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THE APPROXIMATING CHARACTER ON NONLINEARITIES OF SOLUTIONS OF CAUCHY PROBLEM FOR A SINGULAR DIFFUSION EQUATION

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Abstract

In this paper, we consider the Cauchy problem

$$\begin{cases} u_t = (u^{m-1}u_x)_x, & x \in \mathbb{R},\ t > 0,\ -1 < m \le 1, \\ u(x, 0) = u_0, & x \in \mathbb{R}. \end{cases}$$

We will prove that:
1) $|u(x, t, m) - u(x, t, m_0)| \to 0$ uniformly on $[-l, l] \times [\tau, T]$ as $m \to m_0$ for any given $l > 0$, $0 < \tau < T$ and $-1 < m,\ m_0 < 1$,
2) $\int_{\mathbb{R}} |u(x, t, m) - u(x, t, 1)|\, dx \le 2((1 - m)/m)\|u_0\|_{L^1(\mathbb{R})}$.

1. Introduction

We consider the Cauchy problem

$$\begin{cases} u_t = (u^{m-1}u_x)_x, & x \in \mathbb{R}, t > 0, \\ u(x, 0) = u_0, & x \in \mathbb{R}. \end{cases} \tag{1.1}$$

Where, $-1 < m \le 1$ and

$$u_0 \ge 0, \quad 0 < \|u_0\|_{L^1(\mathbb{R})} < +\infty. \tag{1.2}$$

In recent years there has been a considerable interest in the equation in (1.1), such as [4], [13] and [15], and so on. The equation encompasses for different ranges of m a variety of qualitative properties with wide scope of applications. For example, the equation is degenerate parabolic as $m > 1$, so (1.1) only has weak solutions (see [3]) in this case. If $m = 1$, the equation is uniformly parabolic and therefore (1.1) has a unique global smooth solution $u(x, t, 1) = (1/(2\sqrt{\pi t})) \int_{\mathbb{R}} u_0(\xi) e^{-(x-\xi)^2/(4t)}\, d\xi$. If $m < 1$, then u^{m-1} blows up as $u \to 0$. It is usually referred to as the singular diffusion equation and has been proposed in plasma physics and in the heat conduction in solid hydrogen

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(see [12]). In this case, the problem (1.1) with condition (1.2) also has a unique global smooth solution $u(x, t, m)$ (called maximal solution) for any given $-1 < m < 1$ (see [6], [12]) such that

$$u(x, t, m) \in C^{\infty}(Q) \cap C([0, +\infty); L^1(\mathbb{R})), \tag{1.3}$$

$$\frac{1}{m-1}(u^{m-1})_{xx} \geq \frac{-1}{(1+m)t}, \quad \text{for} \quad (x, t) \in Q, \tag{1.4}$$

$$\frac{-u}{(1+m)t} \leq u_t \leq \frac{u}{(1-m)t}, \quad \text{for} \quad (x, t) \in Q, \tag{1.5}$$

and

$$u(x, t, m) \leq c(m, u_0) \cdot t^{-1/(1+m)}, \tag{1.6}$$

where, the constant $c(m)$ depends on m and $\|u_0\|_{L^1(\mathbb{R})}$, $Q = \mathbb{R} \times (0, +\infty)$.

Although the equation of (1.1) arises in many applications, and have been studied by many authors, there are only a few results concerning the approximating character on the nonlinearities of the equations. In 1981, Belinan and Crandall (see [16]) studied the similar problem for degenerate parabolic equations, but their results are not written in terms of explicit estimates. And then, B. Cockburn and G. Gripenberg (see [2]) extended the result of [16] for degenerate parabolic equations in 1999 and obtained an explicit estimate in $L^p(\mathbb{R}^N)$ for any given t. Recently, in 2006 and 2007, the author (see [9], [10]) discussed the problem (1.1) for $m > 1$, and obtain a explicit constant $C^* = O(T^{\gamma})$ such that

$$\int_0^T \int_{\mathbb{R}} |u(x, t, m) - u(x, t, m_0)|^2 \, dx \, dt \leq C^*|m - m_0|, \quad m, m_0 \geq 1.$$

