric G -space with G compact. If $f: X \rightarrow X$ is a weak G -expansive pseudoequivariant homeomorphism with weak G -expansive constant $\delta > 0$, then for every $\gamma > 0$, there is $N > 0$ such that for each $x \in X$ and for each $n \geq N$,

$$f^n(W_\delta^s(x)) \subset W_\gamma^s(f^n(x))$$

and

$$f^{-n}(W_\delta^u(x)) \subset W_\gamma^u(f^{-n}(x)).$$

Proof. We shall prove only the case of a local G -stable set because the other case can be proved similarly. To do it, suppose that there exists $\gamma > 0$ such that for all $N > 0$, there are $x \in X$ and $n \geq N$ satisfying

$$f^n(W_\delta^s(x)) \not\subset W_\gamma^s(f^n(x)).$$

Let $N > 0$. Then there are $x_1 \in X$ and $n \geq N$ satisfying

$$f^n(W_\delta^s(x_1)) \not\subset W_\gamma^s(f^n(x_1)),$$

that is, there exists $y_1 \in W_\delta^s(x_1)$ such that $f^n(y_1) \notin W_\gamma^s(f^n(x_1))$. So there exists $i \geq 0$ such that for every $h \in G$,

$$d(f^i(hf^n(x_1)), f^i(f^n(y_1))) > \gamma.$$

Because f is pseudoequivariant, there exists $i \geq 0$ such that for every $g \in G$,

$$d(gf^{i+n}(x_1), f^{i+n}(y_1)) > \gamma.$$

Take $m_1 = i + n$ and choose $N = m_1 + 1$.

Continuing the process, we can find sequences $m_n > 0$, x_n , and $y_n \in X$ such that

- (1) $y_n \in W_\delta^s(x_n)$;
- (2) $d(hf^{m_n}(x_n), f^{m_n}(y_n)) > \gamma$ for all $h \in G$;
- (3) $\lim_{n \rightarrow \infty} m_n = \infty$.

It follows from $y_n \in W_\delta^s(x_n)$ that for each $i \geq -m_n$, there exists $g_{i+m_n} \in G$ such that

$$d(f^{i+m_n}(g_{i+m_n}x_n), f^{i+m_n}(y_n)) \leq \delta.$$

Since f is pseudoequivariant, for each g_{i+m_n} , there exists $h_{i+m_n} \in G$ such that

$$d(f^i(h_{i+m_n}f^{m_n}(x_n)), f^i(f^{m_n}(y_n))) = d(f^{i+m_n}(g_{i+m_n}x_n), f^{i+m_n}(y_n)).$$

Hence, for each $i \geq -m_n$,

$$d(f^i(h_{i+m_n}f^{m_n}(x_n)), f^i(f^{m_n}(y_n))) \leq \delta.$$

If $f^{m_n}(x_n) \rightarrow x$, $f^{m_n}(y_n) \rightarrow y$, and $h_{i+m_n} \rightarrow h$ as $n \rightarrow \infty$, then

$$d(f^i(hx), f^i(y)) \leq \delta \quad \text{for all } i \in \mathbb{Z}.$$

Since δ is a weak G -expansive constant for f , $G(x) = G(y)$. But $d(hx, y) = \lim_{n \rightarrow \infty} d(hf^{m_n}(x_n), f^{m_n}(y_n)) \geq \gamma > 0$ for all $h \in G$ by (2). Thus $hx \neq y$ for all $h \in G$, and hence $G(x) \neq G(y)$. This is a contradiction. \square

For a homeomorphism f on a compact metric G -space, we define the following:

$$W^s(x) = \left\{ y \in X : \text{there exists a sequence } g_n \in G \text{ such that} \right. \\ \left. \lim_{n \rightarrow \infty} d(f^n(g_n x), f^n(y)) = 0 \right\};$$

$$W^u(x) = \left\{ y \in X : \text{there exists a sequence } g_n \in G \text{ such that} \right. \\ \left. \lim_{n \rightarrow \infty} d(f^{-n}(g_n x), f^{-n}(y)) = 0 \right\}.$$

$W^s(x)$ (resp. $W^u(x)$) is called a G -stable set (resp. G -unstable set).

REMARK 1.10. Let X be a compact metric G -space with G compact. If $f: X \rightarrow X$ is a weak G -expansive pseudoequivariant homeomorphism with weak G -expansive constant $\delta > 0$, then for each ϵ with $0 < \epsilon < \delta$,

$$W^s(x) = \bigcup_{n \geq 0} f^{-n}(W_\epsilon^s(f^n(x)));$$

$$W^u(x) = \bigcup_{n \geq 0} f^n(W_\epsilon^s(f^{-n}(x))).$$

Proof. (C): Let $y \in W^s(x)$ and $0 < \epsilon < \delta$. Then there exists $N > 0$ such that for each $n \geq N$, we can choose $g_n \in G$ satisfying

$$d(f^n(g_n x), f^n(y)) \leq \epsilon.$$

Thus,

$$d(f^i(f^N(g_{i+N}x)), f^i(f^N(y))) \leq \epsilon \quad \text{for all } i \geq 0.$$

