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CHOW-STABILITY AND HILBERT-STABILITY IN MUMFORD'S GEOMETRIC INVARIANT THEORY

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Abstract

In this note, we shall show that Chow-stability and Hilbert-stability in GIT asymptotically coincide. The proof in [5] is simplified in the present form, while a quick review is in [6].

1. Introduction

For moduli spaces of polarized algebraic varieties, a couple of stability concepts are known in algebraic geometry (cf. Mumford et al. [9]): Chow-stability and Hilbert-stability. In this note, we clarify the asymptotic relationship between them. Throughout this note, we fix once for all a very ample holomorphic line bundle L over an irreducible projective algebraic variety M defined over $\mathbb{C}$. Let $n := \dim M > 0$ and let l be a positive integer with $l \geq n+1$. Replacing L by its suitable power, we may assume that $H^i(M, \mathcal{O}(L^j)) = \{0\}$ for all positive integers i and j. Then associated to the complete linear system $|L^l|$, we have the Kodaira embedding

$$\iota_l : M \hookrightarrow \mathbb{P}^*(V_l),$$

where $\mathbb{P}^*(V_l)$ is the set of all hyperplanes in $V_l := H^0(M, \mathcal{O}(L^l))$ through the origin. Let n and d_l be respectively the dimension of M and the degree of $\iota_l(M)$ in the projective space $\mathbb{P}^*(V_l)$. Put $G_l := \mathrm{SL}_{\mathbb{C}}(V_l)$ and $W_l := \{S^{d_l}(V_l)\}^{\otimes n+1}$, where $S^{d_l}(V_l)$ denotes the d_l-th symmetric tensor product of the space V_l. Take an element $M_l \neq 0$ in W_l^* such that the associated element $[M_l]$ in $\mathbb{P}^*(W_l)$ is the Chow point of the irreducible reduced algebraic cycle $\iota_l(M)$ on $\mathbb{P}^*(V_l)$. For the natural action of G_l on W_l^*, let $\hat{G}_l$ denote the isotropy subgroup of G_l at M_l.

DEFINITION 1.1. (a) (M, L^l) is called *Chow-stable* or *Chow-semistable* according as the orbit $G_l \cdot M_l$ is closed in W_l^* with $|\hat{G}_l| < \infty$ or the closure of $G_l \cdot M_l$ in W_l^* is disjoint from the origin.
(b) (M, L) is called *asymptotically Chow-stable* if (M, L^l) is Chow-stable for all $l \gg 1$.

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Let l and k be positive integers. Then the kernel $I_{l,k}$ of the natural homomorphism of $S^k(V_l)$ to $V_{lk} := H^0(M, \mathcal{O}_M(L^{lk}))$ is the degree k component of the homogeneous ideal defining M in $\mathbb{P}^*(V_l)$. Put $m_k := \dim V_{lk}$ and $\gamma_{l,k} := \dim I_{l,k}$. Then $\bigwedge^{\gamma_{l,k}} I_{l,k}$ is a complex line in $F_{l,k} := \bigwedge^{\gamma_{l,k}}(S^k(V_l))$. Take an element $f_{l,k} \neq 0$ in $\bigwedge^{\gamma_{l,k}} I_{l,k}$. For the natural action of G_l on $F_{l,k}$, let $\hat{G}_{l,k}$ be the isotropy subgroup of G_l at $f_{l,k}$.

DEFINITION 1.2. (a) (M, L^l) is called *Hilbert-stable* if the orbit $G_l \cdot f_{l,k}$ is closed in $F_{l,k}$ with $|\hat{G}_{l,k}| < \infty$ for all $k \gg 1$.
(b) (M, L) is called *asymptotically Hilbert-stable* if (M, L^l) is Hilbert-stable for all $l \gg 1$.

A result of Fogarty [4] (see also [9], p.215) states that Chow-stability for (M, L^l) implies Hilbert-stability for (M, L^l). However, little was known for the converse implication.

Consider the maximal connected linear algebraic subgroup H of the group of holomorphic automorphisms of M. To each positive integral multiple L^m of L, we associate the point $[L^m] \in \mathrm{Pic}(M)$ defined by L^m. For the natural H-action on $\mathrm{Pic}(M)$, we denote by $\hat{H}_m$ the identity component of the isotropy subgroup of H at $[L^m]$. Put $\hat{H} := \hat{H}_1$. Since the orbit $\hat{H}_m \cdot [L]$ ($\cong \hat{H}_m/\hat{H}$) sitting in $\{L' \in \mathrm{Pic}(M);\ (L')^m = L^m\}$ reduces to a single point, we have

$$\hat{H} = \hat{H}_m \quad \text{for all} \quad m \in \mathbb{Z}_+.$$

Let $\{k_i;\ i = 0, 1, 2, \dots\}$ be a sequence of integers $\geq n+1$. For a positive integer l, we define a sequence $\{l_i\}$ of positive integers inductively by setting $l_{i+1} := l_i k_i$ and $l_0 := l$. In this paper, we shall show that

Main Theorem. (a) *Assume that $G_{l_i} \cdot f_{l_i,k_i}$ is closed in F_{l_i,k_i} for all integers $i \geq 0$. If $\hat{H} = \{1\}$, then $G_l \cdot M_l$ is closed in W_l^*.*
(b) *(M, L) is asymptotically Chow-stable if and only if (M, L) is asymptotically Hilbert-stable.*

As seen in the beginning of Section 3, (b) follows from (a). Hence, we here sketch the proof of (a) of Main Theorem. Assume $\hat{H} = \{1\}$. Since $G_{l_i} \cdot f_{l_i,k_i}$ is closed in F_{l_i,k_i} for all i, Lemma 3.10 shows that the polynomial Hilbert weight $w_\lambda = w_\lambda(k; l)$ in Section 3 is increasing

$$0 < w_\lambda(K_0; l) < w_\lambda(K_1; l) < \cdots < w_\lambda(K_{i-1}; l) < w_\lambda(K_i; l) < \cdots$$

for K_i, $i = 1, 2, \dots$, in (3.7), where $\lambda\colon \mathbb{C}^* \hookrightarrow G_l$ is an arbitrary algebraic one-parameter subgroup. Since the asymptotic limit

$$w_\lambda(\infty; l) := \lim_{k \to \infty} w_\lambda(k; l)$$

always exist, and since $K_i \to +\infty$ as $i \to \infty$, we have $w_\lambda(\infty; l) > 0$. This means that (M, L^l) is Chow-stable, i.e., $G_l \cdot M_l$ is closed in W_l.