As to the case of $m \leq 1$, the author (see [11]) considered the singular diffusion problem

$$\begin{cases} u_t = (u^{m-1}u_x)_x, & 0 < x < 1, \ t > 0, \\ \left(\dfrac{1}{m}u^m\right)_x \Big|_{x=0,1} = 0, & t \geq 0, \\ u|_{t=0} = u_0(x), & 0 \leq x \leq 1, \end{cases}$$

and proved that there exists a unique global solution $u(x, t, m)$ such that

$$\int_0^{\infty} \int_0^1 |u(x, t, m) - u(x, t, m_0)|^2 \, dx \, dt \leq C^*|m - m_0|,$$

where, $0 < m$, $m_0 \leq 1$ and C^* is a explicit constant. To the knowledge of the author, there are no other correlative results on such problem.

Since $m \leq 1$ in this work, by a solution of the Cauchy problem (1.1) on Q, we mean a function $u(x, t, m)$ belongs to (1.3) and satisfies the equation of (1.1) and

$$\|u(\cdot, t, m) - u_0(\cdot)\|_{L^1(\mathbb{R})} \to 0, \quad \text{as} \quad t \to 0.$$

Our main results of the work read

Theorem. *Let $u(x,t,m)$ be the solutions of* (1.1) *and* (1.2) *for $-1 < m$, $m_0 \leq 1$. If $m_0 \in (-1, 1)$, then for any given $l > 0$ and $0 < \tau < T$,*

$$\lim_{m\to m_0} |u(x, t, m) - u(x, t, m_0)| = 0, \quad \textit{uniformly on} \quad [-l, l] \times [\tau, T]. \tag{1.7}$$

If $m_0 = 1$, then

$$\int_{\mathbb{R}} |u(x, t, m) - u(x, t, 1)|\, dx \leq 2\frac{1-m}{m}\|u_0\|_{L^1(\mathbb{R})}, \quad \textit{for all} \quad t > 0. \tag{1.8}$$

2. Preliminary lemmas

Lemma 1. *Let $u(x, t, m)$ be the solution of* (1.1), *then*

$$|(u^{(m-1)/2}(x, t, m))_x| \leq \sqrt{\frac{1-m}{2(1+m)t}}, \quad \textit{for} \quad m \in (-1, 1). \tag{2.1}$$

Proof. By (1.4),

$$u^{m-1}u_{xx} + (m-2)u^{m-2}(u_x)^2 \geq \frac{-u}{(1+m)t}.$$

Since u satisfies the equation in (1.1), so $u_t = u^{m-1}u_{xx} + (m-1)u^{m-2}(u_x)^2$. Using (1.5) yields

$$\frac{u}{(1-m)t} - u^{m-2}(u_x)^2 \geq \frac{-u}{(1+m)t}.$$

Thus, $u^{m-3}(u_x)^2 \leq 2/((1-m^2)t)$. This yields (2.1). □

Lemma 2. *If $f(x) \in L^1(\mathbb{R})$ and $f'(x)$ is bounded, then $f(x) \to 0$ as $x \to \infty$.*

This is a well known conclusion of real analysis.

Lemma 3. *Let $\phi, \phi_n \in L^p$, $p \geq 1$, $\phi_n \to \phi$ a.e. Then $\|\phi_n - \phi\|_{L^p} \to 0$ if and only if $\|\phi_n\|_{L^p} \to \|\phi\|_{L^p}$.*

This result is also a well known of real analysis ([7], p.187).

Lemma 4. *Let $u(x, t, m)$ be the solution of* (1.1), *then*

$$\int_{\mathbb{R}} u(x, t, m)\, dx = \|u_0\|_{L^1(\mathbb{R})} \quad \textit{for all} \quad t > 0. \tag{2.2}$$

Clearly this lemma means the total mass is conserved. It is a well known result (see [12]).

