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THE TWISTOR SPACES OF A PARA-QUATERNIONIC KÄHLER MANIFOLD

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Abstract

We develop the twistor theory of G-structures for which the (linear) Lie algebra of the structure group contains an involution, instead of a complex structure. The twistor space Z of such a G-structure is endowed with a field of involutions $\mathcal{J} \in \Gamma(\mathrm{End}\, TZ)$ and a $\mathcal{J}$-invariant distribution $\mathcal{H}_Z$. We study the conditions for the integrability of $\mathcal{J}$ and for the (para-)holomorphicity of $\mathcal{H}_Z$. Then we apply this theory to para-quaternionic Kähler manifolds of non-zero scalar curvature, which admit two natural twistor spaces $(Z^\epsilon, \mathcal{J}, \mathcal{H}_Z)$, $\epsilon = \pm 1$, such that $\mathcal{J}^2 = \epsilon \mathrm{Id}$. We prove that in both cases $\mathcal{J}$ is integrable (recovering results of Blair, Davidov and Muškarov) and that $\mathcal{H}_Z$ defines a holomorphic ($\epsilon = -1$) or para-holomorphic ($\epsilon = +1$) contact structure. Furthermore, we determine all the solutions of the Einstein equation for the canonical one-parameter family of pseudo-Riemannian metrics on Z^ϵ. In particular, we find that there is a unique Kähler-Einstein ($\epsilon = -1$) or para-Kähler-Einstein ($\epsilon = +1$) metric. Finally, we prove that any Kähler or para-Kähler submanifold of a para-quaternionic Kähler manifold is minimal and describe all such submanifolds in terms of complex ($\epsilon = -1$), respectively, para-complex ($\epsilon = +1$) submanifolds of Z^ϵ tangent to the contact distribution.

Contents

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1. Introduction

Twistor methods were originally introduced by Penrose with the aim of providing a mathematical framework which could lead to a synthesis of quantum theory and relativity [13, 14]. They have proven very fruitful for the construction and systematic study of various geometric objects governed by non-linear partial differential equations such as Yang-Mills connections, Einstein metrics, harmonic maps and minimal submanifolds.

Given a geometric problem on a real differentiable manifold M endowed with certain geometric structure S, the twistor approach is to try to translate the given problem into a problem of complex geometry an a complex manifold Z, called the twistor space, which is the total space of a bundle over M. In most cases, Z can be defined as the bundle of all complex structures in the tangent spaces of M which are compatible with the geometric structure S and it comes with a natural almost complex structure $\mathcal{J}$, the integrability of which has to be derived from the properties of the structure S.

In the case of a four-dimensional oriented Riemannian manifold M, for instance, the fibre at $p \in M$ of the twistor bundle $Z \to M$ consists of all skew-symmetric complex structures in T_pM, which induce the given orientation [4]. It is identified with the Riemann sphere $\mathbb{C}P^1$ and, thus, carries a natural complex structure. On the other hand, the Levi-Civita connection of M induces a horizontal (i.e. transversal to the fibers) distribution $\mathcal{H}_Z \subset TZ$ and the horizontal spaces carry a tautological complex structure. Putting the complex structures on vertical and horizontal spaces together, one obtains a canonical almost complex structure $\mathcal{J}$ on Z. By the results of Atiyah, Hitchin and Singer, $\mathcal{J}$ is integrable if and only if the Weyl curvature tensor of M is self-dual and, in that case, self-dual Yang-Mills vector bundles on M correspond to certain holomorphic vector bundles on Z. Salamon et al. have extended these constructions from four to higher dimensions, with the role of the self-dual four-dimensional Riemannian manifold played by a quaternionic Kähler manifold [15, 9]. In [3] the twistor method was used to construct (minimal) Kähler submanifolds of quaternionic Kähler manifolds.

A G-structure is called of *twistor type* if the (linear) Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$ of the structure group contains a complex structure, i.e. an element J such that $J^2 = -\mathrm{Id}$. The twistor theory of G-structures of twistor type is developped in [2], see also references therein. This includes the case of quaternionic Kähler manifolds, for which the structure group is $G = \mathrm{Sp}(1)\mathrm{Sp}(n)$.

In this paper, we develop a similar theory for G-structures of *para-twistor type*, i.e. for which $\mathfrak{g}$ contains an involution J, rather than a complex structure. Let $P \to M$ be such a G-structure and denote by $K = Z_G(J)$ the centralizer of the involution J. For any principal connection ω on P, we define the twistor space of (P, ω) as the total space of the bundle $Z = P/K \to P/G = M$, which we endow with a K-structure $P \to Z$, a field of involutions $\mathcal{J} \in \Gamma(\mathrm{End}\, TZ)$ and a $\mathcal{J}$-invariant horizontal distribution $\mathcal{H}_Z$, see Definition 11. We express the integrability of $\mathcal{J}$ and the (para-)holomorphicity of $\mathcal{H}_Z$ as equations for the curvature and torsion of ω, which generalize the self-duality equation for the Weyl curvature of a pseudo-Riemannian metric of signature $(2, 2)$, see Theorem 1.

A *para-quaternionic structure* on a vector space V is a Lie subalgebra $Q \subset \mathrm{End}\, V$ which admits a basis (J_1, J_2, J_3) such that $J_3 = J_1 J_2$ and $J_\alpha^2 = \epsilon_\alpha \mathrm{Id}$, where $(\epsilon_1, \epsilon_2, \epsilon_3) = (-1, 1, 1)$. A pseudo-Riemannian manifold (M, g) of dimension > 4 endowed with a parallel field $M \ni p \mapsto Q_p \subset \mathrm{End}\, T_pM$ of g-skew-symmetric para-quaternionic structures is called a *para-quaternionic Kähler manifold*. The metric g has signature $(2n, 2n)$ and is Einstein [1]. Moreover, para-quaternionic Kähler manifolds are related to certain supersymmetric field theories on space-times with a positive definite rather than a Lorentzian metric [11].

For a para-quaternionic Kähler manifold (M, g, Q), Blair et al. [6, 7] have defined two twistor spaces $Z^\epsilon := \{A \in Q \mid A^2 = \epsilon\}$, $\epsilon = \pm 1$, and endowed them with an integrable structure $\mathcal{J} \subset \mathrm{End}\, TZ^\epsilon$ such that $\mathcal{J}^2 = \epsilon \mathrm{Id}$. We recover these results by considering the twistor space associated to the underlying G-structure, which is of twistor type, as well as of para-twistor type. More precisely, we consider

$$J \in \mathfrak{sl}(2, \mathbb{R}) \subset \mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sp}(\mathbb{R}^{2n}) \subset \mathfrak{gl}(\mathbb{R}^2 \otimes \mathbb{R}^{2n}) = \mathfrak{gl}(4n, \mathbb{R}).$$

Under the assumption that the scalar curvature of g is non-zero, we prove, in addition, that the horizontal distribution $\mathcal{H}_Z$ defines a holomorphic (respectively, para-holomorphic) contact structure on Z and that $(Z^\epsilon, \mathcal{J})$ admits a Kähler-Einstein (respectively, para-Kähler-Einstein) metric and determine all Einstein metrics in the canonical one-parameter family of pseudo-Riemannian metrics, see Theorem 3. It turns out that there is always a second Einstein metric.

Finally, we generalize the twistor construction of Kähler submanifolds of a quaternionic Kähler manifold (see [3]) to the case of Kähler and para-Kähler submanifolds (see Definition 14) of a para-quaternionic Kähler manifold (M, g, Q). We prove that any Kähler or para-Kähler submanifold of a para-quaternionic Kähler manifold (M, g, Q) is minimal (Corollary 7). All such submanifolds can be obtained as projections of complex ($\epsilon = -1$), respectively, para-complex ($\epsilon = +1$) submanifolds of Z^ϵ which are tangent to the contact distribution, see Theorem 4. It follows that the maximal dimension of a Kähler or para-Kähler submanifold of (M, g, Q) is $(1/2) \dim M$ and that maximal Kähler (respectively, para-Kähler) submanifolds of (M, g, Q) correspond to Legendrian submanifolds of the complex (respectively, para-complex) contact manifold $(Z^\epsilon, \mathcal{H}_Z)$.

2. (Almost) para-complex manifolds

2.1. Integrability of an almost para-complex structure.

DEFINITION 1. An (*almost*) *para-complex structure*, *in the weak sense*, on a differentiable manifold M is a field of endomorphisms $J \in \operatorname{End} TM$ such that $J^2 = \mathrm{Id}$. J is called *non-trivial* if $J \neq \pm\mathrm{Id}$. We say that J is an (*almost*) *para-complex structure*, *in the strong sense*, if the ± 1-eigenspace distributions $T^{\pm}M$ of J have the same rank. An almost para-complex structure is called *integrable*, or *para-complex structure* if the distributions $T^{\pm}M$ are integrable, or, equivalently, the Nijenhuis tensor N_J, defined by

$$N_J(X, Y) = [X, Y] + [JX, JY] - J[JX, Y] - J[X, JY], \quad X, Y \in TM, \tag{2.1}$$

vanishes. An (*almost*) *para-complex manifold* (M, J) is a manifold M endowed with an (almost) para-complex structure.

Unless otherwise stated, by an (almost) para-complex structure we shall understand here an (almost) para-complex structure in the weak sense.

REMARK. The difference between weak and strong (almost) para-complex manifolds is that $T_p^{1,0}M = \{X + eJX \mid X \in T_pM\} \subset T_pM \otimes C$, $p \in M$, is a free module over the ring $C := \mathbb{R}[e]$, $e^2 = 1$, of para-complex numbers only in the strong case. In particular, for weak para-complex manifolds, there is no notion of para-holomorphic local coordinates (z^i) on M such that the (dz^i) form a basis of $T_p^{1,0}M$ over C.

Let (V, J) and (U, J_U) be vector spaces endowed with constant para-complex structures. We can decompose the vector space $C^2(U) := U \otimes \bigwedge^2 V^*$ of U-valued two-forms on V according to type

$$C^2(U) = \sum_{p+q=2} C^{p,q}(U), \tag{2.2}$$

where

$$\alpha \in C^{1,1}(U) \quad \text{if} \quad \alpha(JX, JY) = -\alpha(X, Y) \quad \text{for all} \quad X, Y \in V,$$
$$\alpha \in C^{2,0}(U) \quad \text{if} \quad \alpha(JX, Y) = \alpha(X, JY) = J_U\alpha(X, Y) \quad \text{for all} \quad X, Y \in V$$

and

$$\alpha \in C^{0,2}(U) \quad \text{if} \quad \alpha(JX, Y) = \alpha(X, JY) = -J_U\alpha(X, Y) \quad \text{for all} \quad X, Y \in V.$$

Lemma 1. *The projections* $\pi^{p,q}: C^2(U) \to C^{p,q}(U)$, $\alpha \to \alpha^{p,q}$, *are given by*:

$$\alpha^{1,1}(X, Y) = \frac{1}{2}(\alpha(X, Y) - \alpha(JX, JY)),$$

$$\alpha^{2,0}(X, Y) = \frac{1}{4}(\alpha(X, Y) + \alpha(JX, JY) + J_U\alpha(JX, Y) + J_U\alpha(X, JY)),$$

$$\alpha^{0,2}(X, Y) = \frac{1}{4}(\alpha(X, Y) + \alpha(JX, JY) - J_U\alpha(JX, Y) - J_U\alpha(X, JY)).$$

For scalar valued forms ($U = \mathbb{R}$) we will always assume that $J_U = \mathrm{Id}$.

Let J be an almost para-complex structure on a manifold M and ∇ a linear connection which preserves J. The following lemma shows that J is integrable if and only if the (0, 2) component $T^{0,2} = \pi^{0,2}T$ vanishes.

Proposition 1. *Let* ∇ *be a connection which preserves an almost para-complex structure* J *on a manifold* M. *Then the Nijenhuis tensor of* J *is given by* $N_J = -4T^{0,2}$. *In particular,* J *is integrable if and only if* $T^{0,2} = 0$.

Proof. Applying Lemma 1 in the case $U = V = T_pM$, $p \in M$, we have

$$T^{0,2}(X, Y) = \frac{1}{4}(T(X, Y) + T(JX, JY) - JT(JX, Y) - JT(X, JY)), \quad X, Y \in TM.$$

Replacing $T(X, Y)$ by $\nabla_X Y - \nabla_Y X - [X, Y]$ in this formula, we get

$$T^{0,2}(X, Y) = -\frac{1}{4}([X, Y] + [JX, JY] - J[JX, Y] - J[X, JY]) = -\frac{1}{4}N_J(X, Y). \quad \square$$

2.2. Holomorphicity of distributions in almost para-complex manifolds.

DEFINITION 2. Let (M, J) be an almost para-complex manifold of real dimension n. A J-invariant distribution $\mathcal{D} = \mathcal{D}_+ \oplus \mathcal{D}_- \subset T^+M \oplus T^-M = TM$ of rank m is called *para-holomorphic* if it is locally defined by equations $\alpha_+^1 = \cdots = \alpha_+^{k_+} = \alpha_-^1 = \cdots = \alpha_-^{k_-} = 0$, such that $k_+ + k_- = n - m$,

$$\alpha_\pm^i \circ J = \pm\alpha_\pm^i \tag{2.3}$$

and the (1, 1)-component

$$\pi^{1,1}\, d\alpha_+^i = \frac{1}{2}(d\alpha_+^i - J^*\, d\alpha_+^i)$$

vanishes on $\bigwedge^2(\mathcal{D}_+ \oplus T^-M)$ and the (1, 1)-component

$$\pi^{1,1}\, d\alpha_-^i = \frac{1}{2}(d\alpha_-^i - J^*\, d\alpha_-^i)$$

vanishes on $\bigwedge^2(T^+M \oplus \mathcal{D}_-)$.

Let (M, J) be an almost para-complex manifold of real dimension n endowed with a J-invariant distribution $\mathcal{D} \subset TM$ of rank m and a connection ∇ which preserves J and $\mathcal{D}$. Then we can define a two-form with values in $TM/\mathcal{D}$ by

$$S(X, Y) := T(X, Y) \mod \mathcal{D}.$$

Since J induces a para-complex structure on the vector bundle $TM/\mathcal{D}$, we can decompose

$$S = S^{2,0} + S^{1,1} + S^{0,2},$$

see Lemma 1.

Proposition 2. *Let (M, J) be an almost para-complex manifold. A J-invariant distribution $\mathcal{D} = \mathcal{D}_+ \oplus \mathcal{D}_- \subset T^+M \oplus T^-M = TM$ is para-holomorphic if and only if*

$$[\Gamma(\mathcal{D}_\pm), \Gamma(T^\mp M)] \subset \Gamma(T^\mp M \oplus \mathcal{D}_\pm). \tag{2.4}$$

Moreover, if ∇ is a connection which preserves J and $\mathcal{D}$, then (2.4) *is equivalent to*

$$S^{1,1}(JX, \cdot) = -JS^{1,1}(X, \cdot), \tag{2.5}$$

for all $X \in \mathcal{D}$.