This paper is organized as follows. First, Section 2 is given as a preparation for Section 5. Then the proof of Main Theorem will be outlined in Section 3, while two main difficulties (3.6) and Lemma 3.10 will be treated in Sections 4 and 5, respectively.

2. A test configuration and the group action ρ_k

Hereafter, we fix an action of an algebraic torus $T := \mathbb{C}^*$ on $\mathbb{A}^1 := \{s;\ s \in \mathbb{C}\}$ by multiplication of complex numbers

$$T \times \mathbb{A}^1 \to \mathbb{A}^1, \quad (t, s) \mapsto ts.$$

Let $\pi\colon \mathcal{Z} \to \mathbb{A}^1$ be a T-equivariant projective morphism between complex varieties with a relatively very ample invertible sheaf $\mathcal{L}$ on $\mathcal{Z}$ over $\mathbb{A}^1$, where the algebraic group T acts on $\mathcal{L}$, linearly on fibers, lifting the T-action on $\mathcal{Z}$. Now the following concept by Donaldson will play a very important role in our study:

DEFINITION 2.1 (cf. [1]). $\pi\colon \mathcal{Z} \to \mathbb{A}^1$ above is called a *test configuration* of exponent l for (M, L) if, when restricted to fibers $\mathcal{Z}_s := \pi^{-1}(s)$, we have isomorphisms

$$(\mathcal{Z}_s, \mathcal{L}_{|\mathcal{Z}_s}) \cong (M, \mathcal{O}_M(L^l)), \quad 0 \neq s \in \mathbb{A}^1.$$

Let $\pi\colon \mathcal{Z} \to \mathbb{A}^1$ be a test configuration of exponent l for (M, L). To each positive integer k, we assign a vector bundle E_k over $\mathbb{A}^1$ associated to the locally free sheaf $\pi_*\mathcal{L}^k$ over $\mathbb{A}^1$, i.e., $\mathcal{O}_{\mathbb{A}^1}(E_k) = \pi_*\mathcal{L}^k$. For the natural T-action

$$\rho_k\colon T \times E_k \to E_k$$

induced by the T-action on $\mathcal{L}$, we denote by $\rho_{k,0}$ the restriction of the T-action ρ_k to the fiber $(E_k)_0$ over the origin. By this T-action ρ_k, the natural projection of E_k to $\mathbb{A}^1$ is T-equivariant. Note also that, over $\mathbb{A}^1$, we have the relative Kodaira embedding

$$\mathcal{Z} \hookrightarrow \mathbb{P}^*(E_k). \tag{2.2}$$

For the structure of ρ_k, the following equivariant trivialization of the vector bundle E_k is known:

Lemma 2.3 (cf. [3], Lemma 2). *The holomorphic vector bundle E_k over $\mathbb{A}^1$ can be T-equivariantly trivialized by*

$$E_k \cong (E_k)_0 \times \mathbb{A}^1,$$

where $(E_k)_0$ denotes the fiber of E_k over the origin.

Let $\lambda_k : T \to \mathrm{GL}_{\mathbb{C}}((E_k)_0)$ denote the algebraic group homomorphism induced by $\rho_{k,0}$ on $(E_k)_0$. Then the identification in Lemma 2.3 allows us to write the action ρ_k above in the form

$$\rho_k(t, (e, s)) = (\lambda_k(t)(e), ts), \quad (e, s) \in (E_k)_0 \times \mathbb{A}^1. \tag{2.4}$$

3. Proof of Main Theorem

The isotropy subgroup $\tilde{G}_l$ of G_l at $[M_l] \in \mathbb{P}^*(W_l)$ contains $\hat{G}_l$ (cf. Section 1) as a subgroup. Hence (M, L^l) is Chow-stable if and only if

$$|\tilde{G}_l| < \infty \quad \text{and} \quad G_l \cdot M_l \quad \text{is closed in} \quad W_l,$$

because if $\dim \hat{G}_l < \dim \tilde{G}_l$, then $\tilde{G}_l \cdot M_l = \mathbb{C}^* M_l$, and the origin is in the closure of $\tilde{G}_l \cdot M_l$ in W_l^*. Now for all $0 < l \in \mathbb{Z}$, the identity component $\tilde{G}_l^0$ of $\tilde{G}_l$ is isogenous to an algebraic subgroup of $\hat{H}$, while by GIT [9], Proposition 1.5, $\tilde{G}_m^0$ is isogenous to $\hat{H}$ for all multiples $m > 0$ of some fixed integer $\gg 1$. Hence (M, L) is asymptotically Chow-stable if and only if

$$\hat{H} = \{1\} \quad \text{and for} \quad l \gg 1, \quad G_l \cdot M_l \quad \text{is closed in} \quad W_l^*. \tag{3.1}$$

Similarly, if $l > 0$ is a multiple of some fixed integer $\gg 1$, we see that the identity component of $\hat{G}_{l,k}$ with $k \gg 1$ is isogenous to $\hat{H}$. Hence (M, L) is asymptotically Hilbert-stable if and only if

$$\hat{H} = \{1\} \quad \text{and for all} \quad l \gg 1, \quad G_{l,k} \cdot f_{l,k} \quad \text{is closed in} \quad F_{l,k} \quad \text{if} \quad k \gg 1. \tag{3.2}$$

In view of (3.1) and (3.2) above, (b) of Main Theorem follows immediately from Fogarty's result together with (a) of Main Theorem. Hence, we have only to show (a) of Main Theorem.