REMARK. However, the total mass is not always a constant. In fact, the result is not true for $m < -1$ if the space dimension $N = 1$ (see [8]). When $N \geq 2$, J.L. Vázquez proved that the mass can be lost as time grows and neighborhoods of infinity is where the mass is lost (see [14], p.90–92).

Lemma 5. *For the Cauchy problem* (1.1) *and* (1.2), *let $u(x, t, m)$ and $\hat{u}(x, t, m)$ be two solutions corresponding to initial values $u_0(x)$ and $\hat{u}_0(x)$, then*

$$\int_{\mathbb{R}} |u - \hat{u}|(x)\, dx \leq \int_{\mathbb{R}} |u_0 - \hat{u}_0|\, dx.$$

It is also a well known conclusion (see [12]).

Take a function $f(x) \in C_0^\infty(R)$, $0 \leq f(x) \leq 1$ and

$$f(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| \geq 2. \end{cases}$$

For any positive constant l, set

$$f_l(x) = f\left(\frac{x}{l}\right). \tag{2.3}$$

Then there is a positive constant C_0 such that

$$|f_l'(x)| \leq \frac{C_0}{l}, \quad \text{and} \quad |f_l''(x)| \leq \frac{C_0}{l^2}. \tag{2.4}$$

Now for any given $t > 0$, we have

$$\left| \int_0^t \int_{\mathbb{R}} u^{m-1} u_x f_l'(x)\, dx\, d\tau \right| \to 0 \quad \text{as} \quad l \to \infty. \tag{2.5}$$

To prove (2.5), we can use (1.6). In fact, if $m \neq 0$, then there exists a positive constant C_1 such that

$$\left|\int_{t_0}^{t}\int_{\mathbb{R}} u^{m-1}u_x f_l'(x)\,dx\,d\tau\right| \leq \frac{1}{|m|}\int_{t_0}^{t}\int_{l\leq|x|\leq 2l} |u^m f_l''(x)|\,dx\,d\tau$$
$$\leq \frac{C_1}{l^2}\int_{t_0}^{t}\int_{l\leq|x|\leq 2l} t^{-m/(1+m)}\,dx\,d\tau$$
$$\to 0.$$

This is (2.5). If $m = 0$, then $\int_{t_0}^{t}\int_{\mathbb{R}} u^{m-1}u_x f_l'(x)\,dx\,d\tau = \int_{t_0}^{t}\int_{\mathbb{R}} \ln u f_l''(x)\,dx\,d\tau$. We can also use (1.6) to obtain (2.5).

3. Proof of Theorem

We now employ two steps to prove our main results.

STEP 1. Proof of (1.7).

For any $T > 0$, recalling (1.5), (1.6) and (2.1), we deduce that for any $0 < \eta < 1/2$, $l > 0$ and $0 < \tau < T$, u and u_x and u_t are bounded uniformly on $(x, t, m) \in [-2l, 2l]\times[\tau, T]\times[-1+\eta, 1-\eta]$. Thus, for any $m_0 \in [-1+\eta, 1-\eta]$, Arzela's theorem claims that there are subsequence $u(x, t, m_k)$ and a function $\bar{u}(x, t, m_0) \in C([-l, l]\times[\tau, T])$, such that

$$\lim_{m_k\to m_0} |u(x, t, m_k) - \bar{u}(x, t, m_0)| = 0, \quad \text{uniformly on} \quad [-l, l]\times[\tau, T]. \tag{3.1}$$

We next want to prove that the function $\bar{u}(x, t, m_0)$ is indeed the solution of problem (1.1) with (1.2) for $m = m_0$, i.e. $\bar{u} = u(x, t, m_0)$. If it is true, then by the uniqueness, the total sequence $u(x, t, m)$ converges to $u(x, t, m_0)$ as $m \to m_0$, thus, we can drop k in (3.1) and therefore, (3.1) is (1.7) namely.

