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UNIQUENESS FOR THE BREZIS-NIRENBERG PROBLEM ON COMPACT EINSTEIN MANIFOLDS

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Abstract

We consider the positive solution of the following semi-linear elliptic equation on the compact Einstein manifolds M^n with positive scalar curvature R_0

$$\Delta_0 u - \lambda u + f(u)u^{(n+2)/(n-2)} = 0,$$

where Δ_0 is the Laplace-Beltrami operator on M^n. We prove that for $0 < \lambda \leq (n-2)R_0/(4(n-1))$ and $f'(u) \leq 0$, and at least one of two inequalities is strict, the only positive solution to the above equation is constant. The method here is intrinsic.

1. Introduction

Let (M^n, g_0) be the compact Einstein manifold with positive scalar curvature R_0 and $n \geq 3$. In this paper we consider the following nonlinear elliptic equation

$$\begin{cases} \Delta_0 u - \lambda u + f(u)u^{(n+2)/(n-2)} = 0, & \text{on} \quad M^n; \\ u > 0, & \text{on} \quad M^n, \end{cases} \tag{1.1}$$

where Δ_0 is the Laplace-Beltrami operator on M^n related to g_0. In the case of f a constant and $\lambda = (n-2)R_0/(4(n-1))$ with R_0 the scalar curvature of Riemannian manifold M^n, the problem (1.1) is just the Yamabe problem in the conformal geometry. If $M^n = \mathbf{S}^n$, there are infinitely many solutions for the Yamabe problem because the conformal group of the sphere is also infinite. For the Einstein manifold which is conformally distinct from sphere, Obata [11] shown that the Yamabe problem has unique solution, and Schoen pointed out there are more than three solutions for Yamabe problem on $\mathbf{S}^1 \times \mathbf{S}^{n-1}$, $n \geq 3$. Recently, Brezis and Li [4] consider problem (1.1) and specially the following problem by using moving planes and blow-up analysis.

$$\begin{cases} \Delta_0 u - \lambda u + u^p = 0, & \text{on} \quad \mathbf{S}^n; \\ u > 0, & \text{on} \quad \mathbf{S}^n, \end{cases} \tag{1.2}$$

and obtain that

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(A): for $M = \mathbf{S}^n$, $0 < \lambda < n(n-2)/4$ and f decreasing on $(0, +\infty)$, the only positive solution to (1.1) is constant,
(B): for $0 < \lambda \leq n(n-2)/4$ and $1 < p \leq (n+2)/(n-2)$, and at least one of two inequalities is strict, the only positive solution to (1.2) is constant,
(C): for M compact, $n = 3$ and $f = 1$, there exists a constant $\lambda_0 = \lambda_0(M, g) > 0$ such for $0 < \lambda < \lambda_0$, the only positive solution to (1.1) is constant.

In this paper our conclusions rely on the remarkable identity established by intrinsic properties. For related problems, see e.g. [2], [3], [5], [6], [8], [9], [12], [13].

Our main results are as follows.

Theorem 1.1. *Suppose M be the compact Einstein manifold, $0 < \lambda \leq (n-2)R_0/(4(n-1))$ and $f'(u) \leq 0$, and at least one of two inequalities is strict. Then the only positive solution to* (1.1) *is constant.*

As a consequence, we prove the following theorem.

Theorem 1.2. *Suppose M be the compact Einstein manifold, $0 < \lambda \leq (n-2)R_0/(4(n-1))$ and $1 < p \leq (n+2)/(n-2)$, and at least one of two inequalities is strict. Then the only solution of the equation*

$$\begin{cases} \Delta_0 u - \lambda u + u^p = 0, & \text{on} \quad M^n; \\ u > 0, & \text{on} \quad M^n, \end{cases}$$

is the constant solution $u = \lambda^{1/(p-1)}$.

REMARK. Clearly, Theorem 1.1 and Theorem 1.2 can be seen a generalization of Brezis and Li's results (see Theorem 1 in [4]). On the other hand, Theorem 1.2 also answers the Brezis and Li's problem 2 in [4] for compact Einstein manifolds with positive scalar curvature.

2. Proof of Theorems

Let (M^n, g_0) be the compact Einstein manifold with positive scalar curvature R_0. Define the conformal transformation $g = u^{4/(n-2)} g_0$ on M^n, then Δ_0 is related with the scalar curvatur R of g by

$$\Delta_0 u - \frac{(n-2)R_0}{4(n-1)} u + \frac{(n-2)R}{4(n-1)} u^{(n+2)/(n-2)} = 0,$$

which combing with (1.1) gives

$$R = \frac{4(n-1)}{n-2} \left(f(u) + \left(\frac{(n-2)R_0}{4(n-1)} - \lambda \right) u^{-4/(n-2)} \right).$$

Setting $\bar{\lambda} = \lambda - (n-2)R_0/(4(n-1))$, then

$$R = \frac{4(n-1)}{n-2}(f(u) - \bar{\lambda}u^{-4/(n-2)}).$$

In what follows, the Einstein summation convention will be used. Let

$$\varphi = \varphi_{ij}\frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$$

be a symmetric tensor defined on M^n, and

$$\varphi_{ij} = \frac{R}{2}g^{ij} - R_{kl}g^{ik}g^{jl}.$$

It follows from [7] that the operator $\Box$ associated to φ acting on any C^2-function f defined by

$$\Box f = \varphi_{ij} f_{,ij} = \left(\frac{R}{2}g^{ij} - R_{kl}g^{ik}g^{jl}\right) f_{,ij} \tag{2.1}$$

is self-adjoint relative to the L^2 inner product of M^n, that is

$$\int_{M^n} (\Box f)g\, dV_g = \int_{M^n} f(\Box g)\, dV_g.$$

Lemma 2.1. *Let $g_0 = \varphi^{-2}g$, B_0 and B the trace free Ricci tensor of the metric g_0 and g on M^n, respectively. Then we have*

$$B_0 = B + \frac{n-2}{\varphi}\left(D\, d\varphi - \frac{\Delta\varphi}{n}g\right).$$

This formula was studied in particular by Obata [10]. One can find a proof also in Besse's book [1], Theorem 1.159.