For one-dimensional algebraic torus $T := \mathbb{C}^*$, we consider an algebraic one-parameter subgroup

$$\lambda : T \hookrightarrow G_l$$

of the reductive algebraic group $G_l := \mathrm{SL}(V_l)$. Then to each λ as above, we assign a test configuration of exponent l as follows:

DEFINITION 3.3. The *DeConcini-Procesi family* (cf. [13]) associated to λ is the test configuration of exponent l for (M, L) obtained as the T-equivariant projective morphism

$$\pi : \mathcal{Z}(\lambda) \to \mathbb{A}^1,$$

where $\mathcal{Z}(\lambda)$ is the variety defined as the closure of $T \cdot (\iota_l(M) \times \{1\})$ in $\mathbb{P}^*(V_l) \times \mathbb{A}^1$, and the morphism π is induced by the projection of $\mathbb{P}^*(V_l) \times \mathbb{A}^1$ to the second factor. Let $\mathrm{pr}_1 \colon \mathcal{Z}(\lambda) \to \mathbb{P}^*(V_l)$ denote the map induced by the projection of $\mathbb{P}^*(V_l) \times \mathbb{A}^1$ to the first factor. For the open subset $\mathbb{C}^* \subset \mathbb{A}^1$, the holomorphic map $\hbar \colon \mathbb{C}^* \to \mathit{Hilb}_{\mathbb{P}^*(V_l)}$ sending each $t \in \mathbb{C}^*$ to $\hbar(t) := \mathrm{pr}_1(\mathcal{Z}(\lambda)_t) \in \mathit{Hilb}_{\mathbb{P}^*(V_l)}$ extends to a holomorphic map

$$\tilde{\hbar} \colon \mathbb{A}^1 \to \mathit{Hilb}_{\mathbb{P}^*(V_l)},$$

where $\mathcal{Z}(\lambda)_s := \pi^{-1}(s)$, $s \in \mathbb{A}^1$, denotes the scheme-theoretic fiber of π over s. Now we can regard $\mathcal{Z}(\lambda)$ as the pullback, by $\tilde{\hbar}$, of the universal family over $\mathit{Hilb}_{\mathbb{P}^*(V_l)}$. Note also that T acts on $\mathbb{P}^*(V_l) \times \mathbb{A}^1$ by

$$T \times (\mathbb{P}^*(V_l) \times \mathbb{A}^1) \to (\mathbb{P}^*(V_l) \times \mathbb{A}^1), \quad (t, (w, s)) \mapsto (\lambda(t)w, ts),$$

where G_l acts naturally on $\mathbb{P}^*(V_l)$ via the contragradient representation. Then the invertible sheaf

$$\mathcal{L} := \mathrm{pr}_1^* \, \mathcal{O}_{\mathbb{P}^*(V_l)}(1)$$

over $\mathcal{Z}(\lambda)$ is relatively very ample for the morphism π, and allows us to regard π as a projective morphism. Since the bundle space for $\mathcal{O}_{\mathbb{P}^*(V_l)}(-1)$ is identified with the blowing-up of V_l^* at the origin, the G_l-action on V_l^* induces naturally a T-action on $\mathcal{L}$ lifting the T-action on $\mathcal{Z}(\lambda)$. By restricting $\mathcal{L}$ to $\mathcal{Z}(\lambda)_s$, we have isomorphisms

$$(\mathcal{Z}(\lambda)_s, \mathcal{L}_s) \cong (M, \mathcal{O}_M(L^l)), \quad 0 \neq s \in \mathbb{A}^1,$$

where $\mathcal{L}_s := \mathcal{L}_{|\mathcal{Z}(\lambda)_s}$ for each $s \in \mathbb{A}^1$. Hence $\pi \colon \mathcal{Z}(\lambda) \to \mathbb{A}^1$ is a test configuration of exponent l for (M, L).

For the DeConcini-Procesi family $\pi \colon \mathcal{Z}(\lambda) \to \mathbb{A}^1$ as above, let $n_k(\lambda) \in \mathbb{Z}$ denote the weight of the T-action on the complex line

$$\bigwedge^{m_k} (E_k)_0 \quad \left(\cong \bigwedge^{m_k} H^0(\mathcal{Z}(\lambda)_0, \mathcal{L}_0^k) \quad \text{if} \quad k \gg 1 \right),$$

where $(E_k)_0 := (\pi_* \mathcal{L}^k)_0$ denotes the fiber, over the origin, of the locally free sheaf: $\pi_* \mathcal{L}^k \to \mathbb{A}^1$. If $k \gg 1$, then $\dim H^0(\mathcal{Z}(\lambda)_0, \mathcal{L}_0^k)$ is $m_k := \dim H^0(M, \mathcal{O}_M(L^{lk}))$, and we write m_k and $n_k(\lambda)$ as

$$m_k = \sum_{i=0}^{n} \mu_{l,i} k^i, \tag{3.4}$$

$$n_k(\lambda) = \sum_{j=0}^{n+1} \nu_{l,j}(\lambda) k^j, \tag{3.5}$$

where $\mu_{l,i}$, $i = 0, 1, \ldots, n$, and $\nu_{l,j}(\lambda)$, $j = 0, 1, \ldots, n+1$, are rational real numbers independent of the choice of positive integers k.