To do this, we first prove that $\bar{u}(x, t, m_0)$ satisfies the equation of (1.1) for $m = m_0$ in $\mathbb{R}\times(0, T)$.

Let $f_l(x)$ be shown by (2.3). For any $0 < t < T$, we have

$$\int_{\mathbb{R}} u(x, t, m_k) f_l(x)\,dx = \int_{\mathbb{R}} u_0(x) f_l(x)\,dx - I. \tag{3.2}$$

Where $I = \int_0^t\int_{\mathbb{R}} u^{m_k-1}(x, \tau, m_k)u_x(x, \tau, m_k) f_l'(x)\,dx\,d\tau$. Using (2.5) we have

$$\int_{\mathbb{R}} \bar{u}(x, t, m_0)\,dx = \|u_0\|_{L^1(\mathbb{R})} \quad \text{for} \quad 0 < t < T. \tag{3.3}$$

Thus, for any given $t \in (0, T)$, there exists a point $x_0 \in \mathbb{R}$ such that

$$\bar{u}(x_0, t, m_0) > 0.$$

On the other hand, by (2.1), we have

$$(u(x, t, m_k))^{(m_k-1)/2} \le (u(x_0, t, m_k))^{(m_k-1)/2} + \sqrt{\frac{1-m_k}{2(1+m_k)t}}|x-x_0|.$$

It follows from $m_k < 1$ that

$$u(x, t, m_k) \ge \left[(u(x_0, t, m_k))^{(m_k-1)/2} + \sqrt{\frac{1-m_k}{2(1+m_k)t}}|x-x_0|\right]^{2/(m_k-1)}, \quad \text{for} \quad x \in \mathbb{R},\ 0 < t < T.$$

Letting $m_k \to m_0$ yields

$$\bar{u}(x, t, m_0) \ge \left[(\bar{u}(x_0, t, m_0))^{(m_0-1)/2} + \sqrt{\frac{1-m_0}{2(1+m_0)t}}|x-x_0|\right]^{2/(m_0-1)} > 0, \quad \text{for} \quad x \in \mathbb{R},\ 0 < t < T.$$

Because $\bar{u}(x, t, m_0) > 0$ and $\bar{u}(x, t, m_0)$ is continuous, so for any $(x_0, t_0) \in \mathbb{R} \times (0, T)$, there exists a neighborhood of (x_0, t_0), Y, say, $Y \subset (-l, l) \times (\tau, T)$, and two positive constants d and D, such that

$$d \le \bar{u}(x, t, m_0) \le D, \quad \text{for} \quad (x, t) \in Y.$$

Hence, there exists another positive constant θ, such that

$$\frac{d}{2} \le u(x, t, m_k) \le D, \quad \text{for} \quad (x, t) \in Y,\ |m_k - m_0| \le \theta.$$

Because $u(x, t, m_k)$ is smooth and bounded, and satisfies the equation in (1.1) in Y, it follows from a generalization of Nash' theorem ([5], p.204) that there exists a neighborhood $Y_1 \subset Y$ of (x_0, t_0) such that $u(x, t, m_k) \in C^\alpha(\bar{Y}_1)$ for some $\alpha \in (0, 1)$. Where α and $\|u(x, t, m_k)\|_{C^\alpha(\bar{Y}_1)}$ may be estimated independently of m_k. It follows from the standard linear theory ([1], p.77) that there exists a neighborhood $Y_2 \subset Y_1$ of (x_0, t_0) such that $u(x, t, m_k) \in C^{2+\alpha}(\bar{Y}_2)$ for $|m_k - m_0| \le \theta$, with the norm $\|u(x, t, m_k)\|_{C^{2+\alpha}(\bar{Y}_2)}$ uniformly bounded with respect to m_k. Hence the limit function $\bar{u}(x, t, m_0)$ belongs to $C^{2+\alpha}(\bar{Y}_2)$, and is therefore a classical solution of the equation in Y_2 for $m = m_0$. Recalling τ and l are arbitrary positive constants, so we know that $\bar{u}(x, t, m_0)$ is a classical solution of the equation in (1.1) on $\mathbb{R} \times (0, T)$. Furthermore, $\bar{u}(x, t, m_0)$ satisfies (1.4), (1.5), (1.6) and (2.1) on $\mathbb{R} \times (0, T)$.