Proof. First we prove that (2.5) is equivalent to the para-holomorphicity of $\mathcal{D}$. Let $\mathcal{D}$ be a para-holomorphic distribution defined by one-forms $\alpha^i_\pm$ as in Definition 2. The condition on $\pi^{1,1}\, d\alpha^i_\pm$ is equivalent to

$$d\alpha^i_+(X_+, Y_-) = 0, \quad X_+ \in \mathcal{D}_+, \quad Y_- \in T^-M,$$
$$d\alpha^i_-(X_+, Y_-) = 0, \quad X_+ \in T^+M, \quad Y_- \in \mathcal{D}_-.$$

Expressing the exterior derivative in terms of the covariant derivative and torsion we get

$$0 = d\alpha^i_+(X_+, Y_-) = (\nabla_{X_+}\alpha^i_+)Y_- - (\nabla_{Y_-}\alpha^i_+)X_+ + \alpha^i_+(T(X_+, Y_-)).$$

The first two terms on the right-hand side vanish. In fact, since ∇ preserves the distribution $\mathcal{D}$, the covariant derivative $\nabla_X\alpha^i_+$ vanishes on $\mathcal{D}_+ \oplus T^-M$ for all $X \in TM$. The last term can be written as

$$0 = \alpha^i_+(T(X_+, Y_-)) = \alpha^i_+(T^{1,1}(X_+, Y_-)),$$

which implies that $T^{1,1}(X_+, Y_-) \in \mathcal{D}_+ \oplus T^-M$ for all $X_+ \in \mathcal{D}_+$ and $Y_- \in T^-M$. A similar calculation for α_-^i shows that $T^{1,1}(X_+, Y_-) \in T^+M \oplus \mathcal{D}_-$ for all $X_+ \in T^+M$ and $Y_- \in \mathcal{D}_-$. This proves that

$$S^{1,1}(\mathcal{D}_+, T^-M) \subset (T^-M + \mathcal{D})/\mathcal{D},$$
$$S^{1,1}(\mathcal{D}_-, T^+M) \subset (T^+M + \mathcal{D})/\mathcal{D}.$$

In particular, $S^{1,1}(\mathcal{D}, \mathcal{D}) = 0$ and $S^{1,1}(JX, \cdot) = -JS^{1,1}(X, \cdot)$ for all $X \in \mathcal{D}$.

To prove the converse, we assume that the torsion of ∇ satisfies (2.5). Let $\left(\alpha_+^1, \dots, \alpha_+^{k_+}\right)$ and $\left(\alpha_-^1, \dots, \alpha_-^{k_-}\right)$ be local frames of $(\mathcal{D}_+ \oplus T^-M)^\perp$ and $(T^+M \oplus \mathcal{D}_-)^\perp \subset T^*M$, respectively. This implies (2.3). Since $\pi^{1,1}\alpha(T^\pm M, T^\pm M) = 0$ for any two-form α, it is sufficient to check that $\pi^{1,1}\, d\alpha_+^i(\mathcal{D}_+, T^-M) = \pi^{1,1}\, d\alpha_-^i(T^+M, \mathcal{D}_-) = 0$. We calculate for $X_+ \in \mathcal{D}_+$ and $Y_- \in T^-M$:

$$\begin{aligned}\pi^{1,1} d\alpha_+^i(X_+, Y_-) &= d\alpha_+^i(X_+, Y_-) = (\nabla_{X_+}\alpha_+^i)Y_- - (\nabla_{Y_-}\alpha_+^i)X_+ + \alpha_+^i(T(X_+, Y_-))\\ &= \alpha_+^i(T(X_+, Y_-)) = \alpha_+^i(T^{1,1}(X_+, Y_-)) = \alpha_+^i(S^{1,1}(X_+, Y_-))\\ &= \alpha_+^i(S^{1,1}(JX_+, Y_-)) \overset{(2.5)}{=} -\alpha_+^i(JS^{1,1}(X_+, Y_-)) = -\alpha_+^i(S^{1,1}(X_+, Y_-)).\end{aligned}$$

Therefore, $\pi^{1,1} d\alpha_+^i(X_+, Y_-) = 0$. A similar calculation shows that $\pi^{1,1} d\alpha_-^i(\mathcal{D}_-, T^+M) = 0$.

Now we prove the equivalence of (2.4) and (2.5). The condition (2.5) can be written as

$$T(\mathcal{D}_\pm, T^\mp M) \subset T^\mp M \oplus \mathcal{D}_\pm.$$

Using that ∇ preserves the distributions $\mathcal{D}_\pm$ and $T^\pm M$, we calculate for $X_\pm \in \Gamma(\mathcal{D}_\pm)$ and $Y_\pm \in \Gamma(T^\pm M)$

$$\begin{aligned}T^\mp M \oplus \mathcal{D}_\pm \ni T(X_\pm, Y_\mp) &= \nabla_{X_\pm} Y_\mp - \nabla_{Y_\mp} X_\pm - [X_\pm, Y_\mp]\\ &\equiv -[X_\pm, Y_\mp] \mod T^\mp M \oplus \mathcal{D}_\pm.\end{aligned}$$

This proves the equivalence of (2.4) and (2.5). □

Let (M, J) be a para-complex manifold in the strong sense, i.e. the integrable eigendistributions $T^\pm M$ are of the same rank. Recall [10] that a C-valued one-form $\gamma = \alpha + e\beta$ is of *para-complex type* $(1, \underline{0})$, i.e. $J^*\gamma = e\gamma$, if and only if $\beta = \alpha \circ J$. A $(1, 0)$-form γ is *para-holomorphic* if $\bar\partial\gamma := \pi^{1,1}\, d\gamma = 0$, which is equivalent to the para-Cauchy-Riemann equations

$$\partial_-\alpha_+ := \pi^{1,1} d\alpha_+ = \partial_+\alpha_- := \pi^{1,1} d\alpha_- = 0, \tag{2.6}$$

where $\alpha = \alpha_+ + \alpha_-$ is the J-eigenspace decomposition of α.

Proposition 3. *Let (M, J) be a para-complex manifold in the strong sense with eigendistributions $T^{\pm}M$ of rank n and $\mathcal{D} = \mathcal{D}_+ \oplus \mathcal{D}_- \subset T^+M \oplus T^-M = TM$ a J-invariant distribution such that $\mathcal{D}_\pm$ are of the same rank m. Then $\mathcal{D}$ is para-holomorphic if and only if it is locally defined by equations $\gamma^i = 0$ $(i = 1, \dots, k = n - m)$, where the γ^i are para-holomorphic one-forms.*

Proof. Let $\mathcal{D}$ be defined by para-holomorphic one-forms $\gamma^i = \alpha^i_+ + \alpha^i_- + e(\alpha^i_+ - \alpha^i_-)$. The $\alpha^i_\pm$ satisfy (2.6), which imply the equations in the Definition 2.

To prove the converse, we now assume that the distribution $\mathcal{D}$ is para-holomorphic. Thanks to Proposition 2, this means that

$$[\Gamma(\mathcal{D}_\pm), \Gamma(T^{\mp}M)] \subset \Gamma(T^{\mp}M \oplus \mathcal{D}_\pm).$$

In order to construct para-holomorphic one-forms $\gamma^i = \alpha^i_+ + \alpha^i_- + e(\alpha^i_+ - \alpha^i_-)$ which define $\mathcal{D}$, we choose locally linearly independent commuting vector fields $Y_i^\pm \in \Gamma(T^\pm M)$ which generate distributions $N^\pm \subset T^\pm M$ complementary to $\mathcal{D}_\pm$. We define one-forms $\alpha^i_\pm$ vanishing on $\mathcal{D}_\pm \oplus T^\mp M$ by

$$\alpha^i_\pm(Y_j^\pm) = \delta^i_j.$$

It is clear that $\alpha^i_\pm \circ J = \pm\alpha^i_\pm$ and that $\gamma^i := \alpha^i_+ + \alpha^i_- + e(\alpha^i_+ - \alpha^i_-)$ define $\mathcal{D}$. Now we check that the γ^i are para-holomorphic, i.e. $\partial_-\alpha^i_+ = \partial_+\alpha^i_- = 0$. It is sufficient to evaluate this equality on (Z^+, Z^-), where $Z^\pm = X^\pm \in \Gamma(\mathcal{D}_\pm)$ or $Z^\pm = Y_i^\pm$.

$$\partial_-\alpha^i_+(X^+, X^-) = X^+\alpha^i_+(X^-) - X^-\alpha^i_+(X^+) - \alpha^i_+([X^+, X^-]) = 0,$$

since α^i_+ vanishes on $\mathcal{D}_+ \oplus T^-M$ and $[X^+, X^-] \in T^-M \oplus \mathcal{D}_+$ by (2.4). Similarly,

$$\partial_-\alpha^i_+(X^+, Y_j^-) = X^+\alpha^i_+(Y_j^-) - Y_j^-\alpha^i_+(X^+) - \alpha^i_+([X^+, Y_j^-]) = 0.$$

Finally,

$$\partial_-\alpha^i_+(Y_j^+, Y_k^-) = Y_j^+\alpha^i_+(Y_k^-) - Y_k^-\alpha^i_+(Y_j^+) - \alpha^i_+([Y_j^+, Y_k^-]) = 0 - Y_k^-(\delta^i_j) - 0 = 0,$$

since, by construction, $[Y_j^+, Y_k^-] = 0$. Similarly, one can check that $\partial_+\alpha^i_- = 0$. □

3. Para-quaternionic manifolds and para-quaternionic Kähler manifolds

DEFINITION 3. Let $(\epsilon_1, \epsilon_2, \epsilon_3) = (-1, 1, 1)$, or a permutation thereof. An *almost para-quaternionic structure* on a differentiable manifold M (of dimension $4n$) is a rank 3 subbundle $Q \subset \mathrm{End}\, TM$, which is locally generated by three anticommuting fields of endomorphisms $J_1, J_2, J_3 = J_1J_2$, such that $J_\alpha^2 = \epsilon_\alpha \mathrm{Id}$. Such a triple (J_α) will be called a *standard local basis* of Q. A linear connection which preserves Q is called an *almost para-quaternionic connection*. An almost para-quaternionic structure Q is called a *para-quaternionic structure* if M admits a *para-quaternionic connection*, i.e. a torsion-free connection which preserves Q. An (*almost*) *para-quaternionic manifold* is a manifold endowed with an (almost) para-quaternionic structure.

An *almost para-quaternionic Hermitian manifold* (M, g, Q) is a pseudo-Riemannian manifold (M, g) endowed with a para-quaternionic structure Q consisting of skew-symmetric endomorphisms. (M, g, Q), $n > 1$, is called a *para-quaternionic Kähler manifold* if the Levi-Civita connection preserves Q.

Proposition 4 ([1]). *At any point, the curvature tensor R of a para-quaternionic Kähler manifold (M, g, Q) of dimension $4n > 4$ admits a decomposition*

$$R = \nu R_0 + W, \tag{3.1}$$

where $\nu = \mathrm{scal}/(4n(n+2))$ *is the reduced scalar curvature,*

$$R_0(X, Y) := +\frac{1}{2}\sum_\alpha \epsilon_\alpha g(J_\alpha X, Y)J_\alpha + \frac{1}{4}\left(X \wedge Y - \sum_\alpha \epsilon_\alpha J_\alpha X \wedge J_\alpha Y\right), \quad X, Y \in T_pM,$$

is the curvature tensor of the para-quaternionic projective space of the same dimension as M and W is a trace-free Q-invariant algebraic curvature tensor, where Q acts by derivations. In particular, R is Q-invariant.

We define a *para-quaternionic Kähler manifold of dimension* 4 as a pseudo-Riemannian manifold endowed with a parallel skew-symmetric para-quaternionic structure whose curvature tensor admits a decomposition (3.1).

Since the Levi-Civita connection ∇ of a para-quaternionic Kähler manifold preserves the para-quaternionic structure Q, we can write

$$\nabla J_\alpha = -\epsilon_\beta \omega_\gamma \otimes J_\beta + \epsilon_\gamma \omega_\beta \otimes J_\gamma, \tag{3.2}$$

where (α, β, γ) is a cyclic permutation of $(1, 2, 3)$. We shall denote by $\rho_\alpha := g(J_\alpha \cdot, \cdot)$ the *fundamental form* associated with J_α and put $\rho'_\alpha := -\epsilon_\alpha \rho_\alpha$.

Proposition 5. *The locally defined fundamental forms satisfy the following structure equations*

$$\nu\rho'_\alpha := -\epsilon_\alpha \nu \rho_\alpha = \epsilon_3(d\omega_\alpha - \epsilon_\alpha \omega_\beta \wedge \omega_\gamma), \tag{3.3}$$

where (α, β, γ) is a cyclic permutation of $(1, 2, 3)$.

Proof. Using Proposition 4 and the fact that

$$[J_\alpha, J_\beta] = 2\epsilon_3\epsilon_\gamma J_\gamma,$$

we calculate the action of the curvature operator $R(X, Y)$, $X, Y \in TM$, on J_α:

$$[R(X, Y), J_\alpha] = [\nu R_0(X, Y), J_\alpha] = -\frac{\nu}{2}\sum_{\delta=1}^{3}\rho'_\delta(X, Y)[J_\delta, J_\alpha]$$
$$= \epsilon_3\nu(-\epsilon_\beta\rho'_\gamma(X, Y)J_\beta + \epsilon_\gamma\rho'_\beta(X, Y)J_\gamma),$$

where (α, β, γ) is a cyclic permutation of $(1, 2, 3)$. On the other hand, using the equation (3.2), we calculate

$$\begin{aligned}[R(X, Y), J_\alpha] &= [\nabla_X, \nabla_Y]J_\alpha - \nabla_{[X,Y]}J_\alpha \\ &= \nabla_X(-\epsilon_\beta\omega_\gamma(Y)J_\beta + \epsilon_\gamma\omega_\beta(Y)J_\gamma) - \nabla_Y(-\epsilon_\beta\omega_\gamma(X)J_\beta + \epsilon_\gamma\omega_\beta(X)J_\gamma) \\ &\quad - (-\epsilon_\beta\omega_\gamma([X, Y])J_\beta + \epsilon_\gamma\omega_\beta([X, Y])J_\gamma) \\ &= -\epsilon_\beta\, d\omega_\gamma(X, Y)J_\beta + \epsilon_\gamma\, d\omega_\beta(X, Y)J_\gamma - \epsilon_\beta\omega_\gamma(Y)\nabla_X J_\beta + \epsilon_\gamma\omega_\beta(Y)\nabla_X J_\gamma \\ &\quad + \epsilon_\beta\omega_\gamma(X)\nabla_Y J_\beta - \epsilon_\gamma\omega_\beta(X)\nabla_Y J_\gamma.\end{aligned}$$

Applying again the equation (3.2), we finally get

$$[R(X, Y), J_\alpha] = -\epsilon_\beta(d\omega_\gamma - \epsilon_\gamma\omega_\alpha\wedge\omega_\beta)(X, Y)J_\beta + \epsilon_\gamma(d\omega_\beta - \epsilon_\beta\omega_\gamma\wedge\omega_\alpha)(X, Y)J_\gamma.$$

Comparing the two formulas for $[R(X, Y), J_\alpha]$ we obtain the structure equations. □

4. The twistor spaces of a para-quaternionic or para-quaternionic Kähler manifold

4.1. The twistor spaces of a para-quaternionic manifold. In the following, it will be useful to unify complex and para-complex structures in the following definition.