Let $0 \neq M_l^0 \in W_l^*$ be such that the associated $[M_l^0] \in \mathbb{P}^*(W_l)$ is the Chow point for the cycle $\mathcal{Z}(\lambda)_0$ on $\mathbb{P}^*(V_l)$ counted with multiplicities. First, we observe that $\mu_{l,n} = l^n c_1(L)^n[M]/n! > 0$. Next, in Section 4, we shall show that

$$\nu_{l,n+1}(\lambda) = -\frac{a_l}{(n+1)!} \tag{3.6}$$

where a_l denotes the weight of the T-action on $\mathbb{C}^* M_l^0$. We now put

$$w_\lambda(k; l) := n_k(\lambda)/(km_k).$$

REMARK. Besides the embedding $\mathcal{Z}(\lambda)_0 \hookrightarrow \mathbb{P}^*(V_l)$, we also have the embedding

$$\mathcal{Z}(\lambda)_0 \hookrightarrow \mathbb{P}^*((E_1)_0)$$

for the linear subsystem associated to $(E_1)_0$ in the complete linear system $|\mathcal{L}_0|$ on $\mathcal{Z}(\lambda)_0$. In the same manner as the weight a_l above is obtained from the cycle on $\mathcal{Z}(\lambda)_0$ on $\mathbb{P}^*(V_l)$, we similarly obtain a weight a_l' from the cycle $\mathcal{Z}(\lambda)_0$ on $\mathbb{P}^*((E_1)_0)$. Now by Mumford [6], Proposition 2.11,

$$\nu_{l,n+1}(\lambda) = -\frac{a_l'}{(n+1)!}.$$

Then (3.6) above claims that a_l' is replaced by a_l in this last equality.

Proof of (a) of Main Theorem. The argument at the beginning of this section shows that the identity component $\hat{G}_{l_i}^0$ of $\hat{G}_{l_i}$ satisfies

$$\hat{G}_{l_i}^0 \subset \tilde{G}_{l_i}^0,$$

where $\tilde{G}_{l_i}^0$ is isogenous to an algebraic subgroup of $\hat{H}$. Hence the assumption $\hat{H} = \{1\}$ of (a) of Main Theorem implies

$$|\hat{G}_{l_i}| < \infty \quad \text{for all} \quad 0 \leq i \in \mathbb{Z}.$$

Put $K_i := \prod_{j=0}^{i} k_j$ for $0 \leq i \in \mathbb{Z}$, where we put $K_{-1} := 1$ for simplicity. Moreover, we put $l_i := lK_{i-1}$ for $1 \leq i \in \mathbb{Z}$. Applying Lemma 3.10 below to $(l', l'', k', k'') = (l, l_i, K_{i-1}, K_i)$, we obtain

$$w_\lambda(K_i; l) > w_\lambda(K_{i-1}; l), \quad i = 0, 1, 2, \ldots, \tag{3.7}$$

for all algebraic one-parameter subgroup $\lambda\colon \mathbb{C}^* \hookrightarrow G_l$. On the other hand, by Appendix, we have $n_1(\lambda) = 0$, i.e.,

$$w_\lambda(K_{-1}; l) = w_\lambda(1; l) = 0. \tag{3.8}$$

In view of (3.4) and (3.5), we see that

$$\lim_{k\to\infty} w_\lambda(k; l) = \frac{\nu_{l,n+1}(\lambda)}{\mu_{l,n}}. \tag{3.9}$$

By (3.7), (3.8) and (3.9) together with $\mu_{l,n} > 0$, it follows that

$$\nu_{l,n+1}(\lambda) > 0 \quad \text{for all} \quad \lambda.$$

By (3.6), we conclude that (M, L^l) is Chow-stable, as required.

Lemma 3.10. *Let $n+1 \le \hat{k} \in \mathbb{Z}$, and let k', l' be positive integers with $l' \ge n+1$. Assume that $G_{l''} \cdot f_{l'',\hat{k}}$ is closed in $F_{l'',\hat{k}}$ for $k'' := \hat{k}k'$ and $l'' := k'l'$. If $\hat{H} = \{1\}$, then $w_\lambda(k''; l') > w_\lambda(k'; l')$ for all algebraic one-parameter subgroups $\lambda\colon \mathbb{C}^* \hookrightarrow G_{l'}$.*

4. Proof of (3.6)

In this section, we shall prove (3.6) by calculating the term $n_k(\lambda)$ in (3.4) in detail. Hereafter, by considering the Decontini-Procesi family $\mathcal{Z} = \mathcal{Z}(\lambda)$ over $\mathbb{A}^1$, we study the bundles E_k, $k = 1, 2, \ldots$ as in Section 2. A difficulty in calculating $n_k(\lambda)$ comes up when $\mathcal{Z}(\lambda)_0$ sits in a hyperplane of $\mathbb{P}^*(V_l)$. Let N be the, possibly trivial, T-invariant maximal linear subspace of V_l vanishing on $\mathcal{Z}(\lambda)_0$, where we regard $\mathcal{Z}(\lambda)_0$ as a subscheme in $\mathbb{P}^*(V_l)$ ($\cong \{0\} \times \mathbb{P}^*(V_l)$). Then for some T-invariant subspace Q_1 of V_l, we write the vector space V_l as a direct sum

$$V_l = Q_1 \oplus N.$$

By $Q_1 = V_l/N$, we naturally have a T-equivariant inclusion $Q_1 \subset R_1$, where $R_1 := (E_1)_0 = (\pi_*\mathcal{L})_0 \otimes \mathbb{C}$. Then

$$\mathcal{Z}(\lambda)_0 \subset \mathbb{P}^*(Q_1) \subset \mathbb{P}^*(V_l),$$

i.e., $\mathcal{Z}(\lambda)_0$ sits in the T-invariant linear subspace $\mathbb{P}^*(Q_1)$ of $\mathbb{P}^*(V_l)$. By taking the direct sum of the symmetric tensor products for Q_1, we put $Q := \bigoplus_{k=0}^{\infty} S^k(Q_1)$, where $S^k(Q_1)$ denotes $\mathbb{C}$ for $k = 0$. Let $J(Q) \subset Q$ denote the T-invariant homogeneous ideal of $\mathcal{Z}(\lambda)_0$ in $\mathbb{P}^*(Q_1)$. Then by setting $J(Q)_k := J(Q) \cap S^k(Q_1)$, we define

$$Q_k := S^k(Q_1)/J(Q)_k.$$

By Theorem 3 in [7], the natural homomorphism: $S^k(E_1) \to E_k$ is surjective over $\mathbb{A}^1 \setminus \{0\}$ for all positive integers k. We also have the T-equivariant inclusion