In order to prove $\bar{u}(x, t, m_0)$ be the solution of problem (1.1) as $m = m_0$ for $0 < t < T$, we next will show $\bar{u}(x, t, m_0) \in C([0, T); L^1(\mathbb{R}))$. First, recalling (3.3) and

$\bar{u}(x, t, m_0) \in C(\mathbb{R} \times (0, T))$, and using Lemma 3, we know

$$\bar{u}(x, t, m_0) \in C((0, T); L^1(\mathbb{R})). \tag{3.4}$$

So next we need only to show that $\bar{u}(x, t, m_0)$ satisfies the initial condition in (1.1), i.e.

$$\|\bar{u}(x, t, m_0) - u_0(x)\|_{L^1(\mathbb{R})} \to 0 \quad \text{as} \quad t \to 0. \tag{3.5}$$

To prove (3.5), by the result of Lemma 5 and the translation invariance of the equation in (1.1), we have

$$\int_{\mathbb{R}} |u(x+h, t, m_k) - u(x, t, m_k)|\, dx \le \int_{\mathbb{R}} |u_0(x+h) - u_0(x)|\, dx,$$

for every $h \in \mathbb{R}$. Letting $m_k \to m_0$, we know that for any given $\varepsilon > 0$, there exists a positive constant h_0, such that

$$\int_{\mathbb{R}} |\bar{u}(x+h, t, m_0) - \bar{u}(x, t, m_0)|\, dx \le \varepsilon, \quad \text{for} \quad t \in (0, T),\ |h| < h_0. \tag{3.6}$$

On the other hand, letting $m_k \to m_0$ in (3.2) yields

$$\begin{aligned} \int_{\mathbb{R}} \bar{u}(x, t, m_0) f_l(x)\, dx = \int_{\mathbb{R}} u_0(x) f_l(x)\, dx \\ - \int_0^t \int_{\mathbb{R}} \bar{u}^{m_0 - 1}(x, t, m_0) \bar{u}_x(x, t, m_0) f_l'(x)\, dx\, d\tau. \end{aligned} \tag{3.7}$$

Using (3.3), we have

$$\begin{aligned} \int_{|x| \ge 2l} \bar{u}(x, t, m_0)\, dx &= \int_{\mathbb{R}} \bar{u}(x, t, m_0)\, dx - \int_{|x| \le 2l} \bar{u}(x, t, m_0)\, dx \\ &\le \|u_0\|_{L^1(\mathbb{R})} - \int_{\mathbb{R}} \bar{u}(x, t, m_0) f_l(x)\, dx \\ &= \|u_0\|_{L^1(\mathbb{R})} - \int_{\mathbb{R}} u_0(x) f_l(x)\, dx \\ &\quad + \int_0^t \int_{\mathbb{R}} \bar{u}^{m_0 - 1}(x, t, m_0) \bar{u}_x(x, t, m_0) f_l'(x)\, dx\, d\tau \\ &\le \int_{|x| \ge l} u_0(x)\, dx \\ &\quad + \int_0^t \int_{\mathbb{R}} \bar{u}^{m_0 - 1}(x, t, m_0) \bar{u}_x(x, t, m_0) f_l'(x)\, dx\, d\tau, \\ &\qquad \text{for} \quad 0 < t < T. \end{aligned} \tag{3.8}$$