Since f is pseudoequivariant, $f^N(y) \in W_\epsilon^s(f^N(x))$. Therefore,

$$y \in f^{-N}(W_\epsilon^s(f^N(x))) \subset \bigcup_{n \geq 0} f^{-n}(W_\epsilon^s(f^n(x))).$$

(D): Let $y \in f^{-n}(W_\epsilon^s(f^n(x)))$ for some $n \geq 0$. Then $f^n(y) \in W_\epsilon^s(f^n(x))$. It follows from Remark 1.9 that for every $\gamma > 0$ there exists $N > 0$ such that for each $x \in X$ and $m \geq N$,

$$f^{m+n}(y) \in f^m(W_\epsilon^s(f^n(x))) \subset W_\gamma^s(f^{m+n}(x)).$$

So for each $n \geq N$, we can find $g_n \in G$ such that

$$d(f^{m+n}(g_n x), f^{m+n}(y)) \leq \gamma.$$

Since f is pseudoequivariant, $y \in W^s(x)$. The proof is completed. The case of a G -unstable set can be proved similarly. \square

2. Decomposition theorems

First we prepare the following four lemmas to show Theorem A.

Lemma 2.1 ([3]). *Let (X, G, θ) be a compact metric G -space with G compact. Then for any $\epsilon > 0$, there is a finite open cover $\mathcal{U} = \{U_1, \dots, U_s\}$ of X such that $\text{diam}(g\overline{U}_i) \leq \epsilon$ for all $g \in G$ and i with $1 \leq i \leq s$.*

In Lemma 2.1, notice that, for each $g \in G$, the open cover $\{gU : U \in \mathcal{U}\}$ of X satisfies $\text{diam}(hg\overline{U}_i) \leq \epsilon$ for all $h \in G$ and i with $1 \leq i \leq s$.

Lemma 2.2. *Let X be a compact metric G -space with G compact. If \mathcal{U} is a finite open cover of X , then there exists $\delta > 0$ such that for each subset A of X with $\text{diam}(A) \leq \delta$, $A \subset gU$ for some $U \in \mathcal{U}$ and $g \in G$.*

Proof. Suppose not. Then for every $n > 0$ there exists a subset A_n of X such that $\text{diam}(A_n) \leq 1/n$ and $A_n \not\subset gU$ for all $U \in \mathcal{U}$ and $g \in G$. Choose $x_n \in A_n$ for each $n \in \mathbb{N}$. Since X is compact, there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightarrow x$. We fix $g \in G$. Then there is $U \in \mathcal{U}$ with $x \in gU$. Since $X \setminus gU$ is compact, $d(x, X \setminus gU) > 0$. Put $\epsilon = d(x, X \setminus gU)$ and take $n_i > 0$ such that $1/n_i < \epsilon/2$ and $d(x_{n_i}, x) < \epsilon/2$. Then for any $y \in A_{n_i}$,

$$d(y, x) \leq d(y, x_{n_i}) + d(x_{n_i}, x) \leq \frac{1}{n_i} + \frac{\epsilon}{2} < \epsilon.$$

So $y \in gU$. Therefore, $A_{n_i} \subset gU$. This is a contradiction. \square

Lemma 2.3. *Let X be a compact metric G -space with G compact. Then for any $\epsilon > 0$, there exists $\delta > 0$ ($\delta < \epsilon$) such that*

$$d(x, y) < \delta \implies d(gx, gy) < \epsilon \quad \text{for all } g \in G.$$

Proof. Let $\epsilon > 0$. Then it follows from Lemma 2.1 that, for any positive $\epsilon_1 < \epsilon$, there is a finite open cover \mathcal{U} such that $\text{diam}(g\overline{U}) \leq \epsilon_1$ for all $g \in G$ and $U \in \mathcal{U}$. Also, by Lemma 2.2, there is a constant $\delta = \delta(\mathcal{U}) > 0$ such that for any subset A with $\text{diam}(A) \leq \delta$, $A \subset gU$ for some $g \in G$ and $U \in \mathcal{U}$. Let x and y in X with $d(x, y) < \delta$. Then $x, y \in g_0U_0$ for some $g_0 \in G$ and $U_0 \in \mathcal{U}$. Note that $\{g_0U : U \in \mathcal{U}\}$ is an open cover of X . For any $g \in G$, take $g_1 \in G$ such that $g_1 = gg_0$. Then, by Lemma 2.1, $\text{diam}(gg_0\overline{U}) \leq \epsilon_1$, that is, $\text{diam}(g_1\overline{U}) \leq \epsilon_1 < \epsilon$ for all $U \in \mathcal{U}$. Since $gx, gy \in gg_0U_0 = g_1U_0$, $d(gx, gy) < \epsilon$. \square

Lemma 2.4. *Let X be a compact metric G -space with G compact and let f be a pseudoequivariant homeomorphism on X . Then f has the GSP if and only if for any $\epsilon > 0$, we can find $\delta > 0$ such that for every (δ, G) -pseudo orbit $\{x_i\}$ of X for f , there exist $x \in X$ and $h_i \in G$ satisfying*

$$d(f^i(h_i x), x_i) < \epsilon \quad \text{for all } i \in \mathbb{Z}.$$

Proof. Suppose that f has the GSP and let $\epsilon > 0$. Then, by Lemma 2.3, there exists $\epsilon_0 > 0$ ($\epsilon_0 < \epsilon$) such that for each $x, y \in X$,

$$d(x, y) < \epsilon_0 \implies d(gx, gy) < \epsilon \quad \text{for all } g \in G.$$