DEFINITION 4. An *almost ϵ-complex structure*, $\epsilon \in \{-1, 0, 1\}$, on a differentiable manifold M of dimension $2n$ is a field of endomorphisms $J \in \mathrm{End}\,TM$ such that $J^2 = \epsilon\mathrm{Id}$ and, moreover, for $\epsilon = +1$ the eigendistributions $T^{\pm}M$ are of rank n and for $\epsilon = 0$ the two distributions $\ker J$ and $\mathrm{im}\,J$ have rank n. In other words, an almost -1-complex structure is an almost complex structure and an almost $+1$-complex structure is an almost para-complex structure in the strong sense.

An *ϵ-complex manifold* is a differentiable manifold endowed with an integrable (i.e. $N_J = 0$) ϵ-complex structure J.

We shall also use the unifying adjective *ϵ-holomorphic* as a synonym of 'holomorphic' or 'para-holomorphic', depending on whether $\epsilon = -1$ or $\epsilon = +1$, respectively.

Let (M, Q) be an almost para-quaternionic manifold. We associate with (M, Q) a family of bundles $\pi\colon Z^s \to M$, with two-dimensional fibres, depending on a parameter $s \in \mathbb{R}$ as follows:

$$Z^s := \{A \in Q \mid A \neq 0,\ A^2 = s\}.$$

DEFINITION 5. The fibre bundle $\pi\colon Z^s \to M$ is called the *s-twistor space* of the almost para-quaternionic manifold (M, Q).

Proposition 6. *Any almost para-quaternionic connection ∇ on an almost para-quaternionic manifold (M, Q) induces a canonical almost ϵ-complex structure $\mathcal{J}^s = \mathcal{J}^s_\nabla$ on the s-twistor space Z^s, where $\epsilon = \mathrm{sgn}(s) \in \{-1, 0, 1\}$.*

Proof. Let (I, J, K) be a standard basis of Q_m. Then any element $A \in Q_m$ can be written as $A = xI + yJ + zK$ and $A \in Z^s$ if and only if $-x^2 + y^2 + z^2 = s$. Hence, the fibres of Z^s are two-sheeted hyperboloids for $s < 0$, one-sheeted hyperboloids for $s > 0$ and light-cones without origin for $s = 0$. Each fibre $Z^s_m = \pi^{-1}(m)$ is a homogeneous space of the group $\mathrm{SO}(1, 2)$ with one-dimensional stabilizer $\mathrm{SO}(1, 2)_{A_s} = \mathrm{SO}(2)$ if $s < 0$, $\mathrm{SO}(1, 2)_{A_s} = \mathrm{SO}(1, 1)$ if $s > 0$ and $\mathrm{SO}(1, 2)_{A_s} \cong (\mathbb{R}, +)$ if $s = 0$, where $A_s \in Z^s$. First we define the canonical $\mathrm{SO}(1, 2)$-invariant ϵ-complex structure on Z^s_m, as follows. The three-dimensional vector space $Q_m \subset \mathrm{End}\, T_m M$ is a Lie subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. The adjoint action preserves the indefinite scalar product $\langle A, B\rangle = -(1/(4n))\,\mathrm{tr}(AB)$, $4n = \dim M$, in Q and hence identifies the Lie algebra Q with $\mathfrak{so}(Q) = \mathrm{Lie}\,\mathrm{SO}(Q) \cong \mathfrak{so}(1, 2)$. Let $A \in Z^s_m \subset Q_m$. Then $Z^s_m = \mathrm{SO}(Q)A$ and the tangent space to Z^s_m at A is identified with $\mathfrak{so}(Q)A \cong \mathfrak{so}(Q)/\mathfrak{so}(Q)_A = \mathfrak{so}(Q)/\mathbb{R}A$. It is easy to check that the adjoint action of $(1/2)A$ on $\mathfrak{so}(Q)/\mathbb{R}A$ defines an $\mathrm{SO}(Q)$-invariant ϵ-complex structure J^v on Z^s_m. Now we define an almost ϵ-complex structure $\mathcal{J}^s$ on the twistor space Z^s. We have the decomposition

$$T_z Z^s = T^v_z Z^s + H_z \cong T_z(Z^s_m) \oplus T_{\pi z} M, \tag{4.1}$$

where $T^v_z Z^s$ is the vertical space of the bundle $\pi : Z^s \to M$ and H_z is the horizontal space of the connection in the bundle π induced by the para-quaternionic connection ∇ of (M, Q). The latter is identified with $T_{\pi z} M$ via the projection $Z^s \to M$. We denote by J^z the tautological ϵ-complex structure on $T_{\pi z} M$ defined by $z \in Z^s$. With respect to the above decomposition we define

$$\mathcal{J}^s_z = J^v \oplus J^z \tag{4.2}$$

By construction, $\mathcal{J}^s$ is an almost ϵ-complex structure. □

4.2. The twistor spaces of a para-quaternionic Kähler manifold. Let (M, g, Q) be a para-quaternionic Kähler manifold with twistor spaces Z^s. The Levi-Civita connection $\nabla = \nabla^g$ is a para-quaternionic connection and, hence, induces a canonical almost ϵ-complex structure $\mathcal{J}^s = \mathcal{J}^s_\nabla$ on Z^s.

Proposition 7. *The twistor space Z^s of a para-quaternionic Kähler manifold (M, g, Q) admits a canonical almost ϵ-complex structure $\mathcal{J}^\epsilon$, where $\epsilon = \mathrm{sgn}(s)$, and a one-parameter family g^s_t, $t \in \mathbb{R} - \{0\}$, of pseudo-Riemannian metrics such that the* *almost ϵ-complex structure $\mathcal{J}^s$ is skew-symmetric, provided that $s \neq 0$. For $s = 0$ there exists a canonical one-parameter family g_t^0, $t \in \mathbb{R} - \{0\}$, of symmetric bilinear forms with one-dimensional (vertical) kernel such that $\mathcal{J}^s$ is skew-symmetric. Finally, for $s \neq 0$, the projection $\pi : (Z^s, g_t^s) \to (M, g)$ is a pseudo-Riemannian submersion.*

Proof. We denote by $g^v := \langle \cdot , \cdot \rangle|_{Z_m^s}$ the induced metric on the fibres $Z_m^s \subset (Q, \langle \cdot , \cdot \rangle)$. It is nondegenerate for $s \neq 0$ and has one-dimensional kernel for $s = 0$. The ϵ-complex structure J^v on Z_m^s is g^v-skew-symmetric. With respect to the decomposition (4.1), we define

$$(g_t^s)_z = tg^v \oplus g_{\pi z}.$$

The almost ϵ-complex structure $\mathcal{J}^s$ defined above is skew-symmetric with respect to the field of symmetric bilinear forms g_t^s, which is nondegenerate for $s \neq 0$ and has one-dimensional vertical kernel for $s = 0$. The above formula for g_t^s shows that the decomposition of TZ into vertical and horizontal space is g_t^s-orthogonal and that the projection induces an isometry $H_z \to T_{\pi z}M$. This proves that π is a pseudo-Riemannian submersion. □

The scalar multiplication by $|s|^{1/2} \neq 0$ in the vector bundle $Q \to M$ induces an isometry $(Z^\epsilon, g_t^\epsilon) \to (Z^s, g^s_{t/|s|})$, which preserves the almost ϵ-complex structure, where $\epsilon = \mathrm{sgn}(s)$. This shows that it is sufficient to consider only three of the above twistor spaces, namely $Z^+ := Z^{+1}$, $Z^- := Z^{-1}$ and Z^0. We will study the integrability of the almost ϵ-complex structure $\mathcal{J}^\epsilon$ and the holomorphicity of the horizontal distribution $H \subset TZ^\epsilon$, which is $\mathcal{J}^\epsilon$-invariant. For this we extend the G-structure approach developed in [2] to the para-case ($\epsilon = 1$).

4.3. Twistor spaces of para-quaternionic (Kähler) manifolds as bundles associated to G-structures. In this subsection we interpret the twistor spaces Z^ϵ ($\epsilon = -1, 0, 1$) from the point of view of G-structures.

Let (M, Q) be a para-quaternionic manifold. Note that $\tilde{Q}_m := \mathbb{R}\mathrm{Id} + Q_m \subset \mathrm{End}\, T_m M$ is an algebra isomorphic to the algebra of para-quaternions, i.e. to the matrix algebra $\mathbb{R}(2)$. Since any irreducible module of $\mathbb{R}(2)$ is isomorphic to $\mathbb{R}^2$, the $\tilde{Q}_m$-module $T_m M$ is isomorphic to the $\mathbb{R}(2)$-module $\mathbb{R}^2 \otimes \mathbb{R}^n$, $2n = \dim M$, with the action on the first factor.

DEFINITION 6. Let (M, Q) be an (almost) para-quaternionic manifold. A *para-quaternionic coframe* at $m \in M$ is an isomorphism $\phi \colon T_m M \xrightarrow{\sim} \mathbb{R}^2 \otimes \mathbb{R}^n$ which maps $\tilde{Q}_m$ into $\mathbb{R}(2)$, i.e.

$$\phi \circ \tilde{Q}_m \circ \phi^{-1} = \mathbb{R}(2) \otimes \mathrm{Id}.$$

Proposition 8. (i) *The set P of all para-quaternionic coframes together with the natural projection $\pi^P : P \to M$ is a G-structure, i.e. a principal subbundle of the bundle of all coframes with the structure group $G := \mathrm{SL}_2^{\pm}(\mathbb{R}) \otimes \mathrm{GL}_n(\mathbb{R})$, where*

$$\mathrm{SL}_2^{\pm}(\mathbb{R}) = \{A \in \mathrm{GL}_2(\mathbb{R}) \mid \det A = \pm 1\}.$$

(ii) *Let $A \in \mathfrak{sl}_2(\mathbb{R}) \otimes \mathrm{Id} \subset \mathfrak{g} = \mathrm{Lie}\, G$ such that $A^2 = \epsilon \mathrm{Id}$ and G_A the stabilizer* (*i.e. centralizer*) *of A in G. There is a canonical isomorphism of fibre bundles*

$$P/G_A \xrightarrow{\sim} Z^\epsilon.$$

Proof. (i) It is clear that any two para-quaternionic coframes are related by an element of $\mathrm{GL}_2(\mathbb{R}) \otimes \mathrm{GL}_n(\mathbb{R}) = \mathrm{SL}_2^{\pm}(\mathbb{R}) \otimes \mathrm{GL}_n(\mathbb{R})$.

(ii) Let $\phi \in P$ be a coframe at $m \in M$. It induces an algebra isomorphism $\hat{\phi} : \mathbb{R}(2) \to \tilde{Q}_m$, $B \mapsto \phi^{-1} B \phi$. The image $\hat{\phi}(A) \in Q_m$ satisfies $\hat{\phi}(A)^2 = \epsilon \mathrm{Id}$, hence $\hat{\phi}(A) \in Z_m^\epsilon$. If $k \in G_A$ then $\widehat{k\phi}(A) = \phi^{-1} k^{-1} A k \phi = \hat{\phi}(A)$. So the map $P \to Z^\epsilon$, $\phi \mapsto \hat{\phi}(A)$, factorizes to an isomorphism $P/G_A \to Z^\epsilon$ of fibre bundles. □

Assume now that (M, g, Q) is a para-quaternionic Kähler manifold of dimension $4n$, or more generally an almost para-quaternionic Hermitian manifold. On $\mathbb{R}^2 \otimes \mathbb{R}^{2n}$ we fix the standard scalar product $g_{\mathrm{can}} = \omega_{\mathbb{R}^2} \otimes \omega_{\mathbb{R}^{2n}}$, where $\omega_{\mathbb{R}^{2n}}$ denotes the standard symplectic structure of $\mathbb{R}^{2n}$.

DEFINITION 7. Let (M, g, Q) be an almost para-quaternionic Hermitian manifold of dimension $4n$. A *para-quaternionic Hermitian coframe* at $m \in M$ is a linear isometry $\phi : (T_m M, g_m) \xrightarrow{\sim} (\mathbb{R}^2 \otimes \mathbb{R}^{2n}, g_{\mathrm{can}})$ which maps $\tilde{Q}_m$ into $\mathbb{R}(2)$.

Proposition 9. *The set P of all para-quaternionic Hermitian coframes together with the natural projection $\pi^P : P \to M$ is a G-structure with $G = G_0 \cup \xi G_0$, $G_0 :=$ $\mathrm{SL}_2(\mathbb{R}) \otimes \mathrm{Sp}(\mathbb{R}^{2n})$, $\xi = A \otimes B \in \mathrm{SL}_2^{\pm}(\mathbb{R}) \otimes \mathrm{GL}_n(\mathbb{R})$, $\det A = -1$ and $B^* \omega_{\mathbb{R}^{2n}} = -\omega_{\mathbb{R}^{2n}}$. Moreover, the twistor space Z^ϵ is canonically isomorphic to the bundle P/G_A, where $0 \neq A \in \mathfrak{sl}_2(\mathbb{R})$ with $A^2 = \epsilon \mathrm{Id}$.*

5. *G*-structures of para-twistor type and their twistor spaces: obstructions for integrability

5.1. Groups of para-twistor type and para-complex symmetric spaces.

DEFINITION 8. A connected linear Lie group $G \subset \mathrm{GL}(V)$, $V = \mathbb{R}^n$, is called of *para-twistor type* if its Lie algebra contains a para-complex structure, i.e. an element J such that $J^2 = \mathrm{Id}$. (If G is not connected, we shall assume, in addition, that the conjugation by J preserves G.)

Since the endomorphism J is semi-simple, the adjoint operator ad_J is semi-simple and, hence, we have the direct sum $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$, where $\mathfrak{k} = \ker \mathrm{ad}_J = Z_{\mathfrak{g}}(J)$ and $\mathfrak{m} = [J, \mathfrak{g}]$. It follows that

$$\mathfrak{m} = \{A \in \mathfrak{g} \mid \{J, A\} = AJ + JA = 0\}.$$

This implies that $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$ and, hence, that $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ is a symmetric decomposition.

Proposition 10. *The orbit $S := \mathrm{Ad}_G(J) \cong G/K$, $K := Z_G(J)$, is an affine symmetric space and carries a canonical G-invariant para-complex structure J^S.*

Proof. The involutive automorphism $A \mapsto JAJ^{-1} = JAJ$ of G has K as its fixed point set and defines the symmetry of G/K at the point eK.

The formula $J_{\mathfrak{m}} A = JA = (1/2)[J, A]$, $A \in \mathfrak{m}$, defines a K-invariant para-complex structure on $\mathfrak{m}$, which extends to a G-invariant para-complex structure J^S on S. The structure J^S is integrable, since it is parallel under the canonical torsion-free connection of the symmetric space S. □

The projections onto $\mathfrak{k}$ and $\mathfrak{m}$ are given by

$$A \mapsto \frac{1}{2} J\{J, A\} = \frac{1}{2}(A + JAJ), \tag{5.1}$$

$$A \mapsto \frac{1}{2} J[J, A] = \frac{1}{2}(A - JAJ). \tag{5.2}$$

5.2. The space of curvature tensors. Let $G \subset \mathrm{GL}(V)$ be a linear Lie group of para-twistor type with Lie algebra $\mathfrak{g}$, $J \in \mathfrak{g}$ a para-complex structure and $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ the corresponding symmetric decomposition; $\mathfrak{k} = Z_{\mathfrak{g}}(J)$ and $\mathfrak{m} = [J, \mathfrak{g}]$. Recall that $\mathfrak{m}$ carries the para-complex structure $J_{\mathfrak{m}} : A \mapsto JA = (1/2)[J, A]$. For any subspace $U \subset \mathrm{End}\, V$ we denote by

$$\mathcal{R}(U) := \left\{ R \in U \otimes \bigwedge^2 V^* \;\middle|\; R \text{ satisfies the first Bianchi identity} \right\}$$

the vector space of algebraic curvature tensor of type U.