$$Q_k \subset R_k, \quad 0 < k \in \mathbb{Z}, \tag{4.1}$$

where $R_k := (E_k)_0 = (\pi_* \mathcal{L}^k)_0 \otimes \mathbb{C}$. By choosing general elements σ_i, $i = 0, 1, 2, \ldots, n$, in Q_1, we have a surjective holomorphic map

$$\mathrm{pr}_{\mathbb{P}^n} : \mathcal{Z}(\lambda)_0 \to \mathbb{P}^n(\mathbb{C}), \quad z \mapsto (\sigma_0(z) : \sigma_1(z) : \cdots : \sigma_n(z)),$$

so that the fiber $\mathrm{pr}_{\mathbb{P}^n}^{-1}(q)$ over $q := (1 : 0 : 0 : \cdots : 0)$ consists of r points counted with multiplicities, where $r := l^n c_1(L)^n[M]$. For each $k \gg 1$, we consider the subspace $F_k := \mathrm{pr}^*\, H^0(\mathbb{P}^n(\mathbb{C}), \mathcal{O}_{\mathbb{P}^n}(k))$ of Q_k. Then

$$\dim F_k = \frac{(n+k)!}{n!\, k!}$$

is a polynomial in k of degree n with leading coefficient $1/n!$. For some positive integer k_0, there exist elements $\tau_1, \tau_2, \ldots, \tau_r$ in $Q_{k_0} \setminus F_{k_0}$ which separate the points in $\mathrm{pr}_{\mathbb{P}^n}^{-1}(q)$ including infinitely near points. Then for $k \gg 1$, the linear subspaces

$$\tau_1 F_{k-k_0},\ \tau_2 F_{k-k_0}, \ldots,\ \tau_r F_{k-k_0}$$

of Q_k all together span a linear subspace of dimension

$$r \dim F_{k-k_0} = r \frac{(n+k-k_0)!}{n!\,(k-k_0)!} = \frac{r}{n!} k^n + \text{lower order term in } k.$$

In view of (4.1), $r \dim F_{k-k_0} \le \dim Q_k \le \dim R_k = m_k$. Hence

$$\dim R_k / Q_k \le C_1 k^{n-1} \tag{4.2}$$

for some positive constant C_1 independent of k. Put $\delta_k := \dim Q_k$, and let $q_k(\lambda)$ denote the weight of the T-action on $\bigwedge^{\delta_k} Q_k$, where the weight of the T-action on $\bigwedge^{m_k} R_k$ is $n_k(\lambda)$. Then the weight of the T-action on $\bigwedge^{m_k - \delta_k}(R_k/Q_k)$ is $n_k(\lambda) - q_k(\lambda)$. On the other hand, in view of Remark 4.6 below, the weight α for the T-action on every 1-dimensional T-invariant subspace A of R_k/Q_k satisfies

$$|\alpha| \le C_2 k \tag{4.3}$$

for some positive constant C_2 independent of the choice of k. Then we see from (4.2) and (4.3) that

$$|n_k(\lambda) - q_k(\lambda)| \le C_1 C_2 k^n. \tag{4.4}$$

Now a classical result of Mumford [8], Proposition 2.11, asserts that

$$q_k(\lambda) = -\frac{a_l k^{n+1}}{(n+1)!} + \text{lower order term in } k, \tag{4.5}$$

where the weight in [8] and ours have opposite sign. From (3.5), (4.4) and (4.5), we obtain (3.6) as required.

REMARK 4.6. Put $X_0 := \mathcal{Z}(\lambda)_0$. For X_0 sitting in $\mathbb{P}^*(V_l) \times \{0\}$, we choose a sequence of scheme-theoretic intersections

$$X_j := X_0 \cap \Sigma_1 \cap \Sigma_2 \cap \cdots \cap \Sigma_j, \quad j = 1, 2, \ldots, n,$$

where $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$ are n distinct general hyperplanes in $\mathbb{P}^*(V_l) \times \{0\}$. Then there exists an integer i_0 satisfying $i_0 \gg n$ such that

$$H^p(X_j, \mathcal{O}_{X_j}(\mathcal{L}_0^i)) = \{0\}, \quad i \geq i_0 - n,$$

for all $p > 0$ and $j = 0, 1, \ldots, n$. Then by the arguments in the proof of Theorem 2 in [7], the natural homomorphisms

$$H^0(\mathcal{Z}(\lambda)_0, \mathcal{L}_0^i) \otimes H^0(\mathcal{Z}(\lambda)_0, \mathcal{L}_0) \to H^0(\mathcal{Z}(\lambda)_0, \mathcal{L}_0^{i+1}), \quad i \geq i_0,$$

are surjective. In particular, for all positive integers k, the natural homomorphisms: $(R_i)^{\otimes k} \to R_{ik}$ are surjective for all integers $i \geq i_0$.

5. Proof of Lemma 3.10

In this section, we apply Section 2 to $\mathcal{Z} = \mathcal{Z}(\lambda)$ and $l = l'$, where the actions of $T := \mathbb{C}^*$ on $\mathcal{L}$ and $\mathcal{Z}(\lambda)_0$ are induced by the one-parameter group $\lambda : \mathbb{C}^* \hookrightarrow G_{l'}$ in Lemma 3.10, where for each positive integer k, the corresponding T-action ρ_k on E_k induces

$$\lambda_k : T \to \mathrm{GL}_{\mathbb{C}}((E_k)_0)$$

as in (2.4). Recall that $k'' = \hat{k}k'$ and $\hat{k} \geq n + 1$. For each $s \in \mathbb{A}^1 \setminus \{0\}$, let $\mathcal{I}(\mathcal{Z}(\lambda)_s)$ denote the kernel of the naural T-equivariant surjective homomorphism

$$S^{\hat{k}}(E_{k'})_s \to (E_{k''})_s,$$

between fibers over s for bundles $S^{\hat{k}}(E_{k'})$ and $E_{k''}$, where the T-actions on $S^{\hat{k}}(E_{k'})$ and $E_{k''}$ are by $\rho_{k'}$ and $\rho_{k''}$, respectively. By the trivialization in Lemma 2.3 applied to $k = k'$, we can identify each $\mathbb{G}\mathrm{r}_s$, $s \in \mathbb{A}^1$, with $\mathbb{G}\mathrm{r}_0$. Here $\mathbb{G}\mathrm{r}_s$ denotes the complex