Let δ be the constant corresponding to ϵ_0 in the definition of the GSP. Then every (δ, G) -pseudo orbit $\{x_i\}$ of X for f is ϵ_0 -traced by a point $x \in X$, that is, for each i , there exists $g_i \in G$ such that

$$d(f^i(x), g_i x_i) < \epsilon_0 \quad \text{for all } i \in \mathbb{Z}.$$

Since f is pseudoequivariant, for each $g_i \in G$, there exists $h_i \in G$ such that

$$g_i^{-1} f^i(x) = f^i(h_i x).$$

Moreover, $d(g_i^{-1} f^i(x), x_i) < \epsilon$ and hence $d(f^i(h_i x), x_i) < \epsilon$ for all $i \in \mathbb{Z}$.

The converse can be proved similarly. □

We have ([5]) that $f(\Omega_G(f)) = \Omega_G(f)$ and $CR_G(f) = \Omega_G(f)$ for a pseudoequivariant homeomorphism f with GSP on a compact metric G -space X where G is compact.

For $x, y \in X$ and $\delta > 0$, x is said to be (δ, G) -related to y (denoted by $x \overset{\delta}{\sim}_G y$) if there exist finite (δ, G) -pseudo orbits $\{x = x_0, x_1, \dots, x_k = y\}$ and $\{y = y_0, y_1, \dots, y_n = x\}$ for f . If for every $\delta > 0$, x is (δ, G) -related to y , then x is said to be G -related to y (denoted by $x \sim_G y$). A point x is said to be a G -chain recurrent point of f if $x \sim_G x$. $CR_G(f)$ is denoted by the set of all G -chain recurrent points of f . A homeomorphism $f: X \rightarrow X$ is called *topologically G -transitive* provided that for every nonempty open subsets U and V of X , there exist an integer $n > 0$ and $g \in G$ such that $gf^n(U) \cap V \neq \emptyset$.

Proof of Theorem A. Since the pseudoequivariant homeomorphism f satisfies the GSP, $CR_G(f) = \Omega_G(f)$. Thus $\Omega_G(f) = \bigcup_{\lambda} B_{\lambda}$ where each B_{λ} is an equivalence class under the relation \sim_G which is defined in $CR_G(f)$.

Claim 1. *Each B_{λ} is closed in $\Omega_G(f)$.*

Proof. Let $x \in \overline{B_{\lambda}}$. Then we can find a sequence $\{x_i\}$ in B_{λ} which converges to x . Let $\alpha > 0$ be given. Then there exists a finite open cover $\{U_1, \dots, U_s\}$ of X such that

$$\text{diam}(g\overline{U}_i) \leq \frac{\alpha}{2} \quad \text{for all } g \in G \quad \text{and } i \quad \text{with } 1 \leq i \leq s$$

by Lemma 2.1. So $f(x) \in U_i$ for some i . Choose an ϵ_0 -neighborhood $N_{\epsilon_0}(f(x))$ of $f(x)$ such that $N_{\epsilon_0}(f(x)) \subset U_i$. Then since f is uniformly continuous, there exists $\delta_0 > 0$ such that

$$d(x, y) < \delta_0 \implies d(f(x), f(y)) < \epsilon_0.$$

Because $\{x_i\}$ converges to x , there is $J > 0$ such that $d(x_J, x) < \min\{\alpha/2, \delta_0\}$. From the fact that $x_J \in CR_G(f)$, we can find a $(\alpha/2, G)$ -pseudo orbit

$$\{x_J = y_0, y_1, \dots, y_{k-1}, y_k = x_J\}.$$

So $d(gf(y_0), y_1) < \alpha/2$ for some $g \in G$. Also $d(f(y_0), f(x)) < \epsilon_0$ and hence $d(gf(y_0), gf(x)) < \alpha/2$. Thus,

$$d(gf(x), y_1) \leq d(gf(x), gf(y_0)) + d(gf(y_0), y_1) < \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha.$$

Therefore, $\{x, y_1, \dots, y_k = x_J\}$ is an (α, G) -pseudo orbit. It is clear that there is an (α, G) -pseudo orbit from x_J to x by the uniform continuity of f . It follows from $x \overset{\alpha}{\sim}_G x_J$ that $x \overset{\alpha}{\sim}_G x_i$ for all i because each $x_i \in B_\lambda$. Since α is arbitrary, $x \in B_\lambda$. Therefore, B_λ is closed. \square

Claim 2. *Each B_λ is f -invariant.*

Proof. To prove this, we firstly show that $x \sim_G f(x)$ for all $x \in \Omega_G(f)$. Let $\alpha > 0$. Then there is $\delta > 0$ ($\delta < \alpha$) such that

$$d(a, b) < \delta \implies d(f^2(a), f^2(b)) < \alpha.$$

Since $x \in \Omega_G(f)$, there are $n > 0$ and $g \in G$ such that

$$gf^n(N_\delta(x)) \cap N_\delta(x) \neq \emptyset$$

where $N_\delta(x)$ is a δ -neighborhood of x . Then there exists $z \in N_\delta(x)$ such that $gf^n(z) \in N_\delta(x)$. Hence

$$\{f(x), f^2(z), \dots, f^{n-1}(z), x\}$$

is an (α, G) -pseudo orbit and thus, $x \sim_G f(x)$. Since f is a homeomorphism, we can show that $x \sim_G f^{-1}(x)$ for all $x \in \Omega_G(f)$ similarly. Therefore, $f(B_\lambda) = B_\lambda$ for each λ . \square