The projection $\pi_{\mathfrak{m}} : \mathfrak{g} \to \mathfrak{m}$ induces a projection

$$\pi_{\mathfrak{m}} : C^2(\mathfrak{g}) \to C^2(\mathfrak{m}).$$

According to (5.2), the projection $\alpha^{\mathfrak{m}} := \pi_{\mathfrak{m}} \alpha \in C^2(\mathfrak{m})$ of $\alpha \in C^2(\mathfrak{g})$ is given by

$$\alpha^{\mathfrak{m}}(X, Y) = \frac{1}{2}(\alpha(X, Y) - J\alpha(X, Y)J). \tag{5.3}$$

Recall that, since $\mathfrak{m} \subset \operatorname{End} V$ is endowed with the para-complex structure $J_{\mathfrak{m}}$, we have the decomposition (2.2)

$$C^2(\mathfrak{m}) = \sum_{p+q=2} C^{p,q}(\mathfrak{m}).$$

We put $\pi_{\mathfrak{m}}^{p,q} := \pi^{p,q} \circ \pi_{\mathfrak{m}} \colon C^2(\mathfrak{g}) \to C^{p,q}(\mathfrak{m})$ and $\mathcal{R}^{p,q}(\mathfrak{m}) := \mathcal{R}(\mathfrak{m}) \cap C^{p,q}(\mathfrak{m})$.

The action of J as an automorphism of the tensor algebra induces involutions

$$T_J \colon C^2(\mathfrak{g}) \to C^2(\mathfrak{g}), \quad T_J \colon C^2(V) \to C^2(V).$$

We denote the ± 1-eigenspaces of T_J on $C^2(\mathfrak{g})$ by $C^2_{\pm}(\mathfrak{g})$, such that

$$C^2(\mathfrak{g}) = C^2_+(\mathfrak{g}) + C^2_-(\mathfrak{g}),$$

and put $C^2_{\pm}(U) := C^2_{\pm}(\mathfrak{g}) \cap C^2(U)$ and $\mathcal{R}_{\pm}(U) := C^2_{\pm}(\mathfrak{g}) \cap \mathcal{R}(U)$, where $U = \mathfrak{k}, \mathfrak{m}$.

Proposition 11. (i) *The eigenspaces of T_J on $C^2(\mathfrak{g})$ are given by*

$$C^2_+(\mathfrak{m}) = C^{1,1}(\mathfrak{m}), \tag{5.4}$$

$$C^2_-(\mathfrak{m}) = C^{2,0}(\mathfrak{m}) + C^{0,2}(\mathfrak{m}). \tag{5.5}$$

(ii) *The action of T_J on $C^{p,q}(V)$ is given by*

$$\begin{aligned} T_J \alpha^{1,1} &= -J\alpha^{1,1}, \\ T_J \alpha^{2,0} &= J\alpha^{2,0}, \\ T_J \alpha^{0,2} &= J\alpha^{0,2}. \end{aligned}$$

In particular,

$$\begin{aligned} C^{1,1}(V) &= \ker(T_J + L_J), \\ C^{2,0}(V) + C^{0,2}(V) &= \ker(T_J - L_J), \end{aligned}$$

where $L_J \alpha = J \circ \alpha$.

The action of J as a derivation on the tensor algebra induces an endomorphism

$$\alpha \mapsto J \cdot \alpha = [J, \alpha] - \alpha(J \cdot, \cdot) - \alpha(\cdot, J \cdot)$$

of $C^2(\mathfrak{g})$. Similarly, J acts as a derivation on $C^2(V)$.

Proposition 12. (i) *The action of J as a derivation on $C^2(\mathfrak{g})$ is given by*

$$J \cdot \alpha^{p,q} = 2qJ\alpha^{p,q} \quad \textit{for all} \quad \alpha^{p,q} \in C^{p,q}(\mathfrak{m}),$$

$$J \cdot \alpha = -2\alpha(J\,\cdot\,,\ \cdot\,) \quad \textit{for all} \quad \alpha \in C^2_+(\mathfrak{k}),$$
$$J \cdot C^2_-(\mathfrak{k}) = 0.$$

In particular, the vector space of J-invariants is given by

$$C^2(\mathfrak{g})^J = C^2_-(\mathfrak{k}) + C^{2,0}(\mathfrak{m}). \tag{5.6}$$

(ii) *The action of J as a derivation on $C^2(V)$ is given by*

$$J \cdot \alpha^{2,0} = -J\alpha^{2,0},$$
$$J \cdot \alpha^{0,2} = 3J\alpha^{0,2},$$
$$J \cdot \alpha^{1,1} = J\alpha^{1,1}.$$

In particular,

$$C^{2,0}(V) = \ker(D_J + L_J),$$
$$C^{0,2}(V) = \ker(D_J - 3L_J),$$
$$C^{1,1}(V) = \ker(D_J - L_J),$$

where $D_J\alpha = J \cdot \alpha$.

The proposition shows that $\pi^{1,1}_{\mathfrak{m}} C^2(\mathfrak{g})^J = \pi^{0,2}_{\mathfrak{m}} C^2(\mathfrak{g})^J = 0$ and $\pi^{2,0}_{\mathfrak{m}} C^2(\mathfrak{g})^J = C^{2,0}(\mathfrak{m})$.

Proposition 13. *The following holds*
(i) $\mathcal{R}(\mathfrak{g}) = \mathcal{R}_+(\mathfrak{g}) + \mathcal{R}_-(\mathfrak{g})$,
(ii) $\mathcal{R}(\mathfrak{m}) = \mathcal{R}_+(\mathfrak{m}) + \mathcal{R}_-(\mathfrak{m})$,
(iii) $\pi_{\mathfrak{m}}\mathcal{R}_+(\mathfrak{g}) = \pi^{1,1}_{\mathfrak{m}}\mathcal{R}_+(\mathfrak{g}) \supset \mathcal{R}_+(\mathfrak{m}) = \mathcal{R}^{1,1}(\mathfrak{m})$,
(iv) $\pi^{0,2}_{\mathfrak{m}}\mathcal{R}(\mathfrak{g}) = \mathcal{R}^{0,2}(\mathfrak{m})$,
(v) $\pi_{\mathfrak{m}}\mathcal{R}_-(\mathfrak{g}) = (\pi^{2,0}_{\mathfrak{m}} + \pi^{0,2}_{\mathfrak{m}})\mathcal{R}_-(\mathfrak{g}) \supset \mathcal{R}_-(\mathfrak{m}) = \mathcal{R}^{2,0}(\mathfrak{m}) + \mathcal{R}^{0,2}(\mathfrak{m})$.

Proof. (i) and (ii) follow from the fact that $T_J\colon C^2(\mathfrak{g}) \to C^2(\mathfrak{g})$ preserves the subspaces $\mathcal{R}(\mathfrak{m}) \subset \mathcal{R}(\mathfrak{g}) \subset C^2(\mathfrak{g})$ and (iii) follows from the equation (5.4). The equation (5.5) and (iv) imply (v). Therefore it suffices to prove (iv). For $R \in \mathcal{R}(\mathfrak{g})$ and $X, Y, Z \in V$ we calculate

$$\begin{aligned}(\pi^{0,2}_{\mathfrak{m}}R)(X,Y) &= \frac{1}{4}(R^{\mathfrak{m}}(X,Y) + R^{\mathfrak{m}}(JX,JY) - JR^{\mathfrak{m}}(JX,Y) - JR^{\mathfrak{m}}(X,JY))\\ &= \frac{1}{8}(R(X,Y) - JR(X,Y)J + R(JX,JY) - JR(JX,JY)J\\ &\quad - JR(JX,Y) + R(JX,Y)J - JR(X,JY) + R(X,JY)J)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{8}(R(X,Y) - JR(X,Y)J - JR(JX,Y) - JR(X,JY)) \\
&\qquad + \frac{1}{8}(-JR(JX,JY)J + R(JX,JY) + R(X,JY)J + R(JX,Y)J) \\
&= \frac{1}{8}(J(J\cdot R)(X,Y) - (J\cdot R)(JX,JY)J)
\end{aligned}
$$

and, therefore,

$$\sum_{\text{cyclic}} (\pi^{0,2}_{\mathfrak{m}} R)(X,Y)Z = \frac{1}{8} J \sum_{\text{cyclic}} (J\cdot R)(X,Y)Z - \frac{1}{8}\sum_{\text{cyclic}} (J\cdot R)(JX,JY)JZ = 0,$$

where the sum is over cyclic permutations of (X, Y, Z). Here we used the fact that $A \cdot \mathcal{R}(\mathfrak{g}) \subset \mathcal{R}(\mathfrak{g})$ for any $A \in \mathfrak{g}$. □

5.3. *G*-structures with connection and associated *K*-structures. Let $G \subset \mathrm{GL}(V)$, $V = \mathbb{R}^n$, be a linear Lie group.

DEFINITION 9. A *G-structure* on a manifold M is G-principal bundle $\pi : P \to M$ endowed with a displacement form θ, i.e. a G-equivariant V-valued one-form such that $\ker\theta = T^v P := \ker d\pi$.

We shall identify a point $p \in P$ with the coframe

$$p : T_{\pi(p)}M \to V, \quad X \mapsto \theta_p\big((d\pi)^{-1}_p(X)\big).$$

DEFINITION 10. A *principal connection* in a G-principal bundle $\pi : P \to M$ is a G-equivariant $\mathfrak{g}$-valued one-form $\omega : TP \to \mathfrak{g}$ such that $H := \ker\omega$ is a distribution transversal to the vertical distribution $T^v P$.

Recall that the wedge product of two one-forms α, β with values in a Lie algebra is the Lie algebra valued two-form given by

$$[\alpha \wedge \beta](X,Y) := [\alpha(X), \beta(Y)] - [\alpha(Y), \beta(X)].$$

The *curvature* of a connection ω is the $\mathfrak{g}$-valued G-equivariant horizontal two-form

$$\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega].$$

If $\pi : P \to M$ is a G-structure with displacement form θ, then the *torsion* of ω is the V-valued G-equivariant horizontal two-form

$$\Theta := d\theta + [\omega \wedge \theta],$$

where the Lie bracket is taken in the affine Lie algebra $V + \mathfrak{g}$.

If θ is the displacement form of a G-structure $\pi : P \to M$ and ω a principal connection then:

$$\kappa = \theta + \omega : TP \to V \oplus \mathfrak{g}$$

is a *Cartan connection,* i.e. a G-equivariant absolute parallelism which extends the canonical vertical parallelism $T^v P \to \mathfrak{g}$. The *curvature* of the Cartan connection κ is defined as the $(V \oplus \mathfrak{g})$-valued G-equivariant horizontal two-form

$$\Omega_\kappa := d\kappa + \frac{1}{2}[\kappa \wedge \kappa].$$

Notice that the V and $\mathfrak{g}$-components of Ω_κ are exactly the torsion and curvature forms of ω:

$$\Omega_\kappa^V = \Theta, \quad \Omega_\kappa^{\mathfrak{g}} = \Omega.$$

Let now $K \subset G$ be a Lie subgroup with Lie algebra $\mathfrak{k}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ a K-invariant direct decomposition of the vector space $\mathfrak{g}$. Accordingly, any $\mathfrak{g}$-valued form α on P is decomposed as

$$\alpha = \alpha^{\mathfrak{k}} + \alpha^{\mathfrak{m}}.$$

Proposition 14 ([2]). *Let $(\pi : P \to M, \theta, \omega)$ be a G-structure with a connection and $K \subset G$ a Lie subgroup. Then*

$$\pi' : P \to Z := P/K$$

is a K-structure with displacement form

$$\theta' := \theta + \omega^{\mathfrak{m}} : TP \to V' := V \oplus \mathfrak{m}$$

and connection

$$\omega' := \omega^{\mathfrak{k}}.$$

The curvature Ω' and torsion Θ' of ω' are given by

$$\Theta' = (\Theta')^V + (\Theta')^{\mathfrak{m}} = (\Theta - [\omega^{\mathfrak{m}} \wedge \theta]) + \Omega^{\mathfrak{m}} - \frac{1}{2}[\omega^{\mathfrak{m}} \wedge \omega^{\mathfrak{m}}]^{\mathfrak{m}},$$
$$\Omega' = \Omega^{\mathfrak{k}} - \frac{1}{2}[\omega^{\mathfrak{m}} \wedge \omega^{\mathfrak{m}}]^{\mathfrak{k}}.$$

5.4. The twistor space of a *G*-structure of para-twistor type. Let $G \subset \mathrm{GL}(V)$ be a linear Lie group of para-twistor type, $J \in \mathfrak{g}$ a para-complex structure and $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ the corresponding symmetric decomposition; $\mathfrak{k} = Z_{\mathfrak{g}}(J)$ and $\mathfrak{m} = [J, \mathfrak{g}]$. Let $\pi\colon P \to M$ be a G-structure endowed with a principal connection $\omega\colon TP \to \mathfrak{g}$. (P, ω) will be called a *G-structure of para-twistor type*. The vector space $V' := V \oplus \mathfrak{m}$ has the para-complex structure $J' = J \oplus J_{\mathfrak{m}}$. The natural action of $K = Z_G(J)$ on V' preserves this structure and is identified with a subgroup $K \subset \mathrm{GL}(V', C) := \mathrm{Aut}(V', J')$. This implies that the K-structure

$$\pi'\colon P \to Z := P/K$$

is subordinated to a $\mathrm{GL}(V', C)$-structure, i.e. to an almost para-complex structure $\mathcal{J}$ on Z. At the point $z = \pi' p \in Z$, $p \in P$, the almost para-complex structure $\mathcal{J}$ is defined by:

$$\mathcal{J}_z = \hat{p}^{-1} \circ J' \circ \hat{p},$$

where $\hat{p}\colon T_z Z \to V'$ is the coframe associated with $p \in P$. It is easily checked that this definition does not depend on $p \in (\pi')^{-1}(z)$.

Similarly, we can associate a para-complex structure $J_z\colon T_{\pi p} M \to T_{\pi p} M$ with any point $z = Kp \in Z$ by the formula

$$J_z := p \circ J \circ p^{-1},$$

using the isomorphism $p\colon T_{\pi p} M \to V$. This allows to identify the G/K-bundle $\pi_Z\colon Z = P/K \to M = P/G$ with a bundle of para-complex structures on the tangent spaces of M.

We denote by $\mathcal{H}_Z = \pi'_* \ker \omega \subset TZ$ the projection of the horizontal distribution of ω to TZ. We call it the *horizontal distribution* of Z.

DEFINITION 11. Let $(\pi : P \to M, \omega)$ be a G-structure of para-twistor type and $K = Z_G(J)$. Then the induced K-structure $\pi'\colon P \to Z = P/K$ endowed with the induced connection $\omega' = \pi_{\mathfrak{k}} \circ \omega$, the horizontal distribution $\mathcal{H}_Z$ and the almost para-complex structure $\mathcal{J}$ is called the *twistor space* associated to the G-structure of para-twistor type (P, ω) and to the para-complex structure $J \in \mathfrak{g}$.