Since (1.6) is also valid for $\bar{u}(x, t, m_0)$, we can also use (2.5) for $\bar{u}(x, t, m_0)$ and to obtain

$$\int_0^t \int_{\mathbb{R}} \bar{u}^{m_0-1}(x, \tau, m_0)\bar{u}_x(x, \tau, m_0) f_l'(x)\, dx\, d\tau \to 0 \quad \text{as} \quad l \to \infty.$$

Hence, by (3.8), for any given $\varepsilon > 0$, there exists $l_0 > 0$ such that

$$\int_{|x|\geq l} \bar{u}(x, t, m_0)\, dx \leq \varepsilon, \quad \text{for} \quad l \geq l_0,\ t \in (0, T). \tag{3.9}$$

It follows from (3.6) and (3.9) and [17] (p.31, Theorem 2.21) that $\{\bar{u}(\cdot, t, m_0)\}_{0<t\leqslant T}$ is a pre-compact family in $L^1(\mathbb{R})$. Thus for any sequence $t_n \to 0$, we have a subsequence $\{t_{n_k}\}$ and a function $u_0^* \in L^1(\mathbb{R})$, such that

$$\|\bar{u}(\cdot, t_{n_k}, m_0) - u_0^*(\cdot)\|_{L^1(\mathbb{R})} \to 0 \quad \text{as} \quad t_{n_k} \to 0.$$

Hence for any $\phi(x) \in C_0^\infty(\mathbb{R})$, we have

$$\lim_{t_{n_k}\to 0} \int_{\mathbb{R}} (\bar{u}(x, t_{n_k}, m_0) - u_0^*(x))\phi(x)\, dx = 0. \tag{3.10}$$

On the other hand, letting $t = t_{n_k}$ in (3.7), we have

$$\lim_{t_{n_k}\to 0} \int_{\mathbb{R}} \bar{u}(x, t_{n_k}, m_0) f_l\, dx = \int_{\mathbb{R}} u_0 f_l\, dx. \tag{3.11}$$

Clearly, (3.11) is also true for $f_l = \phi(x) \in C_0^\infty(\mathbb{R})$. Thus,

$$\lim_{t_n\to 0} \int_{\mathbb{R}} \bar{u}(x, t_n, m_0)\phi(x)\, dx = \int_{\mathbb{R}} u_0\phi(x)\, dx, \quad \text{for} \quad \phi \in C_0^\infty(\mathbb{R}). \tag{3.12}$$

Combining (3.10) and (3.12) yields $\int_{\mathbb{R}}(u_0 - u_0^*)\phi\, dx = 0$ for all $\phi \in C_0^\infty(\mathbb{R})$. Therefore,

$$u_0^* = u_0,$$

and

$$\lim_{t_{n_k}\to 0} \|\bar{u}(\cdot, t_{n_k}, m_0) - u_0(\cdot)\|_{L^1(\mathbb{R})} = 0.$$

It is easy to see that this is true for any subsequence $t_n \to 0$. Therefore we obtain (3.5). Combining (3.4) and (3.5) yields

$$\bar{u}(x, t, m_0) \in C([0, T); L^1(\mathbb{R})).$$

Now we know the function $\overline{u}(x, t, m_0)$ is indeed the solution of problem (1.1) for $m = m_0$ on Q_T for any $T > 0$. By the uniqueness,

$$\overline{u} = u(x, t, m_0), \quad \text{for} \quad (x, t) \in Q_T.$$

Thus (1.7) holds for $m, m_0 \in [-1 + \eta, 1 - \eta]$. Finally, the arbitresses of $\eta \in (0, 1/2)$ yields that (1.7) holds for all $m, m_0 \in (-1, 1)$.

STEP 2. Proof of (1.8).