Claim 3. *$Per_G(f)$ is dense in $\Omega_G(f)$.*

Proof. Let $\alpha > 0$ be a G -expansive constant for f and take $\epsilon < \alpha/2$. Since f has the GSP, there exists $\delta > 0$ ($\delta < \epsilon$) such that every (δ, G) -pseudo orbit is ϵ -traced by a point in X . Since f is uniformly continuous, there exists a positive constant $\gamma < \delta$ such that if $d(a, b) < \gamma$, then $d(f(a), f(b)) < \delta$. Let $p \in \Omega_G(f)$. Then for every γ -neighborhood $N_\gamma(p)$ of p , there exist an integer $n > 0$ and $g \in G$ such that

$$gf^n(N_\gamma(p)) \cap N_\gamma(p) \neq \emptyset.$$

Choose a point $y \in gf^n(N_\gamma(p)) \cap N_\gamma(p)$. Since $f^{-n}(g^{-1}y) \in N_\gamma(p)$,

$$d(f(p), f(f^{-n}(g^{-1}y))) < \delta.$$

Hence

$$\{\dots, x_0 = p, x_1 = f^{-n+1}(g^{-1}y), x_2 = f^{-n+2}(g^{-1}y), \dots, x_{n-1} = f^{-1}(g^{-1}y), x_n = p, \dots\}$$

is a (δ, G) -pseudo orbit for f . Since f has the GSP, it follows from Lemma 2.4 that, for each $i \in \mathbb{Z}$, there exist $x \in X$ and $g_i \in G$ such that

$$d(f^i(g_i x), x_i) < \epsilon \quad \text{for all } i \in \mathbb{Z}.$$

Thus,

$$\begin{aligned} d(f^k(f^n(g_{k+n}x)), f^k(g_k x)) &\leq d(f^k(f^n(g_{k+n}x)), x_{k+n}) + d(x_{k+n}, f^k(g_k x)) \\ &= d(f^k(f^n(g_{k+n}x)), x_{k+n}) + d(x_k, f^k(g_k x)) \\ &< 2\epsilon < \alpha \end{aligned}$$

for all k . Since α is a G -expansive constant for f ,

$$G(f^n(x)) = G(x),$$

and hence

$$g_0 x \in \text{Per}_G(f) \cap N_\epsilon(p)$$

where $N_\epsilon(p)$ is an ϵ -neighborhood of p . Therefore, $\text{Per}_G(f)$ is dense in $\Omega_G(f)$. \square

Claim 4. *Each B_λ is open in $\Omega_G(f)$.*

Proof. Let $\alpha > 0$ be a G -expansive constant for f and let $\epsilon < \alpha$. Denote

$$N_\delta(B_\lambda) = \{y \in \Omega_G(f) : d(y, B_\lambda) < \delta\}$$

where δ is the constant corresponding to ϵ in the definition of the GSP for $f|_{\Omega_G(f)}$. Then for a point $p \in N_\delta(B_\lambda) \cap \text{Per}_G(f)$, there exists $y \in B_\lambda$ such that

$$d(y, p) < \delta.$$

Since $f|_{\Omega_G(f)}$ has the GSP, it follows from Remark 1.10 that

$$W^u(p) \cap W^s(y) \neq \emptyset$$

and

$$W^s(p) \cap W^u(y) \neq \emptyset.$$

Here, $W^s(p)$ and $W^u(p)$ are defined on $\Omega_G(f)$. So, there exists $y_0 \in B_\lambda$ (in particular,

y_0 belongs to the α -limit set $\alpha(y)$ such that $y_0 \sim p$, that is, $p \in B_\lambda$. Therefore,

$$B_\lambda \supset \overline{N_\delta(B_\lambda) \cap \text{Per}_G(f)} \supset N_\delta(B_\lambda) \cap \overline{\text{Per}_G(f)} = N_\delta(B_\lambda),$$

that is, B_λ is open in $\Omega_G(f)$. \square

Since X is compact and $\Omega_G(f)$ is a closed subset of X , $\Omega_G(f)$ can be covered by finitely many B_λ 's, that is, $\Omega_G(f) = \bigcup_{i=1}^n B_i$.