Notice that the almost para-complex structure $\mathcal{J}$ and the horizontal distribution $\mathcal{H}_Z$ are invariant under the parallel transport in TZ defined by the connection ω'. Therefore, we can apply Propositions 1 and 2.

Theorem 1. *Let $(\pi : P \to M, \omega)$ be a G-structure of para-twistor type, where ω is a principal connection with curvature form Ω and torsion form Θ and $(Z, \mathcal{J}, \mathcal{H}_Z)$ the corresponding twistor space. Then*

(i) *The almost para-complex structure* $\mathcal{J}$ *on* Z *is integrable if and only if*

$$\pi^{0,2} \circ \Theta = 0 \quad \textit{and} \quad \pi^{0,2}_{\mathfrak{m}} \circ \Omega = 0, \tag{5.7}$$

(ii) *The horizontal distribution* $\mathcal{H}_Z \subset TZ$ *is para-holomorphic if and only if*

$$\pi^{1,1}_{\mathfrak{m}} \circ \Omega = 0,$$

where we consider the values of the horizontal forms Θ *and* Ω *at* $p \in P$ *as*

$$\Theta_p \colon \bigwedge^2 T_{\pi' p} Z \to V \quad \textit{and} \quad \Omega_p \colon \bigwedge^2 T_{\pi' p} Z \to \mathfrak{g}.$$

Proof. Since G is of para-twistor type, $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ is a symmetric decomposition and, in particular, $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$. By Proposition 14, the torsion of the connection ω' in the K-principal bundle $\pi' \colon P \to Z$ is given by

$$\Theta' = (\Theta')^V + (\Theta')^{\mathfrak{m}} = (\Theta - [\omega^{\mathfrak{m}} \wedge \theta]) + \Omega^{\mathfrak{m}}.$$

The second term $[\omega^{\mathfrak{m}} \wedge \theta]_p \colon \bigwedge^2 T_{\pi' p} Z \to V' = V \oplus \mathfrak{m}$, $p \in P$, on the right-hand side is of type (2, 0) since

$$\theta' = \theta + \omega^{\mathfrak{m}} \colon T_{\pi' p} Z \to V'$$

is of type (1, 0):

$$\theta' \circ \mathcal{J}_{\pi' p} = (J \oplus J_{\mathfrak{m}}) \circ \theta'.$$

Therefore the integrability condition $\pi^{0,2}\Theta' = 0$ of Proposition 1 reduces to (5.7).

To prove (ii), we notice that the coframe $\hat{p} \colon T_{\pi' p} Z \to V' = V \oplus \mathfrak{m}$ maps the horizontal space $(\mathcal{H}_Z)_{\pi' p}$ to V. Therefore the tensor

$$S = T \mod \mathcal{H}_Z$$

corresponds to $(\Theta')^{\mathfrak{m}} = \Omega^{\mathfrak{m}}$ and $S^{1,1}$ corresponds to $\pi^{1,1}_{\mathfrak{m}} \circ \Omega$. The two-form $\Omega^{\mathfrak{m}}$ on P vanishes on the vertical distribution $T^v P = \kappa^{-1}(\mathfrak{g})$. This implies that $\pi^{1,1}_{\mathfrak{m}} \circ \Omega$ vanishes on $\hat{p}^{-1}(\mathfrak{m})$. Therefore the para-holomorphicity condition (2.5) of Proposition 2 reduces to $\pi^{1,1}_{\mathfrak{m}} \circ \Omega|_{\mathcal{H}_Z \times \mathcal{H}_Z} = 0$, which is equivalent to $\pi^{1,1}_{\mathfrak{m}} \circ \Omega = 0$. □

Since any $p \in P$ is an isomorphism $p \colon T_{\pi p} M \to V$ we can identify the horizontal two-forms Θ and Ω with G-equivariant functions

$$T \colon P \to \bigwedge^2 V^* \otimes V \quad \text{and} \quad R \colon P \to \bigwedge^2 V^* \otimes \mathfrak{g}.$$

In particular, $T + \pi_{\mathfrak{m}} \circ R \colon P \to \bigwedge^2 V^* \otimes V' = C^2(V') = \oplus C^{p,q}(V')$. Now we can reformulate the theorem in terms of T and $R_{\mathfrak{m}} := \pi_{\mathfrak{m}} \circ R$.

Corollary 1. *Under the assumptions of the previous theorem, the following is true.*
(i) *The almost para-complex structure is integrable if and only if T and $R_{\mathfrak{m}}$ take values in $C^{2,0}(V') \oplus C^{1,1}(V')$.*
(ii) *The horizontal distribution is para-holomorphic if and only if $R_{\mathfrak{m}}$ takes values in $C_{-}(\mathfrak{m}) = C^{2,0}(\mathfrak{m}) \oplus C^{0,2}(\mathfrak{m})$.*
Both conditions are satisfied if and only if $R_{\mathfrak{m}}$ is of type $(2,0)$ and T is of type $(2,0)+(1,1)$.

Now we choose a local section $p_0 \colon M \to P$ and identify P locally with $M \times G$. We denote by $T^{(p_0)}$ and $R^{(p_0)}$ the restrictions of T and R to $M = M \times \{e\} \subset M \times G$. Then

$$T_{(x,g)} = g_* T_x^{(p_0)} = g T_x^{(p_0)}(g^{-1} \cdot , g^{-1} \cdot)$$

and

$$R_{(x,g)} = g_* R_x^{(p_0)} = g R_x^{(p_0)}(g^{-1} \cdot , g^{-1} \cdot) g^{-1}.$$

This implies, for all $u, v \in V$,

$$\begin{aligned} \pi_{\mathfrak{m}} R_{(x,g)}(u, v) &= \pi_{\mathfrak{m}} g R_x^{(p_0)}(g^{-1}u, g^{-1}v) g^{-1} \\ &= g \pi_{g^{-1}\mathfrak{m} g} R_x^{(p_0)}(g^{-1}u, g^{-1}v) g^{-1} = g_*(\pi_{g^{-1}\mathfrak{m} g} R_x^{(p_0)})(u, v). \end{aligned}$$

For any para-complex structure $I = gJg^{-1} \in S = G/K$ we have the vector spaces $\mathfrak{m}(I) = [I, \mathfrak{g}] = g\mathfrak{m}g^{-1}$ and $V'(I) = V \oplus \mathfrak{m}(I)$ with the para-complex structures $gJ_{\mathfrak{m}}g^{-1}$ and $I' = gJ'g^{-1}$, respectively.

The above calculation implies that the (p, q) component of T or $R_{\mathfrak{m}}$, with respect to (J, J'), vanishes if and only if the (p, q) component of $T^{(p_0)}$ or $\pi_{\mathfrak{m}(I)} \circ R^{(p_0)}$, with respect to (I, I'), vanishes for all $I \in S$. We will use he symbol $\pi^{p,q}_{\mathfrak{m}(I)} := \pi^{p,q}_I \circ \pi_{\mathfrak{m}(I)}$, where $\pi^{p,q}_I \colon C^2(\mathfrak{m}(I)) \to C^{p,q}_I(\mathfrak{m}(I))$ is the projection onto the (p, q)-component with respect to (I, I') for any $I \in S$. Similarly we define $\pi^{p,q}_I \colon C^2(V) \to C^{p,q}_I(V)$ as the projection onto the (p, q)-component with respect to I.

This motivates the definition of the following two G-submodules of $\mathcal{R}(\mathfrak{g})$:

$$\begin{aligned} \mathcal{R}_{\text{int}}(\mathfrak{g}) &:= \left\{ R \in \mathcal{R}(\mathfrak{g}) \;\middle|\; \pi^{0,2}_{\mathfrak{m}(I)} R = 0 \text{ for all } I \in S \right\}, \\ \mathcal{R}_{\text{hol}}(\mathfrak{g}) &:= \left\{ R \in \mathcal{R}(\mathfrak{g}) \;\middle|\; \pi^{1,1}_{\mathfrak{m}(I)} R = 0 \text{ for all } I \in S \right\}. \end{aligned}$$

We also define a G-submodule $\mathcal{T}_{\text{int}}(\mathfrak{g}) \subset C^2(V)$ by

$$\mathcal{T}_{\text{int}}(\mathfrak{g}) := \left\{ T \in C^2(V) \;\middle|\; \pi^{0,2}_I T = 0 \text{ for all } I \in S \right\}.$$

Corollary 2. *Under the assumptions of* Theorem 1, *the following is true.*
(i) *The almost para-complex structure $\mathcal{J}$ is integrable if and only if the functions $T^{(p_0)}$ and $R^{(p_0)}$, associated to a local frame p_0, take values in the G-modules $\mathcal{T}_{\mathrm{int}}(\mathfrak{g})$ and $\mathcal{R}_{\mathrm{int}}(\mathfrak{g})$, respectively.*
(ii) *The horizontal distribution is para-holomorphic if and only if $R^{(p_0)}$ takes values in $\mathcal{R}_{\mathrm{hol}}(\mathfrak{g})$.*
Both conditions are satisfied if and only if $\pi_{\mathfrak{m}(I)}R$ is of type $(2,0)$ and T is of type $(2,0)+(1,1)$ for all $I \in S$.

Corollary 3. *Under the assumptions of* Theorem 1, *the almost para-complex structure $\mathcal{J}$ on the twistor space Z is integrable and the horizontal distribution $\mathcal{H}_Z$ is para-holomorhic if for all $x \in M$ there exists a frame $p \in \pi^{-1}(x)$ such that the curvature $R^{(p)} \in \mathcal{R}(\mathfrak{g})$ takes values in the G-module*

$$\mathcal{R}(\mathfrak{g})^{D_S} = \{R \in \mathcal{R}(\mathfrak{g}) \mid I \cdot R = 0 \text{ for all } I \in S\}$$

and the torsion $T^{(p)}$ satisfies $\pi^{0,2}T^{(p)} = 0$.

Proof. This follows from (5.6) and the previous corollary. □

Corollary 4. *Let G be a group of para-twistor type such that $\pi_{\mathfrak{m}(I)}\mathcal{R}(\mathfrak{g}) \subset C^{2,0}(\mathfrak{m}(I))$, for all $I \in S$, for example if $\mathcal{R}(\mathfrak{g}) = \mathcal{R}(\mathfrak{g})^{D_S}$. Then for any G-structure $(\pi : P \to M, \omega)$ with a torsion-free connection ω, the almost para-complex structure $\mathcal{J}$ on the twistor space Z is integrable and the horizontal distribution $\mathcal{H}_Z$ is para-holomorhic.*

6. Integrability and holomorphicity results for the twistor spaces of a para-quaternionic Kähler manifold

Theorem 2. *Let (M, g, Q) be a para-quaternionic Kähler manifold and $(Z^\epsilon, \mathcal{J}^\epsilon, \mathcal{H}_{Z^\epsilon})$ its twistor space, where $\epsilon = \pm$, see* Sections 4 *and* 5.4. *Then for $\epsilon = -1$ the almost complex structure $\mathcal{J}^\epsilon$ is integrable and the horizontal distribution is holomorphic. Similarly, for $\epsilon = 1$ the almost para-complex structure $\mathcal{J}^\epsilon$ is integrable and the horizontal distribution is para-holomorphic.*

Proof. By Proposition 9, the para-quaternionic Kähler structure defines a G-structure $\pi : P \to M$, where $G \subset \mathrm{GL}(\mathbb{R}^2 \otimes \mathbb{R}^{2n})$ is the normalizer of the connected Lie group $G_0 := \mathrm{SL}_2(\mathbb{R}) \otimes \mathrm{Sp}(\mathbb{R}^{2n})$ in $\mathrm{SO}(2n, 2n)$. Any para-quaternionic coframe $p \in P$ defines an isometry p: $(T_{\pi p}M, g_{\pi p}) \xrightarrow{\sim} (\mathbb{R}^2 \otimes \mathbb{R}^{2n}, g_{\mathrm{can}})$, which maps $Q_{\pi p}$ to $\mathfrak{sl}(2,\mathbb{R}) = \mathfrak{sl}(2,\mathbb{R}) \otimes \mathrm{Id}$, see Definition 7. The linear group G is of para-twistor type and also of twistor type, i.e. there exists elements $I, J \in \mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sp}(\mathbb{R}^{2n})$ such that $I^2 = -\mathrm{Id}$ and $J^2 = \mathrm{Id}$. In fact, we can choose $I = p \circ J_1 \circ p^{-1}$ and $J = p \circ J_2 \circ p^{-1}$. The symmetric space

$$\begin{aligned}\mathfrak{sl}(2,\mathbb{R}) = \mathfrak{sl}(2,\mathbb{R}) \otimes \mathrm{Id} \supset S^- &= \mathrm{Ad}_G(I) = G/Z_G(I) = \mathrm{GL}_2(\mathbb{R})/Z_{\mathrm{GL}_2(\mathbb{R})}(I)\\ &= \mathrm{GL}_2(\mathbb{R})/\mathrm{GL}_1(\mathbb{C}) = \mathrm{SL}_2^{\pm}(\mathbb{R})/\mathrm{SO}(2)\end{aligned}$$

is the two-sheeted hyperboloid in the three-dimensional Minkowski space $\mathfrak{sl}(2,\mathbb{R}) \cong \mathbb{R}^{2,1}$, whereas the symmetric space

$$\begin{aligned}\mathfrak{sl}(2,\mathbb{R}) \supset S^+ &= \mathrm{Ad}_G(J) = G/Z_G(J) = \mathrm{GL}_2(\mathbb{R})/Z_{\mathrm{GL}_2(\mathbb{R})}(J)\\ &= \mathrm{GL}_2(\mathbb{R})/\mathrm{GL}_1(C) = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(1,1)\end{aligned}$$

is the one-sheeted hyperboloid.

To finish the proof, in the case $\epsilon = +1$ we apply Corollary 4, in the case $\epsilon = -1$ [2] Theorem 7.3, since, by Proposition 4, the space of curvature tensors

$$\mathcal{R}(\mathfrak{g}) = \mathcal{R}(\mathfrak{g})^{\mathfrak{sl}(2,\mathbb{R})} = \mathcal{R}(\mathfrak{g})^{D_S},$$

for $S = S^{\pm}$. □

7. The canonical ϵ-Kähler-Einstein metric and contact structure on the twistor space Z^ϵ of a para-quaternionic Kähler manifold

DEFINITION 12. An *ϵ-Kähler manifold* is a pseudo-Riemannian manifold (M, g) together with a parallel skew-symmetric ϵ-complex structure J. An ϵ-Kähler manifold (M, g, J) is called a *Kähler manifold* if $\epsilon = -1$ and a *para-Kähler manifold* if $\epsilon = +1$. The parallel symplectic form $\omega = g(J\,\cdot,\,\cdot)$ is called the *Kähler form*.

REMARKS. The metric of a para-Kähler manifold has signature (n, n), since the ± 1-eigendistributions $T^{\pm}M$ of J are isotropic. Moreover, they are parallel and ω-Lagrangian.

Conversely, a *bi-Lagrangian manifold* [8], i.e. a symplectic manifold (M, ω) with two complementary Lagrangian integrable distributions $T^{\pm}M$, has the structure of a para-Kähler manifold, where $J|_{T^{\pm}M} = \pm\mathrm{Id}$ and $g = \omega(J\,\cdot,\,\cdot)$.