Grassmannian of all complex $\gamma_{l'',\hat{k}}$-planes through the origin in $S^{\hat{k}}(E_{k'})_s$. The holomorphic map sending each $s \in \mathbb{A}^1 \setminus \{0\}$ to $\mathcal{I}(\mathcal{Z}(\lambda)_s)$ regarded as an element in $(\mathbb{G}\mathrm{r}_s \cong)$ $\mathbb{G}\mathrm{r}_0$ extends naturally to a holomorphic map: $\mathbb{A}^1 \to \mathbb{G}\mathrm{r}_0$, where the image of the origin under this holomorphic map will be denoted by $\mathcal{I}(\mathcal{Z}(\lambda)_0)$ by abuse of terminology. For the inclusion $\mathcal{Z}(\lambda) \hookrightarrow \mathbb{P}^*(E_{k'})$ in (2.2), the action of each $t \in T$ maps $\mathcal{Z}(\lambda)_s$ onto $\mathcal{Z}(\lambda)_{ts}$, and we have

$$\mathcal{I}(\mathcal{Z}(\lambda)_{ts}) = \rho_{k'}(t)(\mathcal{I}(\mathcal{Z}(\lambda)_s)). \tag{5.1}$$

Here, via the T-action on $\mathcal{Z}(\lambda)_0$, T acts on $S^{\hat{k}}(E_{k'})_0$ preserving $\mathcal{I}(\mathcal{Z}(\lambda)_0)$. At $s = 1$, the fiber $\mathcal{Z}(\lambda)_s := \pi^{-1}(s)$ over s is thought of as $\iota_{l'}(M)$ sitting in $\mathbb{P}^*(V_{l'})$. Hence by the notation in Section 1, we have

$$\mathcal{I}(\mathcal{Z}(\lambda)_s)_{|s=1} = I_{l'',\hat{k}}$$

by identifying $E_{k'|s=1}$, $E_{k''|s=1}$ with $V_{l''}$, $V_{l''\hat{k}}$, respectively. Consider the closed disc $\Delta := \{s \in \mathbb{A}^1;\ |s| \le 1\}$ of $\mathbb{A}^1$. Since $\bigwedge^{\gamma_{l'',\hat{k}}} \mathcal{I}(\mathcal{Z}(\lambda)_s)$ in $\bigwedge^{\gamma_{l'',\hat{k}}} S^{\hat{k}}(E_{k'})_s$ is a complex line, for each $s \in \Delta$, we can choose an element $\psi_{l'',\hat{k}}(s) \neq 0$ in the line in such a way that $\psi_{l'',\hat{k}}(s)$ depends on s holomorphically. Then $\psi_{l'',\hat{k}}(1)$ is regarded as a nonzero element in the line $\bigwedge^{\gamma_{l'',\hat{k}}} I_{l'',\hat{k}}$ in $\bigwedge^{\gamma_{l'',\hat{k}}} S^{\hat{k}}(V_{l''})$. By the trivialization in Lemma 2.3 applied to $k = k'$, we hereafter identify $(E_{k'})_s$, $s \in \Delta$, with $(E_{k'})_0$. Consequently, this identification allows us to regard $\psi_{l'',\hat{k}}(s)$ as an element in $\Psi := \bigwedge^{\gamma_{l'',\hat{k}}} S^{\hat{k}}(E_{k'})_0$ for each $s \in \Delta$, and $G_{l''}$ is viewed as $\mathrm{SL}_{\mathbb{C}}((E_{k'})_0)$. For each t, $t' \in \mathbb{C}^*$, by taking an unramified cover of $\mathbb{C}^*$ of degree $m_{k'}$, we can write

$$t = \tilde{t}^{m_{k'}} \quad \text{and} \quad t' = (\tilde{t}')^{m_{k'}},$$

for $\tilde{t}$, $\tilde{t}' \in \mathbb{C}^*$, where $m_{k'}$ is the rank of the vector bundle $E_{k'}$. The closedness of $G_{l''} \cdot f_{l'',\hat{k}}$ in $F_{l'',\hat{k}}$ in the assumption of Lemma 3.10 means that the orbit $\mathrm{SL}_{\mathbb{C}}((E_{k'})_0) \cdot \psi_{l'',\hat{k}}(1)$ is closed in Ψ. Now by the Hilbert-Mumford stability criterion,

$$\hat{\lambda}(\mathbb{C}^*) \cdot \psi_{l'',\hat{k}}(1) \quad \text{is closed in} \quad \Psi, \tag{5.2}$$

where $\hat{\lambda} : \mathbb{C}^* \to \mathrm{SL}_{\mathbb{C}}((E_{k'})_0)$ is an algebraic group homomorphism defined by

$$\hat{\lambda}(\tilde{t}) := \frac{\lambda_{k'}(t)}{\det \lambda_{k'}(\tilde{t})}, \quad \tilde{t} \in \mathbb{C}^*,$$

for $\lambda_{k'}$ as in Section 2. To each $\psi_{l'',\hat{k}}(s)$, $s \in \Delta$, we can naturally assign an element $[\psi_{l'',\hat{k}}(s)]$ in the complex Grassmannian $\mathbb{G}\mathrm{r}_0$. Here $[\psi_{l'',\hat{k}}(s)]$ corresponds to the subspace $\mathcal{I}(\mathcal{Z}(\lambda)_s)$ in $S^{\hat{k}}(E_{k'})_0$ via the identification $S^{\hat{k}}(E_{k'})_s \cong S^{\hat{k}}(E_{k'})_0$ in terms of the trivialization in Lemma 2.3 applied to $k = k'$. Obviously,

$$[\psi_{l'',\hat{k}}(s)] \to [\psi_{l'',\hat{k}}(0)] \quad \text{as} \quad s \to 0.$$