Claim 5. *Each B_i is G -invariant.*

Proof. Let $x \in B_i$, $g \in G$, and $\delta > 0$. We shall show that $gx \in B_i$. Since $x \in B_i$, there exists a (δ, G) -pseudo orbit $\{x_0 = x, x_1, \dots, x_{n-1}, x_n = x\}$. Then $d(g_0 f(x), x_1) < \delta$ for some $g_0 \in G$. Since f is pseudoequivariant, we can take $h \in G$ such that $g_0 f(x) = hf(gx)$. Thus $\{gx, x_1, \dots, x_{n-1}, x_n = x\}$ is a (δ, G) -pseudo orbit. By Lemma 2.3, there exists $\gamma > 0$ ($\gamma < \delta$) such that

$$d(x, y) < \gamma \implies d(gx, gy) < \delta \quad \text{for all } g \in G.$$

Let $\{x_0 = x, x_1, \dots, x_{n-1}, x_n = x\}$ be a (γ, G) -pseudo orbit. Then

$$d(g_{n-1} f(x_{n-1}), x) < \gamma \quad \text{for some } g_{n-1} \in G$$

and hence $d(gg_{n-1} f(x_{n-1}), gx) < \delta$. Thus $\{x_0 = x, x_1, \dots, x_{n-1}, gx\}$ is a (δ, G) -pseudo orbit. Since δ is arbitrary, $gx \sim_G x$. Therefore, $gx \in B_i$. \square

Claim 6. *$f|_{B_i}$ has the GSP.*

Proof. Let $0 < \epsilon < \min\{d(B_i, B_j) : i \neq j, 1 \leq i, j \leq n\}$ be given. Since $f|_{\Omega_G(f)}$ has the GSP, there exists $\delta < \epsilon$ such that every (δ, G) -pseudo orbit $\{x_k\} \subset B_i$ is ϵ -traced by a point $x \in \Omega_G(f)$. This means that, for each k , there exists $g_k \in G$ such that

$$d(f^k(x), g_k x_k) < \epsilon.$$

Since B_i is G -invariant and $x_0 \in B_i$, $g_0 x_0 \in B_i$. Therefore $x \in B_i$. \square

Claim 7. *$f|_{B_i}$ is topologically G -transitive.*

Proof. Let U and V be nonempty open subsets of B_i . Take $x \in U$ and $y \in V$. Then $x \sim_G y$. Let $N_\epsilon(x)$ and $N_\epsilon(y)$ be ϵ -neighborhoods of x and y respectively such that $N_\epsilon(x) \subset U$ and $N_\epsilon(y) \subset V$. Choose a positive $\epsilon_1 < \epsilon$ such that

$$d(a, b) < \epsilon_1 \implies d(ga, gb) < \epsilon \quad \text{for all } g \in G.$$

Since $f|_{B_i}$ has the GSP, there exists $\delta_1 > 0$ such that every (δ_1, G) -pseudo orbit in B_i is ϵ_1 -traced by a point in B_i . Thus, a (δ_1, G) -pseudo orbit $\{x_0 = x, \dots, x_n = y\} \subset B_i$ from x to y is ϵ_1 -traced by a point $z \in B_i$. In particular,

$$d(z, g_0x) < \epsilon_1 \quad \text{and} \quad d(f^n(z), g_ny) < \epsilon_1 \quad \text{for some} \quad g_0, g_n \in G.$$

Since $d(g_0^{-1}z, x) < \epsilon$ and $d(g_n^{-1}f^n(z), y) < \epsilon$,

$$g_0^{-1}z \in N_\epsilon(x) \subset U$$

and

$$g_n^{-1}f^n(z) \in N_\epsilon(y) \subset V.$$

Since $f^n(g_0^{-1}z) \in f^n(U)$ and f is pseudoequivariant,

$$g_1f^n(z) \in f^n(U) \quad \text{for some} \quad g_1 \in G.$$

Choose $g \in G$ such that $gg_1 = g_n^{-1}$. Then $g_n^{-1}f^n(z) \in gf^n(U)$. Therefore, $gf^n(U) \cap V \neq \emptyset$. \square

We next prepare the following three lemmas to complete Theorem B.

Lemma 2.5. *Let $f: X \rightarrow X$ be a pseudoequivariant homeomorphism on a compact metric G -space X with G compact. Then*

$$W^i(x) = W^i(p) \quad \text{for any} \quad x \in W^i(p) \quad (i = s, u).$$

Proof. We shall prove only the case $i = s$. Let $y \in W^s(x)$ and let $\epsilon > 0$. Since $y \in W^s(x)$, there exists $N_1 \in \mathbb{N}$ such that $n \geq N_1$ implies that

$$d(f^n(h_nx), f^n(y)) < \frac{\epsilon}{2} \quad \text{for some} \quad h_n \in G.$$

Let $\delta > 0$ be the constant satisfying the following:

$$d(x, y) < \delta \implies d(gx, gy) < \frac{\epsilon}{2} \quad \text{for all} \quad g \in G.$$

Since $x \in W^s(p)$, there exists $N_2 \in \mathbb{N}$ such that $n \geq N_2$ implies that

$$d(f^n(g'_np), f^n(x)) < \delta \quad \text{for some} \quad g'_n \in G.$$

Hence for some $h'_n \in G$ with $h'_nf^n(x) = f^n(h_nx)$,

$$d(h'_nf^n(g'_np), h'_nf^n(x)) < \frac{\epsilon}{2}.$$

Since $h'_n f^n(g'_n p) = f^n(g_n p)$ for some $g_n \in G$,

$$d(f^n(g_n p), f^n(h_n x)) < \frac{\epsilon}{2}.$$

Take $N = \max\{N_1, N_2\}$. Then $n \geq N$ implies that

$$d(f^n(g_n p), f^n(y)) \leq d(f^n(g_n p), f^n(h_n x)) + d(f^n(h_n x), f^n(y)) < \epsilon.$$