An integrable skew-symmetric ϵ-complex structure on a pseudo-Riemannian manifold is *parallel*, and hence defines an ϵ-Kähler structure, if and only if the Kähler form ω is closed, see [10] Theorem 1.

DEFINITION 13. An ϵ-holomorphic distribution $\mathcal{D}$ of real codimension 2 on an ϵ-complex manifold Z is called an ϵ-holomorphic *contact structure* if the Frobenius form $[\,\cdot,\,\cdot\,]\colon \bigwedge^2 \mathcal{D} \to TZ/\mathcal{D}$ is non-degenerate.

Theorem 3. *Let $(Z^\epsilon, \mathcal{J}^\epsilon)$ be the ϵ-twistor space of a para-quaternionic Kähler manifold (M, g, Q) with non-zero reduced scalar curvature ν. Then*

(i) *the canonical metric* $g_t = g_t^\epsilon$ *on* Z^ϵ *is* ϵ*-Kähler-Einstein if and only if* $t = -\epsilon/\nu$. *Moreover,* g_t *is Einstein if and only if* $t = -\epsilon/\nu$ *or* $t = -\epsilon/(\nu(n+1))$.
(ii) *The horizontal distribution* $\mathcal{H}_Z \subset TZ^\epsilon$ *is an* ϵ*-holomorphic contact structure.*

Proof. (i) By Theorem 2 and Proposition 7 the ϵ-complex structure $\mathcal{J}^\epsilon$ is integrable and g_t-skew-symmetric for all t. By the above remark, to check when $(Z^\epsilon, \mathcal{J}^\epsilon, g_t)$ is ϵ-Kähler it is sufficient to check when the Kähler form $\omega_t = g_t(\mathcal{J}^\epsilon\,\cdot\,,\,\cdot\,)$ is closed.

The twistor bundle $Z^\epsilon = P/G_A \to M$, see Proposition 9, is a bundle associated with the principal bundle

$$P' := P/Z_G(\mathrm{GL}_2) \to M = P'/\mathrm{SO}_3^\epsilon,$$

where $\mathrm{SO}_3^\epsilon = \mathrm{SO}(2,1)$ for $\epsilon = +1$ and $\mathrm{SO}_3^\epsilon = \mathrm{SO}(1,2) \cong \mathrm{SO}(2,1)$ for $\epsilon = -1$. In other words, P' is the SO_3^ϵ-principal bundle of standard bases $p = (J_1, J_2, J_3)$ of Q_x^ϵ, $x \in M$, where $J_1^2 = \epsilon\mathrm{Id}$, $J_2^2 = \mathrm{Id}$ and $J_3^2 = -\epsilon\mathrm{Id}$. We have a natural projection

$$\pi_{P'}\colon P' \to Z^\epsilon = P'/\mathrm{SO}_2^\epsilon, \quad (J_1, J_2, J_3) \mapsto J_1,$$

where $\mathrm{SO}_2^\epsilon = \mathrm{SO}(1,1)$ for $\epsilon = +1$ and $\mathrm{SO}_2^\epsilon = \mathrm{SO}(2)$ for $\epsilon = -1$ is the stabilizer of $(1,0,0)^t \in \mathbb{R}^3$.

The closure of ω_t is equivalent to the closure of its pull back $\omega_t' = \pi_{P'}^*\omega_t$ to P'. The two-form ω_t' can be written as

$$\omega_t' = g_t'(\mathcal{J}_1\,\cdot\,,\,\cdot\,), \quad g_t' = tg^v + \pi_{P'}^* g. \tag{7.1}$$

Here $\pi_{P'}^* g$ is the pull back of the metric g on M and g^v is the metric on the vertical bundle T^vP', which corresponds to a suitably normalized ad-invariant scalar product $\langle\,\cdot\,,\,\cdot\,\rangle$ on $\mathfrak{so}_3^\epsilon = \mathrm{Lie}\,\mathrm{SO}_3^\epsilon$, extended by zero to the horizontal bundle $\mathcal{H}$ associated with the Levi-Civita connection of M. The normalization of the scalar product $\langle\,\cdot\,,\,\cdot\,\rangle$ on $\mathfrak{so}_3^\epsilon = \mathrm{ad}(\mathfrak{sl}_2(\mathbb{R})) \cong \mathfrak{sl}_2(\mathbb{R}) = \mathrm{span}\{J_1^0, J_2^0, J_3^0\}$ is given by

$$-\epsilon\langle \mathrm{ad}_{J_\alpha^0}, \mathrm{ad}_{J_\beta^0}\rangle = -4\epsilon_a\delta_{\alpha\beta} = 4\langle J_\alpha^0, J_\beta^0\rangle, \tag{7.2}$$

where (J_1^0, J_2^0, J_3^0) is the standard ϵ-quaternionic basis of $\mathfrak{sl}_2(\mathbb{R})$, with the relations

$$(J_\alpha^0)^2 = \epsilon_\alpha\mathrm{Id}, \quad (\epsilon_1, \epsilon_2, \epsilon_3) = (\epsilon, 1, -\epsilon). \tag{7.3}$$

The above scalar product on $\mathfrak{so}_3^\epsilon$ has signature $(2,1)$ if $\epsilon = +1$ and $(1,2)$ if $\epsilon = -1$. The factor 4 is chosen such that the canonical projection $(P', g_t') \to (Z^\epsilon = P'/\mathrm{SO}_2^\epsilon, g_t)$

is a pseudo-Riemannian submersion. Notice that the vertical vectors

$$\mathrm{ad}_{J_2},\ \mathrm{ad}_{J_3} \in T^v_p P' \cong \mathfrak{so}^\epsilon_3 = \mathrm{ad}(\mathfrak{sl}_2(\mathbb{R})), \quad p \in P',$$

are mapped to

$$\mathrm{ad}_{J_2}\, J_1 = -2J_3, \quad \mathrm{ad}_{J_3}\, J_1 = -2\epsilon J_2 \in T^v Z^\epsilon \subset Q^\epsilon_x \cong \mathfrak{sl}_2(\mathbb{R}), \quad x = \pi_{P'}(p).$$

The field $p \mapsto (\mathcal{J}_\alpha)_p$ is defined at $p = (J_1, J_2, J_3)$ as the following endomorphism of $T_p P' = T^v P' \oplus \mathcal{H}_p \cong \mathfrak{so}^\epsilon_3 \oplus T_x M,\ x = \pi_{P'}(p)$,

$$\mathcal{J}_\alpha|_{\mathcal{H}_p} : T_x M \to T_x M, \quad X \mapsto J_\alpha X, \quad \mathcal{J}_\alpha|_{T^v P'} = \frac{1}{2}\,\mathrm{ad}_{J^0_\alpha}\,.$$

It is sufficient to check $d\omega'_t = 0$ on three vectors, each of which are horizontal or vertical. Moreover, it is sufficient to consider the fundamental vertical fields (V_1, V_2, V_3), which correspond to (J^0_1, J^0_2, J^0_3) and basic horizontal fields $X, Y, Z, \dots$ on P', i.e. horizontal lifts of vector fields $X_M,\ Y_M,\ Z_M$ on M.

Lemma 2. *With the above notations we have*
(i) $[V_1, V_2] = 2V_3,\ [V_3, V_1] = -2\epsilon V_2,\ [V_2, V_3] = -2V_1$,
(ii) *the functions* $g'_t(V_\alpha, V_\beta)$ *and* $\omega'_t(V_\alpha, V_\beta)$ *are constant for all* $\alpha, \beta \in \{1, 2, 3\}$,
(iii) $[V_\alpha, X] = 0$,
(iv) $[X, Y]^v = -(\nu/2)\sum_\alpha \epsilon_\alpha g'_t(\mathcal{J}_\alpha X, Y)V_\alpha = -(\nu/2)\sum_\alpha \epsilon_\alpha g(J_\alpha X_M, Y_M)V_\alpha$, *where* $[X, Y]^v$ *is evaluated at the point* $p = (J_1, J_2, J_3) \in P'$,
(v) $\mathcal{L}_{V_\alpha} g'_t = 0,\ \mathcal{L}_{V_1}\mathcal{J}_1 = 0,\ \mathcal{L}_{V_2}\mathcal{J}_1 = -2\mathcal{J}_3,\ \mathcal{L}_{V_3}\mathcal{J}_1 = -2\epsilon\mathcal{J}_2$ *and*
(vi) $(\mathcal{L}_X g'_t)(U, V) = 0$ *for all* $U, V \in T^v P'$.

Proof. (i) follows from the ϵ-quaternionic relations

$$[J^0_1, J^0_2] = 2J^0_3, \quad [J^0_3, J^0_1] = -2\epsilon J^0_3, \quad [J^0_2, J^0_3] = -2J^0_1.$$

(ii) Since the metric g^v corresponds to the ad-invariant scalar product (7.2), the functions

$$g'_t(V_\alpha, V_\beta) = tg^v(V_\alpha, V_\beta) = -4\epsilon t\langle J^0_\alpha, J^0_\beta\rangle = 4\epsilon t\epsilon_\alpha\delta_{\alpha\beta}$$

are constant. Similarly, the functions $\omega'_t(V_\alpha, V_\beta)$ are constant, because, for all fundamental vector fields V_α, the vector field $\mathcal{J}_1 V_\alpha$ is again a fundamental vector field.

(iii) The vector field $[V_\alpha, X]$ is horizontal, since the principal action preserves the horizontal distribution. On the other hand, it is mapped to $[0, X_M] = 0$ under the projection $P' \to M$. This shows that $[V_\alpha, X] = 0$.

(iv) follows from Proposition 4, since $[X, Y]^v = -\Omega'(X, Y)$, where Ω' stands for the curvature form of the principal bundle $P' \to M$.

(v) $\mathcal{L}_{V_\alpha} g'_t = 0$ follows from the ad-invariance of g^v, cf. (7.1). The remaining equations are obtained from (i) using

$$\mathcal{J}_1 V_1 = 0, \quad \mathcal{J}_1 V_2 = V_3, \quad \mathcal{J}_1 V_3 = \epsilon V_2.$$

Finally, (ii) and (iii) easily imply (vi). □

Part (i) and (ii) of the lemma, yields

$$d\omega'_t(V_1, V_2, V_3) = V_1\omega'_t(V_2, V_3) - \omega'_t([V_1, V_2], V_3) + \text{cycl.} = 0.$$

Using part (i), (ii), (v) and (vi) of the lemma, we calculate

$$\begin{aligned} d\omega'_t(V_\alpha, V_\beta, X) &= -\omega'_t([V_\beta, X], V_\alpha) - \omega'_t([X, V_\alpha], V_\beta) = -(\mathcal{L}_X\omega'_t)(V_\alpha, V_\beta) \\ &= -g'_t((\mathcal{L}_X\mathcal{J}_1)V_\alpha, V_\beta) = -g'_t([X, \mathcal{J}_1 V_\alpha] - \mathcal{J}_1[X, V_\alpha], V_\beta) \\ &= g'_t(X, [V_\beta, \mathcal{J}_1 V_\alpha]) + g'_t(X, [\mathcal{J}_1 V_\beta, V_\alpha]) = 0. \end{aligned}$$

By (iii), (iv) and (v) of the lemma, we compute

$$\begin{aligned} d\omega'_t(V_1, X, Y) &= V_1\omega'_t(X, Y) - \omega'_t([X, Y], V_1) \\ &= g'_t((\mathcal{L}_{V_1}\mathcal{J}_1)X, Y) + \frac{\nu}{2}\sum_{\alpha=1}^{3}\epsilon_\alpha g(J_\alpha X_M, Y_M)\omega'_t(V_\alpha, V_1) \\ &= 0 + \frac{\nu}{2}\sum_{\alpha=1}^{3}\epsilon_\alpha g(J_\alpha X_M, Y_M)g'_t(\mathcal{J}_1 V_\alpha, V_1) = 0, \end{aligned}$$

since $\mathcal{J}_1 T^v P' = \mathrm{span}\{V_2, V_3\}$. Similarly, we calculate

$$\begin{aligned} d\omega'_t(V_2, X, Y) &= V_2\omega'_t(X, Y) - \omega'_t([X, Y], V_2) \\ &= g'_t((\mathcal{L}_{V_2}\mathcal{J}_1)X, Y) + \frac{\nu}{2}\sum_{\alpha=1}^{3}\epsilon_\alpha g(J_\alpha X_M, Y_M)\omega'_t(V_\alpha, V_2) \\ &= -2g'_t(\mathcal{J}_3 X, Y) + \frac{\nu}{2}\sum_{\alpha=1}^{3}\epsilon_\alpha g(J_\alpha X_M, Y_M)g'_t(\mathcal{J}_1 V_\alpha, V_2) \\ &= -2g(J_3 X_M, Y_M) + \frac{\nu t}{2}\epsilon_3 g(J_3 X_M, Y_M)g^v(\mathcal{J}_1 V_3, V_2) \\ &= -2g(J_3 X_M, Y_M) + \frac{\nu t}{2}(-\epsilon)g(J_3 X_M, Y_M)g^v(\epsilon V_2, V_2) \\ &= -2g(J_3 X_M, Y_M) - 2\epsilon\nu t g(J_3 X_M, Y_M), \end{aligned}$$

since $g^v(V_2, V_2) = -4\epsilon\langle J_2^0, J_2^0\rangle = 4\epsilon\epsilon_2 = 4\epsilon$. In the same way, we obtain

$$\begin{aligned} d\omega_t'(V_3, X, Y) &= -2\epsilon g(J_2 X_M, Y_M) + \frac{\nu t}{2} g(J_2 X_M, Y_M) g^v(\mathcal{J}_1 V_2, V_3) \\ &= -2\epsilon g(J_2 X_M, Y_M) + \frac{\nu t}{2} g(J_2 X_M, Y_M) g^v(V_3, V_3) \\ &= -2\epsilon g(J_2 X_M, Y_M) - 2\nu t g(J_2 X_M, Y_M), \end{aligned}$$

since $g^v(V_3, V_3) = -4\epsilon\langle J_3^0, J_3^0\rangle = 4\epsilon\epsilon_3 = -4$. This shows that $d\omega_t'(U, X, Y) = 0$ for all vertical vector fields U if and only if $\nu t = -\epsilon$.

It remains to check that $d\omega_t'(X_p, Y_p, Z_p)$ vanishes on three horizontal vectors

$$X_p, Y_p, Z_p \in \mathcal{H}_p, \quad p \in P'.$$

Let $t \mapsto \tilde{c}(t) = (J_1(t), J_2(t), J_3(t)) \in P'$ be the horizontal lift of a curve $t \mapsto c(t) \in M$ such that $\tilde{c}(0) = p$ and $\tilde{c}'(0) = X_p$. Notice that the horizontality of $\tilde{c}$ means that $t \mapsto J_\alpha(t)$ is parallel along c.