For an Einstein metric g_0, its trace free Ricci tensor is nothing but zero. Let $\varphi = u^{2/(n-2)}$, then the above lemma shows that, in the local coordinate system,

$$B_{ij} = (n-2)\left(\frac{1}{n}\Delta(u^{2/(n-2)})g_{ij} - (u^{2/(n-2)})_{,ij}\right)u^{-2/(n-2)}, \tag{2.2}$$

where the covariant derivatives are with respect to g, and

$$R_{ij} = B_{ij} + \frac{R}{n}g_{ij}. \tag{2.3}$$

(2.2) can be written as

$$(u^{2/(n-2)})_{,ij} = \frac{1}{n}\Delta(u^{2/(n-2)})g_{ij} - \frac{1}{n-2}u^{2/(n-2)}B_{ij}. \tag{2.4}$$

Substituting f in (2.1) with $u^{2/(n-2)}$ and using (2.4), we have

$$\begin{aligned}
\Box(u^{2/(n-2)}) &= \left(\frac{R}{2}g^{ij} - R_{kl}g^{ik}g^{jl}\right)(u^{2/(n-2)})_{,ij} \\
&= \frac{R}{2}\Delta(u^{2/(n-2)}) - R_{kl}(u^{2/(n-2)})_{,ij}g^{ik}g^{jl} \\
&= \frac{R}{2}\Delta(u^{2/(n-2)}) - R_{kl}\left(\frac{1}{n}\Delta(u^{2/(n-2)})g_{ij} - \frac{1}{n-2}u^{2/(n-2)}B_{ij}\right)g^{ik}g^{jl} \\
&= \frac{(n-2)R}{2n}\Delta(u^{2/(n-2)}) + \frac{1}{n-2}u^{2/(n-2)}R_{kl}B_{ij}g^{ik}g^{jl}.
\end{aligned} \tag{2.5}$$

Therefore, (2.5) together with (2.3) gives

$$\begin{aligned}
\Box(u^{2/(n-2)}) &= \frac{(n-2)R}{2n}\Delta(u^{2/(n-2)}) + \frac{1}{n-2}u^{2/(n-2)}\left(B_{kl} + \frac{R}{n}g_{kl}\right)B_{ij}g^{ik}g^{jl} \\
&= \frac{(n-2)R}{2n}\Delta(u^{2/(n-2)}) + \frac{1}{n-2}u^{2/(n-2)}|B|^2 + \frac{R}{n(n-2)}u^{2/(n-2)}B_{ij}g^{ij} \\
&= \frac{(n-2)R}{2n}\Delta(u^{2/(n-2)}) + \frac{1}{n-2}u^{2/(n-2)}|B|^2.
\end{aligned}$$

Note that

$$\int_{M^n} \Box(u^{2/(n-2)})\, dV_g = 0, \quad dV_g = u^{2n/(n-2)}\, dV_{g_0}.$$

Integrating the above equality and using the divergence theorem, we obtain

$$\begin{aligned}
\int_{M^n} u^{2/(n-2)}|B|^2\, dV_g &= \frac{(n-2)^2}{2n}\int_{M^n} \langle\nabla(u^{2/(n-2)}), \nabla R\rangle\, dV_g \\
&= \frac{(n-2)^2}{2n}\int_{M^n} u^2\langle\nabla_0(u^{2/(n-2)}), \nabla_0 R\rangle\, dV_{g_0} \\
&= \frac{2(n-1)(n-2)}{n}\int_{M^n} u^2\langle\nabla_0(u^{2/(n-2)}), \nabla_0(f(u) - \bar{\lambda}u^{-4/(n-2)})\rangle\, dV_{g_0} \\
&= \frac{4(n-1)}{n}\left(\int_{M^n} f'(u)u^{n/(n-2)}|\nabla_0 u|^2\, dV_{g_0}\right. \\
&\qquad \left. + \frac{4\bar{\lambda}}{n-2}\int_{M^n} u^{-2/(n-2)}|\nabla_0 u|^2\, dV_{g_0}\right).
\end{aligned} \tag{2.6}$$

Under the assumption of Theorem 1.1, (2.6) shows that

$$0 \le \int_{M^n} u^{2/(n-2)}|B|^2\,dV_g$$
$$= \frac{4(n-1)}{n}\left(\int_{M^n} f'(u)u^{n/(n-2)}|\nabla_0 u|^2\,dV_{g_0} + \frac{4\bar{\lambda}}{n-2}\int_{M^n} u^{-2/(n-2)}|\nabla_0 u|^2\,dV_{g_0}\right) \le 0,$$

and u must be a constant.

Let $f(u) = u^{\alpha}$, $\alpha \le 0$, we get Theorem 1.2 holds. The proof of theorems is completed finally.

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