Therefore, $W^s(x) \subset W^s(p)$. Similarly, one can prove $W^s(p) \subset W^s(x)$. \square

Lemma 2.6. *Let $f: X \rightarrow X$ be a pseudoequivariant homeomorphism on a compact metric G -space X with G compact and let $x \in W^i(p)$. Then*

$$gx \in W^i(p) \text{ for every } g \in G,$$

and hence

$$G(W^i(p)) = W^i(p) \quad (i = s, u).$$

Proof. Let $x \in W^s(p)$, $g \in G$ and let $\epsilon > 0$. Then there is $\delta > 0$ such that if $d(x, y) < \delta$, then $d(gx, gy) < \epsilon$ for all $g \in G$. Since for each $n \in \mathbb{Z}$, we have $g_n \in G$ such that

$$\lim_{n \rightarrow \infty} d(f^n(g_n p), f^n(x)) = 0,$$

that is, there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies d(f^n(g_n p), f^n(x)) < \delta.$$

Hence, for $h'_n \in G$ with $h'_n f^n(x) = f^n(gx)$,

$$d(h'_n f^n(g_n p), h'_n f^n(x)) < \epsilon.$$

Let $h'_n f^n(g_n p) = f^n(h_n p)$. Then

$$d(f^n(h_n p), f^n(gx)) < \epsilon.$$

Therefore, $gx \in W^s(p)$. Similarly, one can prove the statement for the case $i = u$. \square

Lemma 2.7. *Let $f: X \rightarrow X$ be a pseudoequivariant homeomorphism on a compact metric G -space X with G compact. Then for any $\epsilon > 0$, there exists a positive*

constant $\delta < \epsilon$ satisfying the following: if $x \in W_\delta^u(y)$, then for all $g \in G$,

$$(1) \quad gx \in W_\epsilon^u(y)$$

and

$$(2) \quad gy \in W_\epsilon^u(x).$$

Proof. Let $\epsilon > 0$. Then, by Lemma 2.3, there exists a positive constant $\delta < \epsilon$ such that

$$d(x, y) < \delta \implies d(gx, gy) < \epsilon \quad \text{for all } g \in G.$$

Let $x \in W_\delta^u(y)$ and let $g \in G$. Then for each $n \geq 0$, there exists $g_n \in G$ such that

$$d(f^{-n}(x), f^{-n}(g_n y)) < \delta.$$

(1) Take $g'_n \in G$ such that $g'_n f^{-n}(x) = f^{-n}(gx)$. Then

$$d(f^{-n}(gx), g'_n f^{-n}(g_n y)) < \epsilon.$$

Since f is pseudoequivariant, $gx \in W_\epsilon^u(y)$.

(2) Take $g'_n \in G$ such that $g'_n f^{-n}(g_n y) = f^{-n}(gy)$. Then

$$d(g'_n f^{-n}(x), f^{-n}(gy)) < \epsilon.$$

Since f is pseudoequivariant, $gy \in W_\epsilon^u(x)$ for all $g \in G$. □

Proof of Theorem B. Let $\epsilon > 0$ be a constant which is less than the G -expansive constant for $f|_B$ and let $\delta > 0$ ($\delta < \epsilon$) be the constant corresponding to ϵ in the definition of the GSP. Let $X_p = \overline{W^u(p)} \cap B$ for $p \in B \cap \text{Per}_G(f)$. We can see directly from Lemmas 2.3 and 2.6 that X_p is G -invariant, that is, if $x \in X_p$, then $gx \in X_p$ for all $g \in G$.

Claim 1. X_p is open in B .

Proof. Since $p \in \text{Per}_G(f)$, we have an integer $m > 0$ and $g_1 \in G$ such that $g_1 f^m(p) = p$. Denote $N_\delta(X_p) = \{y \in B : d(y, X_p) < \delta\}$. Let $q \in N_\delta(X_p) \cap \text{Per}_G(f)$. Then there is $x \in W^u(p) \cap B$ with $d(q, x) < \delta$. Note that $g_2 f^n(q) = q$ for some integer $n > 0$ and $g_2 \in G$. Since $f|_B$ has the GSP, the (δ, G) -pseudo orbit

$$\{\dots, f^{-2}(x), f^{-1}(x), q, f(q), f^2(q), \dots\}$$

is ϵ -traced by a point $x' \in B$, that is, for each $t \in \mathbb{Z}$, there exists $h_t \in G$ such that

- (a) $d(x', h_0 q) < \epsilon$;
- (b) $d(f^t(x'), h_t f^t(q)) < \epsilon$ ($t > 0$);
- (c) $d(f^{-t}(x'), h_{-t} f^{-t}(x)) < \epsilon$ ($t > 0$).

Hence, it follows from Remark 1.10 that $x' \in W^s(q) \cap W^u(x) \cap B$.