Moreover, in view of (5.1), we obtain

$$\lambda_{k'}(t)\cdot\psi_{l'',\hat{k}}(s)\in\mathbb{C}^*\cdot\psi_{l'',\hat{k}}(ts),\quad s\in\Delta,$$

for all $t\in\mathbb{C}^*$ satisfying $|t|\le 1$. For some $\varepsilon\in\mathbb{R}$ with $0<\varepsilon\ll 1$, we put $D_\varepsilon:=\{t\in\mathbb{C}^*;\ |t|<\varepsilon\}$. Then

$$\hat{\lambda}(\tilde{t})\cdot\psi_{l'',\hat{k}}(1)=\frac{\lambda_{k'}(t)\cdot\psi_{l'',\hat{k}}(1)}{\det\lambda_{k'}(\tilde{t})}=\tilde{t}^{\beta}\psi(t),\quad t\in D_\varepsilon, \tag{5.3}$$

for some nonvanishing holomorphic map $\Delta_\varepsilon\ni s\mapsto\psi(s)\in\Psi$, where by Δ_ε, we mean the subset $\{s\in\mathbb{C};\ |s|\le\varepsilon\}=D_\varepsilon\cup\{0\}$ of Δ. Now by (5.2) and (5.3), we obtain

$$\beta<0. \tag{5.4}$$

On the other hand, since the map ψ is continuous, (5.3) implies

$$\lim_{t\to 0}\tilde{t}^{-\beta}\hat{\lambda}(\tilde{t})\cdot\psi_{l'',\hat{k}}(1)=\psi(0). \tag{5.5}$$

If $t,\ t'\in D_\varepsilon$, then from (5.3), it follows that

$$\hat{\lambda}(\tilde{t})\hat{\lambda}(\tilde{t}')\cdot\psi_{l'',\hat{k}}(1)=\hat{\lambda}(\tilde{t}\tilde{t}')\cdot\psi_{l'',\hat{k}}(1)=(\tilde{t}\tilde{t}')^{\beta}\psi(tt').$$

Hence $\hat{\lambda}(\tilde{t})\{\tilde{t}'^{-\beta}\hat{\lambda}(\tilde{t}')\cdot\psi_{l'',\hat{k}}(1)\}=\tilde{t}^{\beta}\psi(tt')$. Let $t'\to 0$. Then this together with (5.5) implies

$$\hat{\lambda}(\tilde{t})\cdot\psi(0)=\tilde{t}^{\beta}\psi(0). \tag{5.6}$$

In view of (5.4) and (5.6), the argument as in [1], 2.3, applied to $S^{\hat{k}}(E_{k'})_0\to(E_{k''})_0$ allows us to obtain

$$0>\beta=\frac{\hat{k}m_{k''}n_{k'}(\hat{\lambda})}{m_{k'}}-n_{k''}(\hat{\lambda})=k''m_{k''}\left\{\frac{n_{k'}(\hat{\lambda})}{k'm_{k'}}-\frac{n_{k''}(\hat{\lambda})}{k''m_{k''}}\right\},$$

where $n_{k'}(\hat{\lambda})$ and $n_{k''}(\hat{\lambda})$ are the weights of the $\mathbb{C}^*$-actions on $\bigwedge^{m_{k'}}(E_{k'})_0$ and $\bigwedge^{m_{k''}}(E_{k''})_0$, respectively, induced by $\hat{\lambda}$. Since $\lambda_{k'}$ is induced by λ, the definition of $\tilde{t}$ and $\hat{\lambda}$ shows that

$$\frac{n_{k'}(\hat{\lambda})}{k'm_{k'}}-\frac{n_{k''}(\hat{\lambda})}{k''m_{k''}}=m_{k'}\{w_\lambda(k';l')-w_\lambda(k'';l')\}$$

and hence $w_\lambda(k';l')<w_\lambda(k'';l')$, as required. This completes the proof of Lemma 3.10.

6. Appendix

The purpose of this section is to study the T-action $\rho_{k,0}$ on $(E_k)_0$ with $k = 1$ for the DeConcini-Procesi family $\mathcal{Z} = \mathcal{Z}(\lambda)$ over $\mathbb{A}^1$. Let

$$\tilde{\mathrm{pr}}_1 : \mathbb{P}^*(V_l) \times \mathbb{A}^1 \to \mathbb{P}^*(V_l), \quad \tilde{\pi} : \mathbb{P}^*(V_l) \times \mathbb{A}^1 \to \mathbb{A}^1$$

be the projections to respective factors. Put $\tilde{\mathcal{L}} := \tilde{\mathrm{pr}}_1^* \mathcal{O}_{\mathbb{P}^*(V_l)}(1)$. Then for every $e \in V_l$, the map sending each $s \in \mathbb{A}^1$ to $(e, s) \in V_l \times \mathbb{A}^1$ defines a holomorphic section, denoted by $\tau(e)$, in $H^0(\mathbb{A}^1, \tilde{\pi}_* \tilde{\mathcal{L}})$. The pullback $\iota^* \tau(e)$ by the inclusion map

$$\iota : \mathcal{Z}(\lambda) \hookrightarrow \mathbb{P}^*(V_l) \times \mathbb{A}^1$$

is naturally regarded as a holomorphic section of E_1 over $\mathbb{A}^1 = \{s \in \mathbb{C}\}$, where s is the affine coordinate for $\mathbb{A}^1$. Note that, for $s \neq 0$, the map

$$V_l \ni e \mapsto \{\iota^* \tau(e)\}(s) \in (E_1)_s$$

is a linear isomorphism. Here E_k with $k = 1$ is written as E_1, and $(E_1)_s$ denotes the fiber of the vector bundle E_1 over s. In terms of the T-action on V_l via the one-parameter group $\lambda : T \to \mathrm{SL}(V_l)$, write the vector space V_l as a direct sum