Let $t \mapsto Y(t) \in \mathcal{H}_{\tilde{c}(t)}$ be the horizontal lift of the vector field

$$t \mapsto Y_M(t) := \|_{c(0)}^{c(t)}\, d\pi_{P'} Y_p \in T_{c(t)}M,$$

which is parallel along the base curve c. The initial value of Y is $Y(0) = Y_p$. It suffices to prove that

$$(\nabla'_{X_p}\omega_t')(Y_p, Z_p) = g_t'((\nabla'_{X_p}\mathcal{J}_1)Y_p, Z_p) = 0,$$

where ∇' is the Levi-Civita connection of g_t'. We have to check that the horizontal component of

$$(\nabla'_{X_p}\mathcal{J}_1)Y_p = \nabla'_{X_p}(\mathcal{J}_1 Y) - \mathcal{J}_1\nabla'_{X_p} Y$$

vanishes. Therefore, we calculate

$$\begin{aligned} d\pi_{P'}(\nabla'_{X_p}(\mathcal{J}_1 Y) - \mathcal{J}_1\nabla'_{X_p} Y) &= \nabla_{c'(0)}(J_1(t) Y_M(t)) - J_1(0)\nabla_{c'(0)} Y_M(t) \\ &= (\nabla_{c'(0)} J_1(t)) Y_M(0) = 0. \end{aligned}$$

Here we have used two facts: first, that $t \mapsto \mathcal{J}_1 Y(t)$ is a basic horizontal vector field along $\tilde{c}$, which projects onto

$$d\pi_{P'}\mathcal{J}_1 Y(t) = J_1(t) Y_M(t)$$

and, second, that $d\pi_{P'}\nabla'_X Y = \nabla_{X_M} Y_M$ for any two basic horizontal vector fields X, Y (e.g. along a horizontal curve), where ∇ is the Levi-Civita connection in M. The latter is a standard fact about pseudo-Riemannian submersions. This proves that g_t is ϵ-Kähler-Einstein if and only if $t = -\epsilon/\nu$. The above argument proves also the following proposition.

Proposition 15. *For any horizontal vectors X, Y, Z on P' and $\alpha = 1, 2, 3$, we have*

$$g'_t((\nabla_X \mathcal{J}_\alpha)Y, Z) = 0.$$

Next we study the Einstein equations for the family g'_t. We recall the definition of the O'Neill tensor and the O'Neill formulas for the covariant derivative of a pseudo-Riemannian submersion $\pi : E \to M$ with totally geodesic fibres, see [12, 5]. The O'Neill tensor $A \in \Omega^1(\mathrm{End}\, TE)$ is a one-form with values in skew-symmetric endomorphisms. It is given by

$$A_U = 0, \quad A_X Y = -A_Y X = (\nabla_X Y)^v = \frac{1}{2}[X, Y]^v, \quad A_X U = (\nabla_X U)^h, \tag{7.4}$$

where U is a vertical vector field and X, Y are horizontal vector fields. The superscripts v and h stand for the vertical and horizontal components, respectively. If X is a basic horizontal vector field then, in addition

$$A_X U = (\nabla_X U)^h = \nabla_U X. \tag{7.5}$$

The covariant derivatives in E are given by

$$\nabla_U V = \nabla^F_U V, \tag{7.6}$$

$$\nabla_U X = (\nabla_U X)^h, \tag{7.7}$$

$$\nabla_X U = (\nabla_X U)^v + A_X U, \tag{7.8}$$

$$\nabla_X Y = A_X Y + (\nabla_X Y)^h. \tag{7.9}$$

Here ∇^F and ∇^M denote the covariant derivative in the fibres F and in the base M, respectively. For basic horizontal vector fields X, Y, we have $[U, X]^h = 0$ for any vertical (and hence projectable) vector field U. Moreover, we have

$$(\nabla_X U)^v = [X, U], \tag{7.10}$$

$$\pi_* \nabla_X Y = \nabla^M_{\pi_* X} \pi_* Y. \tag{7.11}$$

In particular, $\nabla_X Y$ is a projectable vector field on E.

Proposition 16 (cf. [12]). *Let $\pi : E \to M$ be a pseudo-Riemannian submersion with totally geodesic fibres F. Then the Ricci and scalar curvatures of E are given by*:

$$\mathrm{Ric}(U, V) = \mathrm{Ric}^F(U, V) + \sum_i \epsilon_i \langle A_{X_i} U, A_{X_i} V \rangle, \tag{7.12}$$

$$\mathrm{Ric}(X, U) = \langle (\mathrm{div}\, A)X, U \rangle = \sum_i \epsilon_i \langle (\nabla_{X_i} A)_{X_i} X, U \rangle, \tag{7.13}$$

(7.14) $$\mathrm{Ric}(X, Y) = \mathrm{Ric}^M(\pi_* X, \pi_* Y) - 2\sum_i \epsilon_i \langle A_X X_i, A_Y X_i\rangle,$$

(7.15) $$\mathrm{scal} = \pi^*\mathrm{scal}^M + \mathrm{scal}^F - \sum_{i,j} \epsilon_i \epsilon_j \langle A_{X_i} X_j, A_{X_i} X_j\rangle.$$

Proposition 17. *The divergence* div $A \in \Gamma(\mathrm{End}\, TP')$ *of the O'Neill tensor of the principal bundle* $P' \to M$ *preserves the horizontal distribution. In particular,*

$$\mathrm{Ric}(X, U) = g'_t((\mathrm{div}\, A)X, U) = 0.$$

Proof. By (7.4) and Lemma 2 (iv), the value of the O'Neill tensor on two basic horizontal vector fields X, Y is given by

(7.16) $$A_X Y = \frac{1}{2}[X, Y]^v = -\frac{\nu}{4}\sum_\alpha \epsilon_\alpha g'_t(\mathcal{J}_\alpha X, Y)V_\alpha.$$

It is sufficient to prove that $g'_t((\nabla_X A)_Y Z, U) = 0$. This follows from the remark that $\nabla_X V_\alpha = A_X V_\alpha$ is horizontal, by (7.5), and Proposition 15. □

The skew-symmetry of A_X and (7.16) imply

(7.17) $$A_X U = \frac{\nu}{4}\sum_\alpha \epsilon_\alpha g'_t(U, V_\alpha)\mathcal{J}_\alpha X.$$

In fact,

$$g'_t(A_X U, Y) = -g'_t(U, A_X Y) = \frac{\nu}{4}\sum_\alpha \epsilon_\alpha g'_t(\mathcal{J}_\alpha X, Y) g'_t(U, V_\alpha).$$

Proposition 18. *Let* P' *be the total space of the principal bundle* $P' \to M$ *of admissible frames of* Q *over a para-quaternionic Kähler manifold* (M, g, Q). *Then the Ricci curvature of the metric* g'_t *on* P' *is given by*:

(7.18) $$\mathrm{Ric}(U, V) = -\epsilon\left(\frac{1}{2t} + \nu^2 n t\right) g'_t(U, V), \quad U, V \in T^v P',$$

(7.19) $$\mathrm{Ric}(U, X) = 0,$$

(7.20) $$\mathrm{Ric}(X, Y) = \left(\nu(n+2) + \frac{3\epsilon\nu^2 t}{2}\right) g'_t(X, Y), \quad X, Y \in \mathcal{H} = (T^v P')^\perp.$$

Proof. We calculate the Ricci curvature using the formulas in Proposition 16. The fibre F is identified with the Lie group SO(2, 1) with a bi-invariant pseudo-Riemannian metric g_t', which is related with the Killing form B by

$$g_t' = \frac{\epsilon t}{2} B, \tag{7.21}$$

see (7.2). Therefore

$$\mathrm{Ric}^F = -\frac{1}{4} B = \frac{-\epsilon}{2t} g_t'. \tag{7.22}$$

We compute the second term in equation (7.12) using (7.17):

$$\begin{aligned}\sum_i \epsilon_i \langle A_{X_i} U, A_{X_i} V\rangle &= \sum_i \epsilon_i \frac{\nu^2}{16} \sum_\alpha \epsilon_\alpha^2 g_t'(U, V_\alpha) g_t'(V, V_\alpha) g_t'(\mathcal{J}_\alpha X_i, \mathcal{J}_\alpha X_i)\\ &= \frac{\nu^2}{16} \sum_{i,\alpha} \epsilon_i^2 (-\epsilon_\alpha) g_t'(U, V_\alpha) g_t'(V, V_\alpha) = -\epsilon \nu^2 n t g_t'(U, V).\end{aligned}$$

This implies the first equation (7.18). The second equation (7.19) was already established in Proposition 17. Since M is an Einstein manifold with scalar curvature $\mathrm{scal}^M = 4n(n+2)\nu$,

$$\mathrm{Ric}^M = \frac{\mathrm{scal}}{4n} g = \nu(n+2) g. \tag{7.23}$$

We compute the second term in equation (7.14) using (7.16):

$$\begin{aligned}-2\sum_i \epsilon_i \langle A_X X_i, A_Y X_i\rangle &= -2 \sum_{i,\alpha} \epsilon_i \frac{\nu^2}{16} g_t'(\mathcal{J}_\alpha X, X_i) g_t'(\mathcal{J}_\alpha Y, X_i) g_t'(V_\alpha, V_\alpha)\\ &= -\frac{\nu^2}{8} \sum_\alpha g_t'(\mathcal{J}_\alpha X, \mathcal{J}_\alpha Y) g_t'(V_\alpha, V_\alpha)\\ &= -\frac{\nu^2}{8} \sum_\alpha (-\epsilon_\alpha) g_t'(X, Y)(4\epsilon t \epsilon_\alpha)\\ &= \frac{3\epsilon \nu^2 t}{2} g_t'(X, Y).\end{aligned}$$

This proves the proposition. □

Corollary 5. *Let P' be the total space of the principal bundle $P' \to M$ of admissible frames of Q over a para-quaternionic Kähler manifold (M, g, Q) with reduced scalar curvature ν. Then the metric g_t' is Einstein if and only if*

$$t = \frac{-\epsilon}{\nu} \quad \textit{or} \quad t = \frac{-\epsilon}{\nu(2n+3)}.$$

The corresponding Einstein constant is, respectively,

$$c = \left(n + \frac{1}{2}\right)\nu \quad \textit{and} \quad c = \frac{4n^2 + 14n + 9}{4n + 6}\nu.$$

Next we calculate the Ricci curvature of the metric g_t^ϵ on the twistor spaces $Z^\epsilon = P'/\mathrm{SO}_2^\epsilon$, $\epsilon = \pm 1$.

Proposition 19.

$$(7.24) \qquad (A_X Y)_{J_1} = \frac{\nu}{2}(g(J_2\pi_* X, \pi_* Y)J_3 - g(J_3\pi_* X, \pi_* Y)J_2) \in T_{J_1}^v Z = \mathrm{span}\{J_2, J_3\},$$

$$(7.25) \qquad A_X J_2 = -\epsilon_2 \frac{\nu t}{2} \widetilde{J_3\pi_* X} = -\frac{\nu t}{2} \widetilde{J_3\pi_* X},$$

$$(7.26) \qquad A_X J_3 = \epsilon_3 \frac{\nu t}{2} \widetilde{J_2\pi_* X} = -\epsilon \frac{\nu t}{2} \widetilde{J_2\pi_* X},$$

where X and Y are horizontal vectors and $\tilde{X}_M \in T_{J_1} Z^\epsilon$ denotes the horizontal lift of the vector $X_M \in T_{\pi(J_1)} M$.

$$(7.27) \qquad \mathrm{Ric}(U, V) = -\epsilon\left(\frac{1}{t} + \nu^2 nt\right) g_t^\epsilon(U, V),$$

$$(7.28) \qquad \mathrm{Ric}(X, U) = 0,$$

$$(7.29) \qquad \mathrm{Ric}(X, Y) = (\nu(n+2) + \epsilon\nu^2 t) g_t^\epsilon(X, Y),$$

where U and V are vertical vectors.

Proof. The equations (7.24)–(7.26) are obtained from (7.16), (7.17) and (7.21). We calculate the Ricci curvature using the formulas in Proposition 16. In fact, the projection $\pi: Z^\epsilon \to M$ is a pseudo-Riemannian submersion with totally geodesic fibre $F = \mathrm{SO}_3^\epsilon/\mathrm{SO}_2^\epsilon$, where $\mathrm{SO}_3^\epsilon \cong \mathrm{SO}(2, 1)$ and $\mathrm{SO}_2^{\epsilon=+1} = \mathrm{SO}(1, 1)$ and $\mathrm{SO}_2^{\epsilon=-1} = \mathrm{SO}(2)$. Here $Z_x^\epsilon \subset Q_x^\epsilon = \mathrm{span}\{J_1, J_2, J_3\}$, where (J_1, J_2, J_3) is an admissible basis such that $J_\alpha^2 = \epsilon_\alpha \mathrm{Id}$ and $(\epsilon_1, \epsilon_2, \epsilon_3) = (\epsilon, 1, -\epsilon)$. In both cases, the Lie algebra $\mathfrak{so}_2^\epsilon = \mathbb{R}\,\mathrm{ad}(J_1)$. The fibre F is a two-dimensional symmetric space, with symmetric decomposition

$$\mathfrak{so}_3^\epsilon = \mathfrak{so}_2^\epsilon + \mathfrak{m}, \quad \mathfrak{m} = \mathbb{R}\,\mathrm{ad}(J_2) + \mathbb{R}\,\mathrm{ad}(J_3).$$

The curvature tensor is given by

$$R(\mathrm{ad}(J_2), \mathrm{ad}(J_3)) = -\,\mathrm{ad}_{[J_2, J_3]}|_{\mathfrak{m}} = 2\,\mathrm{ad}_{J_1}|_{\mathfrak{m}}$$

and the sectional curvature of the metric $g^F = g_t^\epsilon|_F = tg^v$ is $-\epsilon/t$. In particular,

$$(7.30) \qquad \mathrm{Ric}^F = -\epsilon g^v = -\frac{\epsilon}{t} g^F.$$

Next we compute the second term in equation (7.12) using (7.25):

$$\begin{aligned}\sum_i \epsilon_i g_t^\epsilon(A_{X_i}J_2,\ A_{X_i}J_2) &= \frac{\nu^2 t^2}{4}\sum_i \epsilon_i g(J_3\pi_* X_i,\ J_3\pi_* X_i)\\ &= \nu^2 t^2 n(-\epsilon_3) = \epsilon\nu^2 t^2 n\\ &= -\epsilon\nu^2 t n g_t^\epsilon(J_2,\ J_2),\end{aligned}$$

since $g_t^\epsilon(J_2,\ J_2) = -t\epsilon_2 = -t$. The same calculation for $(U,\ V) = (J_2,\ J_3)$ and $(U,\ V) = (J_3,\ J_3)$ shows that for any two vertical vectors $U,\ V,$ we have

$$\sum_i \epsilon_i g_t^\epsilon(A_{X_i}U,\ A_{X_i}V) = -\epsilon\nu^2 t n g_t^\epsilon(U,\ V).$$

This proves (7.27).

Now we calculate the second term in equation (7.14) using (7.24).

$$\begin{aligned}&-2\sum_i \epsilon_i g_t^\epsilon(A_X X_i,\ A_Y X_i)\\ &= -\frac{\nu^2}{2}\sum_i \epsilon_i g(J_2\pi_* X,\ \pi_* X_i)g(J_2\pi_* Y,\ \pi_* X_i)g_t^\epsilon(J_3,\ J_3)\\ &\quad -\frac{\nu^2}{2}\sum_i \epsilon_i g(J_3\pi_* X,\ \pi_* X_i)g(J_3\pi_* Y,\ \pi_* X_i)g_t^\epsilon(J_2,\ J_2)\\ &= -\frac{\nu^2}{2}g(J_2\pi_* X,\ J_2\pi_* Y)g_t^\epsilon(J_3,\ J_3) - \frac{\nu^2}{2}g(J_3\pi_* X,\ J_3\pi_* Y)g_t^\epsilon(J_2,\ J_2)\\ &= -\frac{\nu^2}{2}[(-\epsilon_2)(-t\epsilon_3) + (-\epsilon_3)(-t\epsilon_2)]g_t^\epsilon(X,\ Y) = \epsilon\nu^2 t g_t^\epsilon(X,\ Y).\end{aligned}$$

This proves (7.29).