Since f is pseudoequivariant and $p \in \text{Per}_G(f)$, for each $k \in \mathbb{Z}$, we have $g_{kmn} \in G$ such that $f^{kmn}(g_{kmn}p) = p$. Since $W^u(x) = W^u(p) = W^u(g_{kmn}p)$ by Lemmas 2.5 and 2.6,

$$f^{kmn}(x') \in f^{kmn}(W^u(g_{kmn}p)) = W^u(f^{kmn}(g_{kmn}p)) = W^u(p).$$

Since $q \in W^s(x')$, for each $k \in \mathbb{Z}$, one can find $h_{kmn} \in G$ such that

$$\lim_{k \rightarrow \infty} d(h_{kmn} f^{kmn}(x'), f^{kmn}(q)) = 0.$$

Take $i_{kmn} \in G$ such that $i_{kmn}(h_{kmn})^{-1} f^{kmn}(q) = q$. Then

$$\lim_{k \rightarrow \infty} d(i_{kmn} f^{kmn}(x'), i_{kmn}(h_{kmn})^{-1} f^{kmn}(q)) = \lim_{k \rightarrow \infty} d(i_{kmn} f^{kmn}(x'), q) = 0.$$

Hence, $q \in \overline{W^u(p) \cap B} = X_p$ because $i_{kmn} f^{kmn}(x') \in W^u(p)$ for each $k \in \mathbb{Z}$ by Lemma 2.6. Therefore,

$$X_p \supset \overline{N_\delta(X_p) \cap \text{Per}_G(f)} \supset N_\delta(X_p) \cap \overline{\text{Per}_G(f)} = N_\delta(X_p),$$

that is, X_p is open in B . □

Note that $f(X_p) = f(\overline{W^u(p) \cap B}) = \overline{f(W^u(p)) \cap f(B)} = \overline{W^u(f(p)) \cap B} = X_{f(p)}$. Since $X_p = X_{g_1 p}$ for any $g \in G$ and $g_1 f^m(p) = p$,

$$f^m(X_p) = X_{f^m(p)} = X_{g_1 f^m(p)} = X_p.$$

Take the smallest integer $a > 0$ such that $a \leq m$ and $f^a(X_p) = X_p$.

Claim 2. $B = \bigcup_{j=0}^{a-1} f^j(X_p)$.

Proof. Let $y \in B$. Since $f|_B$ is topologically G -transitive, for each $1/n$ -neighborhood $N_{1/n}(y)$ of y , there are $k > 0$ and $h_n \in G$ such that $h_n N_{1/n}(y) \cap f^k(X_p) \neq \emptyset$. So $h_n N_{1/n}(y) \cap (\bigcup_{j=0}^{a-1} f^j(X_p)) \neq \emptyset$ for each $n \in \mathbb{N}$. We may assume that $h_n \rightarrow h \in G$ because G is compact. Since $\bigcup_{j=0}^{a-1} f^j(X_p)$ is closed in B , $h y \in \bigcup_{j=0}^{a-1} f^j(X_p)$. Since $G(f^j(X_p)) = G(X_{f^j(p)}) = X_{f^j(p)} = f^j(X_p)$, we have $y \in \bigcup_{j=0}^{a-1} f^j(X_p)$. □

Claim 3. $X_p = X_q$ for $q \in X_p \cap \text{Per}_G(f)$.

Proof. Let $q \in X_p \cap \text{Per}_G(f)$ and suppose m and n are G -periodic numbers of p and q respectively. Since $N_\delta(X_p) = X_p$ for the constant $\delta > 0$ in the above of Claim 1, $W_\delta^u(q) \subset X_p$. We firstly show that $p \in X_q$. Suppose that $p \notin X_q$. Then $d(K, X_q) > 0$ where $K = X_p \setminus X_q$. Since $q \in X_p = \overline{W^u(p) \cap B}$, there exists $z \in W^u(p) \cap B$ such that $d(z, q) < d(K, X_q)$. Since $z \in X_p$ and $z \notin K$, $z \in X_q$. Furthermore, for each $j \in \mathbb{Z}$, there exists $g'_{mnj} \in G$ such that

$$\lim_{j \rightarrow \infty} d(f^{-mnj}(z), f^{-mnj}(g'_{mnj}p)) = 0.$$

For each $j \in \mathbb{Z}$, choose $g_{mnj} \in G$ with $g_{mnj}f^{-mnj}(g'_{mnj}p) = p$. Then we have

$$\lim_{j \rightarrow \infty} d(g_{mnj}f^{-mnj}(z), p) = 0.$$

So $g_{mnj}f^{-mnj}(z) \notin X_q$ for sufficiently large j . Hence,

$$h_{mnj}z \notin f^{mnj}(X_q) = X_q$$

for $h_{mnj} \in G$ with $g_{mnj}f^{-mnj}(z) = f^{-mnj}(h_{mnj}z)$. Thus, $z \notin X_q$. This is a contradiction. Therefore, $p \in X_q$.