$$N = \bigoplus_{i=1}^{p} N_i, \tag{6.1}$$

where $N_i = \{e \in V_l;\ \lambda(t)e = t^{\alpha_i} e \text{ for all } t \in T\}$ for some mutually distinct integers α_i, $i = 1, 2, \ldots, p$. For each i, consider the $\mathbb{C}[s]$-module $N_i[s] := N_i \otimes_{\mathbb{C}} \mathbb{C}[s]$, where by $\mathbb{C}[s]$, we mean the ring of polynomials in s with coefficients in $\mathbb{C}$. Let $\{e_1, e_2, \ldots, e_{n_i}\}$ be a basis for the vector space N_i, where $n_i := \dim N_i$. For every $e \in N_i[s]$, by writing e as a sum $\sum_{j=1}^{n_i} f_j(s) e_j \in N_i[s]$ for some polynomials $f_j(s) \in \mathbb{C}[s]$ in s, we put

$$\tau(e) := \sum_{j=1}^{n_i} f_j(s) \tau(e_j) \in H^0(\mathbb{A}^1, \tilde{\pi}_* \tilde{\mathcal{L}}).$$

From the T-action on V_l via λ, we have a natural fiberwise T-action on the trivial bundle $V_l \times \mathbb{A}^1$ over $\mathbb{A}^1$. This then induces a fiberwise T-action on the vector bundle E_1 over $\mathbb{A}^1$, while the restriction of this induced T-action to the fiber $(E_1)_0$ is exactly $\rho_{1,0}$.

Proposition 6.2. *There exists a non-decreasing sequence of nonnegative integers $\beta_{i1} \leq \beta_{i2} \leq \cdots \leq \beta_{in_i}$ together with $\mathbb{C}[s]$-generators $\{e_{ij};\ j = 1, 2, \ldots, n_i\}$ for the $\mathbb{C}[s]$-module $N_i[s]$ such that*

$$\iota^* \tau(e_{ij}) = s^{\beta_{ij}} \sigma_{ij}, \quad i = 1, 2, \ldots, p,\ j = 1, 2, \ldots, n_i, \tag{6.3}$$

for some holomorphic sections σ_{ij} *to* E_1 *over* $\mathbb{A}^1$, *where*

$$\{\sigma_{ij}(0);\ i = 1, 2, \ldots, p,\ j = 1, 2, \ldots, n_i\} \tag{6.4}$$

forms a basis for the vector space $(E_1)_0$.

Proof. By induction on $j = 1, 2, \ldots, n_i$, we define e_{ij} and σ_{ij} from $\{e_{i1}, e_{i2}, \ldots, e_{ij-1}\}$ and $\{\sigma_{i1}, \sigma_{i2}, \ldots, \sigma_{ij-1}\}$ as follows. Let B_{j-1} be the $\mathbb{C}[s]$-submodule of $N_i[s]$ generated by $\{e_{i1}, e_{i2}, \ldots, e_{ij-1}\}$, where we put $B_{j-1} = \{0\}$ for $j = 1$. Let $\mathcal{Y}_{ij}$ denote the set of all $\mathbb{C}[s]$-submodules $Y \subset N_i[s]$ generated by $n_i - j + 1$ elements such that

$$Y + B_{j-1} = N_i[s], \tag{6.5}$$

where $Y + B_{j-1}$ is the $\mathbb{C}[s]$-submodule of $N_i[s]$ generated by Y and B_{j-1}. For each $Y \in \mathcal{Y}_{ij}$, let $\beta(Y)$ denote the maximal nonnegative integer β such that all $\iota^*\tau(e)$, $e \in Y$, are divisible by s^β in $H^0(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}(E_1))$. In view of the inequality $j \le n_i$, the maximum

$$\beta_{ij} := \max_{Y \in \mathcal{Y}_{ij}} \beta(Y)$$

exists because, otherwise, (6.5) would imply that $\iota^*\tau(N_i) \subset \iota^*\tau(B_{j-1})$ modulo s^β for all positive integers β, in contradiction to $n_i > j - 1$. By the definition of β_{ij}, it now easily follows that $\beta_{i1} \le \beta_{i2} \le \cdots \le \beta_{in_i}$. Take an element Y_{ij} of $\mathcal{Y}_{ij}$ such that $\beta(Y_{ij}) = \beta_{ij}$. Then the maximality of β_{ij} allows us to obtain $e_{ij} \in Y_{ij}$ and $\sigma_{ij} \in H^0(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1}(E_1))$ satisfying $\iota^*\tau(e_{ij}) = s^{\beta_{ij}}\sigma_{ij}$ such that $\sigma_{ij}(0)$ is $\mathbb{C}$-linearly independent from $\sigma_{i1}(0)$, $\sigma_{i2}(0), \ldots, \sigma_{ij-1}(0)$ in $(E_1)_0$. Since this induction procedure stops at $j = n_i$, we obtain both (6.3) and the required condition for (6.4). □

Now the vector bundle E_1 is generated by the global sections $\{\sigma_{ij};\ i = 1, 2, \ldots, p,\ j = 1, 2, \ldots, n_i\}$ over $\mathbb{A}^1$. Then by (6.1) and (6.3),

$$\rho_{1,0}(t,\ \sigma_{ij}(0)) = t^{\alpha_i}\sigma_{ij}(0)$$

for all i and j. In particular, $n_1(\lambda) = \sum_{i=1}^p n_i\alpha_i$. Since λ is an algebraic one-parameter subgroup in $G_l = \mathrm{SL}(V_l)$, by the definition of N_i, it follows from (6.1) that

$$1 = \det(\lambda(t)) = t^{\sum_{i=1}^p n_i\alpha_i}$$

for all $t \in T$, i.e., $n_1(\lambda) = 0$. Note that this equality follows also from Lemma 2.3 by the equivariant isomorphism $(E_1)_1 \cong (E_1)_0$ (see also [11]).

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