To prove (1.8), we notice that

$$\begin{aligned}(u(x, t, m) - u(x, t, 1))_t &= \left(\frac{1}{m}u^m(x, t, m) - u(x, t, 1)\right)_{xx} \\ &= \frac{1}{m}(u^m(x, t, m) - u(x, t, 1))_{xx} + \frac{1-m}{m}u(x, t, 1)_{xx}.\end{aligned}$$

Let $w = u^m(x, t, m) - u(x, t, 1)$ and set

$$p(s) = \begin{cases} 1, & s \geq 1, \\ e^{(-1/s^2)e^{-1/(1-s)^2}}, & 0 < s < 1, \\ 0, & s \leq 0. \end{cases} \tag{3.13}$$

Then $p(s) \in C^\infty(\mathbb{R})$ and $p'(s) \geq 0$. Let

$$p_\varepsilon(w) = p\left(\frac{w}{\varepsilon}\right).$$

Thus,

$$\begin{aligned}\int_{\mathbb{R}} (u(x, t, m) - u(x, t, 1))_t \, p_\varepsilon(w)\, dx &= -\frac{1}{m}\int_{\mathbb{R}} (u^m(x, t, m) - u(x, t, 1))_x^2 \, p'_\varepsilon(w)\, dx \\ &\quad + \frac{1-m}{m}\int_{\mathbb{R}} u(x, t, 1)_{xx}\, p_\varepsilon(w)\, dx \\ &\leq \frac{1-m}{m}\int_{\mathbb{R}} u(x, t, 1)_t \, p_\varepsilon(w)\, dx.\end{aligned}$$

For any given $t > 0$, let

$$\mathbb{R}_1 = \{x \in \mathbb{R},\ u^m(x, t, m) \geq u(x, t, 1)\}, \quad \mathbb{R}_2 = \mathbb{R} - \mathbb{R}_1.$$

Letting $\varepsilon \to 0$, using Lemma 3.1 in [12] yields

$$\frac{d}{dt}\int_{\mathbb{R}_1} (u^m(x, t, m) - u(x, t, 1))\, dx \leq \frac{1-m}{m}\frac{d}{dt}\int_{\mathbb{R}_1} u(x, t, 1)\, dx.$$

Thus for any $0 \leq \tau < t$, we have

$$\begin{aligned}&\int_{\mathbb{R}_1}(u(x,t,m)-u(x,t,1))\,dx-\frac{1-m}{m}\int_{\mathbb{R}_1}u(x,t,1)\,dx\\&\leq\int_{\mathbb{R}_1}(u(x,\tau,m)-u(x,\tau,1))\,dx-\frac{1-m}{m}\int_{\mathbb{R}_1}u(x,\tau,1)\,dx.\end{aligned}$$

Similarly,

$$\begin{aligned}&\int_{\mathbb{R}_2}(u(x,t,1)-u(x,t,m))\,dx-\frac{m-1}{m}\int_{\mathbb{R}_2}u(x,t,1)\,dx\\&\leq\int_{\mathbb{R}_2}(u(x,\tau,1)-u(x,\tau,m))\,dx-\frac{m-1}{m}\int_{\mathbb{R}_2}u(x,\tau,1)\,dx.\end{aligned}$$

Combining the two inequalities gives

$$\begin{aligned}\int_{\mathbb{R}}|u(x,t,1)-u(x,t,m)|\,dx\leq&\int_{\mathbb{R}}|u(x,\tau,1)-u(x,\tau,m)|\,dx\\&+\frac{1-m}{m}\left[\int_{\mathbb{R}_1}u(x,t,1)\,dx+\int_{\mathbb{R}_2}u(x,\tau,1)\,dx\right].\end{aligned}$$

Letting $\tau \to 0$ and recalling $u(x,t,m), u(x,t,1) \in C([0,\infty); L^1(\mathbb{R}))$ and $\int_{\mathbb{R}} u(x,t,1)\,dx = \|u_0\|_{L^1(\mathbb{R})}$ for any $t > 0$, we have

$$\int_{\mathbb{R}}|u(x,t,1)-u(x,t,m)|\,dx\leq 2\frac{1-m}{m}\|u_0\|_{L^1(\mathbb{R})}.$$

This is (1.8).

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