Let $y \in W^u(q)$ and let $0 < \delta_1 < \delta_2 < \delta_3 = \delta$ such that

$$d(x, y) < \delta_i \implies d(gx, gy) < \delta_{i+1} \quad \text{for all } g \in G \quad (i = 1, 2).$$

Then there exists $N \in \mathbb{N}$ such that if $k \geq N$, then $d(f^{-k}(y), f^{-k}(h_kq)) < \delta_1$ for some $h_k \in G$. Choose $j \in \mathbb{N}$ with $mnj \geq N$. Then

$$d((f^{-i} \circ f^{-mnj})(y), (f^{-i} \circ f^{-mnj})(h_{mnj+i}q)) < \delta_1 \quad \text{for all } i \geq 0,$$

that is,

$$f^{-mnj}(y) \in W_{\delta_1}^u(f^{-mnj}(q)).$$

By Lemma 2.7 (2), $gf^{-mnj}(q) \in W_{\delta_2}^u(f^{-mnj}(y))$ for all $g \in G$. Since $q \in \text{Per}_G(f)$, we have $q \in W_{\delta_2}^u(f^{-mnj}(y))$. Again, by Lemma 2.7 (2), $gf^{-mnj}(y) \in W_\delta^u(q)$ for all $g \in G$. In particular, $f^{-mnj}(y) \in W_\delta^u(q)$. This means that $y \in f^{mnj}(W_\delta^u(q))$ for some $j \geq 0$. So $W^u(q) \subset \bigcup_{j \geq 0} f^{mnj}(W_\delta^u(q))$. Therefore,

$$X_q = \overline{W^u(q) \cap B} \subset \overline{\bigcup_{j \geq 0} f^{mnj}(W_\delta^u(q)) \cap B} \subset \overline{X_p \cap B} = X_p \cap B = X_p.$$

Similarly, we have $X_p \subset X_q$. □

Claim 4. $X_p \cap f^j(X_p) = \emptyset$ for $0 < j < a$.

Proof. Suppose $X_p \cap f^j(X_p) \neq \emptyset$ for some j . Since $X_p \cap f^j(X_p)$ is open in B , we can find $q \in X_p \cap f^j(X_p) \cap \text{Per}_G(f)$. Then $X_q = X_p = f^j(X_p)$, which is a contradiction to the choice of the integer a . \square

Claim 5. $f^a|_{X_p}$ is topologically G -mixing.

Proof. Let U and V be non-empty open subsets of X_p and let $q \in V \cap \text{Per}_G(f)$. Then $f^{aj}(q) \in X_p \cap \text{Per}_G(f)$ for all $j \in \mathbb{Z}$. Since $X_p = X_{f^{aj}(q)}$ for all $j \in \mathbb{Z}$,

$$U \cap W^u(f^{aj}(q)) = U \cap (W^u(f^{aj}(q)) \cap B) \neq \emptyset \quad \text{for all } j \in \mathbb{Z}.$$

Let $n > 0$ be a G -periodic number of q . Then for each j such that $0 \leq j \leq n-1$, there exists $z_j \in U \cap W^u(f^{aj}(q))$. Since f is pseudoequivariant, we may take this statement: for each $t \in \mathbb{Z}$, there exists $h_t \in G$ such that

$$\lim_{t \rightarrow \infty} d(f^{-ant}(z_j), f^{aj}(h_t f^{-ant}(q))) = 0.$$

For each $t \in \mathbb{Z}$, choose $g_t \in G$ such that $g_t f^{aj}(h_t f^{-ant}(q)) = f^{aj}(q)$. Then we have

$$\lim_{t \rightarrow \infty} d(g_t f^{-ant}(z_j), f^{aj}(q)) = 0,$$

and thus

$$\lim_{t \rightarrow \infty} g_t f^{-ant}(z_j) = f^{aj}(q).$$

Since $f^{aj}(q) \in f^{aj}(V)$, for each j with $0 \leq j \leq n-1$, we may choose $N_j > 0$ such that for all $t \geq N_j$,

$$g_t f^{-ant}(z_j) \in f^{aj}(V).$$

Let $M = \max\{N_j : 0 \leq j \leq n-1\}$. For each $t \geq M$, we get $t = ns + j$. If $s \geq M$, then

$$f^{-at}(i_s z_j) = f^{-aj}(g_s f^{-ans}(z_j)) \in V$$

for each $i_s \in G$ such that $f^{-ans-aj}(i_s z_j) = f^{-aj}(g_s f^{-ans}(z_j))$. Hence,

$$i_s z_j \in f^{at}(V) \quad \text{if } s \geq M \quad (\text{that is, } t \geq nM).$$

Thus, it follows from $z_j \in U$ that there exists $k_t \in G$ such that

$$k_t f^{at}(V) \cap U \neq \emptyset \quad \text{for each } t \geq nM.$$

Therefore, $f^a|_{X_p}$ is topologically G -mixing. \square

ACKNOWLEDGEMENT. The authors would like to thank the referee for very detailed comments in improving the exposition of the paper.

References

- [1] N. Aoki: *On homeomorphisms with pseudo-orbit tracing property*, Tokyo J. Math. **6** (1983), 329–334.
- [2] N. Aoki and K. Hiraide: *Topological theory of dynamical systems, Recent advances*, North-Holland, Amsterdam, 1994.
- [3] T. Choi and J. Kim: *G-expansive homeomorphisms on compact metric G-spaces*, preprint.
- [4] R. Das: *Expansive self-homeomorphisms on G-spaces*, Period. Math. Hungar. **31** (1995), 123–130.
- [5] E. Shah: *Dynamical properties of maps on topological spaces and G-spaces*, Ph.D. Thesis, 2005.

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