To prove that $\mathrm{Ric}(X,\ U) = 0,$ by Proposition 16 we have to check that $\operatorname{div} A$ preserves the horizontal distribution $\mathcal{H}_Z \subset TZ^\epsilon$. It sufficient to prove that

$$g_t^\epsilon((\nabla_X A)_Y Z,\ \mathcal{J}^\epsilon U) = 0$$

for all basic horizontal vector fields $X,\ Y,\ Z$ and vertical vector fields U. We compute this using the fact that $\nabla\mathcal{J}^\epsilon = 0$ and (7.10):

$$\begin{aligned}g_t^\epsilon((\nabla_X A)_Y Z,\ \mathcal{J}^\epsilon U) &= X g_t^\epsilon(A_Y Z,\ \mathcal{J}^\epsilon U) - g_t^\epsilon(A_Y Z,\ \mathcal{J}^\epsilon \nabla_X U)\\ &= X g_t^\epsilon(A_Y Z,\ \mathcal{J}^\epsilon U) - g_t^\epsilon(A_Y Z,\ \mathcal{J}^\epsilon [X,\ U]).\end{aligned}$$

Lemma 3. *For any basic horizontal vector fields X, Y and vertical vector field U we have*

$$g_t^\epsilon(A_X Y, \mathcal{J}^\epsilon U) = \frac{\epsilon \nu t}{2} g(U\pi_* X, \pi_* Y) = -\frac{1}{2} U\omega_t(X, Y), \tag{7.31}$$

where $\omega_t = g_t^\epsilon(\mathcal{J}^\epsilon \cdot, \cdot)$ is the ϵ-Kähler form and the value $U_{J_1} \in T_{J_1}^v Z = \mathrm{span}\{J_2, J_3\} \subset Q_x$, $x = \pi(J_1)$, of the vertical vector field U at the point $J_1 \in Z^\epsilon$ is considered as an endomorphism of $T_x M$.

Proof. The first equation follows from (7.24) and the formulas $\mathcal{J}^\epsilon J_2 = J_3$, $\mathcal{J}^\epsilon J_3 = \epsilon J_2$. For the second equality we use that $[U, X]$ and $[U, Y]$ are vertical and that ω_t is closed:

$$\begin{aligned}\mathcal{L}_U(\omega_t(X, Y)) &= (\mathcal{L}_U \omega_t)(X, Y) = (d\iota_U \omega_t)(X, Y) \\ &= -\omega_t(U, [X, Y]) = -2g_t^\epsilon(\mathcal{J}^\epsilon U, A_X Y).\end{aligned}$$ □

The following corollary finishes the proof of Theorem 3 (i).

Corollary 6. *Let Z^ϵ, $\epsilon = \pm 1$, be the twistor spaces of a para-quaternionic Kähler manifold. Then the metric g_t^ϵ is Einstein if and only if*

$$t = -\frac{\epsilon}{\nu} \quad or \quad t = -\frac{\epsilon}{\nu(n+1)}.$$

The corresponding Einstein constant is, respectively,

$$c = (n+1)\nu \quad and \quad c = \frac{n^2 + 3n + 1}{n+1}\nu.$$

(ii) By Theorem 2, we know that the horizontal distribution $\mathcal{H}_Z \subset TZ^\epsilon$ is holomorphic if $\epsilon = -1$ and para-holomorphic if $\epsilon = -1$. We show that it is a para-holomorphic contact structure if $\epsilon = +1$. The case $\epsilon = -1$ is similar. We have to check that the Frobenius form

$$\mathcal{H}_Z^{1,0} \times \mathcal{H}_Z^{1,0} \ni (Z, W) \mapsto ([Z, W] \bmod \mathcal{H}_Z^{1,0}) \in T^{1,0} Z^\epsilon / \mathcal{H}_Z^{1,0}$$

of $\mathcal{H}_Z^{1,0}$ is nondegenerate.

Let X and Y be basic horizontal vector fields on P' and $Z = X + e\mathcal{J}_1 X$ and $W = Y + e\mathcal{J}_1 Y$ the corresponding sections of $\mathcal{H}^{1,0} \subset \mathcal{H} \otimes C = \mathcal{H} + e\mathcal{H}$ the $(+e)$-eigenbundle of the C-linear extension of $\mathcal{J}_1$ on $\mathcal{H} \otimes C$. Notice that $\mathcal{J}_1^2|_{\mathcal{H}} = \epsilon \mathrm{Id} = \mathrm{Id}$, since $\epsilon = +1$. Let us calculate, with the help of part (iv) of Lemma 2, the vertical component of $[Z, W]$

at any point $p = (J_1, J_2, J_3) \in P'$:

$$
\begin{aligned}
[Z, W]^v &= -\frac{\nu}{2} \sum_\alpha \epsilon_\alpha (g(J_\alpha X_M, Y_M) + g(J_\alpha J_1 X_M, J_1 Y_M)) V_\alpha \\
&\quad - e\frac{\nu}{2} \sum_\alpha \epsilon_\alpha (g(J_\alpha X_M, J_1 Y_M) + g(J_\alpha J_1 X_M, Y_M)) V_\alpha \\
&= -\nu(\rho_2(X_M, Y_M)V_2 - \rho_3(X_M, Y_M)V_3) \\
&\quad + e\nu(\rho_3(X_M, Y_M)V_2 - \rho_2(X_M, Y_M)V_3) \\
&= -\nu(\rho_2(X_M, Y_M) - e\rho_3(X_M, Y_M))(V_2 + eV_3),
\end{aligned}
$$

where $\rho_\alpha = g(J_\alpha \cdot, \cdot)$. This shows that the Frobenius form of $\mathcal{H}^{1,0} \subset TP' \otimes C$ is nondegenerate. Let us denote by $\tilde{X}_M$ and $\tilde{Y}_M$ the horizontal lifts of X_M and Y_M to vector fields on Z^ϵ. We put $\tilde{Z} := X_M + e\mathcal{J}^\epsilon X_M$ and $\tilde{W} := Y_M + e\mathcal{J}^\epsilon Y_M$. Thanks to the above formula, we can calculate the vertical component of $[\tilde{Z}, \tilde{W}]$ at the point $z = J_1 \in Z^\epsilon$, which is the image of $p = (J_1, J_2, J_3) \in P'$ under the natural projection $P' \to Z^\epsilon = P'/\mathrm{SO}_2^\epsilon$.

$$
\begin{aligned}
[\tilde{Z}, \tilde{W}]^v &= -\nu(\rho_2(X_M, Y_M) - e\rho_3(X_M, Y_M))([J_2, J_1] + e[J_3, J_1]) \\
&= 2\nu(\rho_2(X_M, Y_M) - e\rho_3(X_M, Y_M))(J_3 + eJ_2).
\end{aligned}
$$

This shows that $\mathcal{H}_Z \subset TZ^\epsilon$ is a para-holomorphic contact structure if $\epsilon = +1$. □

8. Twistor construction of minimal submanifolds of para-quaternionic Kähler manifolds

8.1. Kähler and para-Kähler submanifolds of para-quaternionic Kähler manifolds.

DEFINITION 14. Let (M, g, Q) be a para-quaternionic Kähler manifold of dimension $4n$. An *ϵ-Kähler submanifold* ($\epsilon = \pm 1$) of M is a triple (N, J^ϵ, g_N), where N is a $2m$-dimensional g-nondegenerate submanifold of M, $g_N = g|_N$ is the induced pseudo-Riemannian metric and J^ϵ is a parallel section of the para-quaternionic bundle $Q|_N$ such that $J^\epsilon TN = TN$ and $(J^\epsilon)^2 = \epsilon \mathrm{Id}$. For $\epsilon = -1$ (M, J^ϵ, g_N) is called also a *Kähler submanifold* and for $\epsilon = +1$ it is called a *para-Kähler submanifold*.

We shall include J^ϵ into a local frame $(J_1 = J^\epsilon, J_2, J_3 = J_1 J_2 = -J_2 J_1)$ of $Q|_N$ such that $J_2^2 = \mathrm{Id}$. Such frames (J_α) will be called *adapted* to the ϵ-Kähler submanifold $N \subset M$.

Proposition 20. *Let (M, g, Q) be a para-quaternionic Kähler manifold of dimension $4n$ with non-zero reduced scalar curvature ν and N a g-nondegenerate submanifold* *of M endowed with a section $J^\epsilon \in \Gamma(N, Q)$ such that $(J^\epsilon)^2 = \epsilon \mathrm{Id}$ and $J^\epsilon TN = TN$. Let (J_α) be a standard local basis of Q such that $J_1|_N = J^\epsilon$. Then the triple (N, J^ϵ, g_N) is an ϵ-Kähler submanifold if and only if $\omega_2|_N = \omega_3|_N = 0$ or, equivalently, $J_2 TN \perp TN$. In particular, the dimension of an ϵ-Kähler submanifold $N \subset M$ is at most $2n$.*

Proof. It is clear that J_1 is parallel if and only if $\omega_2|_N = \omega_3|_N = 0$, see (3.2). Moreover, if $\omega_2|_N = \omega_3|_N = 0$, then, by the structure equation (3.3), we have that $\rho_2|_N = \rho_3|_N = 0$. Conversely, assume that $J_2 TN \perp TN$, i.e. $\rho_2|_N = \rho_3|_N = 0$. Differentiating the structure equations for ρ_2 and ρ_3, we get

$$\nu d\rho'_\alpha = -\epsilon_\alpha \nu \rho'_\beta \wedge \omega_\gamma + \epsilon_\alpha \nu \omega_\beta \wedge \rho_{\gamma'}.$$

Restricting this equation for $\alpha = 2, 3$ to the the submanifold N yields

$$\rho_1 \wedge \omega_2|_N = \rho_1 \wedge \omega_3|_N = 0.$$

This shows that $\omega_2|_N = \omega_3|_N = 0$, i.e. that $J^\epsilon \in \Gamma(N, Q)$ is parallel. □

Proposition 21. *The shape operator A of an ϵ-Kähler submanifold (N, J^ϵ, g_N) of a para-quaternionic Kähler manifold (M, g, Q) anticommutes with $J := J^\epsilon|_{TN}$.*

Proof. Let ξ be a normal vector field on N. Then the shape operator $A^\xi \in \Gamma(\mathrm{End}\, TN)$ is defined by

$$g(A^\xi X, Y) = -g(\nabla_X \xi, Y) = -g(\nabla_Y \xi, X) = g(\xi, \nabla_Y X).$$

Thus

$$\begin{aligned} g(A^\xi JX, Y) &= g(\xi, \nabla_Y(JX)) = g(\xi, J\nabla_Y X) = -g(J\xi, \nabla_Y X) = -g(J\xi, \nabla_X Y) \\ &= g(\xi, J\nabla_X Y) = g(\xi, \nabla_X(JY)) = g(A^\xi X, JY) = -g(JA^\xi X, Y). \end{aligned}$$ □

Corollary 7. *Any ϵ-Kähler submanifold of a para-quaternionic Kähler manifold is minimal.*

Proof. Since A^ξ anticommutes with J, we have $A^\xi = -JA^\xi J^{-1}$. Hence $\mathrm{tr}\, A^\xi = -\mathrm{tr}\, A^\xi = 0$. □

8.2. Twistor construction of Kähler and para-Kähler submanifolds of para-quaternionic Kähler manifolds. Let (M, g, Q) be a para-quaternionic Kähler manifold and $\pi_Z \colon Z^\epsilon \to M$ its ϵ-twistor space with the horizontal distribution $\mathcal{H}_Z$. For any ϵ-Kähler submanifold (N, J^ϵ, g_N) the section $J^\epsilon \colon N \to Z^\epsilon \subset Q$ defines an embedding of N into Z^ϵ. The image $\tilde{N} = J^\epsilon(N) \subset Z^\epsilon$ is called *the canonical lift* of

N in the twistor space Z^ϵ. The following theorem gives the description of ϵ-Kähler submanifolds of M in terms of ϵ-complex horizontal submanifolds of Z^ϵ, i.e. submanifolds $L \subset Z^\epsilon$ such that $\mathcal{J}^\epsilon TL = TL$ and $TL \subset \mathcal{H}_Z$.

Theorem 4. *Let* (N, J^ϵ, g_N) *be an* ϵ*-Kähler submanifold of a para-quaternionic Kähler manifold* (M, g, Q) *and* $\tilde{N} = J^\epsilon(N) \subset Z^\epsilon$ *its canonical lift. Then*
(i) $\tilde{N} \subset Z^\epsilon$ *is an* ϵ*-complex horizontal submanifold which is nondegenerate with respect to the canonical one-parameter family of metrics* g_t^ϵ *on* Z^ϵ*. Moreover, in the case* $\epsilon = +1$ *the restriction of* $\mathcal{J}^\epsilon$ *to* $\tilde{N}$ *is a para-complex structure in the strong sense.*
(ii) *Conversely, let* $L \subset Z^\epsilon$ *be an* ϵ*-complex horizontal submanifold which is nondegenerate with respect to* g_t^ϵ *and such that* $\pi_Z|_L\colon L \to \pi_Z(L) \subset M$ *is a diffeomorphism. Then its projection* $(N = \pi_Z(L), J^\epsilon, g_N)$ *is a (minimal)* ϵ*-Kähler submanifold of* M*, where*

$$J^\epsilon = d\pi_Z \circ \mathcal{J}^\epsilon \circ (d\pi_Z)^{-1}\colon TN \to TN, \quad g_N = g|_N.$$

Proof. (i) Since J^ϵ is parallel, the submanifold $\tilde{N} = J^\epsilon(N) \subset Z^\epsilon$, is horizontal. Its tangent bundle $T\tilde{N} \subset \mathcal{H}_Z$ is $\mathcal{J}^\epsilon$-invariant, since

$$d\pi_Z \circ \mathcal{J}^\epsilon = J^\epsilon \circ d\pi_Z,$$

on the horizontal distribution $\mathcal{H}_Z$, by the definition of $\mathcal{J}^\epsilon$, see (4.2). In the case $\epsilon = +1$, J^ϵ is a para-complex structure in the strong sense, because J^ϵ is skew-symmetric for the metric g_N. Since $(T_z\tilde{N}, \mathcal{J}_z^\epsilon|_{\tilde{N}}) \cong (T_xN, J_x^\epsilon)$, $x = \pi_Z(z)$, $\mathcal{J}^\epsilon$ restricts to a para-complex structure in the strong sense on $\tilde{N}$.

(ii) The ϵ-complex structure $J^\epsilon \in \Gamma(N, Q^\epsilon)$ is parallel, since $L = \tilde{N}$ is horizontal. This proves that (N, J^ϵ, g_N) is an ϵ-Kähler submanifold of M. □

REMARK. The nondegeneracy assumption on the metric $g_t^\epsilon|L$ is essential even if we assume that $\dim L = 2n$. Indeed there exist $2n$-dimensional J^ϵ-invariant isotropic subspaces $U \subset T_xM$.

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