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ON A GENERALIZATION OF THE "DIV-CURL LEMMA"

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Abstract

We present a generalization of the div-curl lemma to a Banach space framework which is not included in the almost existing generalizations. An example is shown where this generalization is needed.

1. The div-curl lemma and its generalization

In this note we present a generalization of the famous "div-curl lemma", which was first formulated by [1] in a Hilbert space setting. This lemma is widely used in the analysis of nonlinear partial differential equations. In [5] the result was generalized to an Banach space framework.

Here we present a further generalization to a setting, where on allows every component v_i^k, w_i^k of the vectors v^k, w^k to lie in different L^{p_i} spaces (for details see below). This is of special interest in problems arising from limiting procedures in the hydrodynamic equations for plasmas and semiconductors [3], where both the original version and the version presented in [5] are not sufficient for the analysis.

We denote by

$$(\text{curl}\, w)_{i,j} = w_{i,x_j} - w_{j,x_i} \tag{1}$$

the curl (matrix) of a vector field. The superscript $(\ )'$ indicates the conjugate index with $1 = 1/p + 1/p'$. Then, the generalized version of the "div-curl lemma" reads

Theorem 1.1. *Let $U \in \mathbb{R}^n$ be a bounded, open, smooth domain. Let $1 < p_i < \infty$ for $i = 1, \dots, n$ and let us denote $p_{\min} = \min_{1\le i\le n} p_i$ and $p_{\max} = \max_{1\le i\le n} p_i$. Let $v^k(x)$, $w^k(x) \in \mathbb{R}^n$ for $k \in \mathbb{N}$ satisfiing*

- *$\{v_i^k\}_{k=1}^\infty$ and $\{w_i^k\}_{k=1}^\infty$ are bounded sequences in $L^{p_i'}(U)$ and $L^{p_i}(U)$, respectively, with $1/p_{\min} - 1/n < 1/p_{\max}$.*
- *$\{\text{div}\, v^k\}_{k=1}^\infty$ lies in a compact set of $W^{-1,t}(U)$, where $t \ge \max_{1\le i\le n}(p_i') = (p_{\min})'$.*
- *$\{(\text{curl}\, w^k)_{i,j}\}_{k=1}^\infty$ lies in a compact set of $W^{-1,s_{i,j}}(U)$ for $1 \le i, j \le n$, where $\min_{1\le j\le n} s_{j,i} \ge p_i$ for $1 \le i \le n$.*

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Then, the following convergence holds in $\mathcal{D}'(U)$

$$v^k \cdot w^k \rightharpoonup v \cdot w, \tag{2}$$

where v, w *denote the weak limits* (*of subsequences*) *of* $\{v^k\}_{k=1}^{\infty}$ *and* $\{w^k\}_{k=1}^{\infty}$, *respectively.*

REMARK 1.1. Note that in the case $n \le p_{\min}$ the condition $1/p_{\min} - 1/n < 1/p_{\max}$ is always satisfied.

REMARK 1.2. The classical "div-curl lemma" is obtained by setting $p_i = p_i' = 2$ for $1 \le i \le n$. Then $\{\operatorname{div} v^k\}_{k=1}^{\infty}$ and $\{(\operatorname{curl} w^k)_{i,j}\}_{k=1}^{\infty}$ have to be in a compact set of $H^{-1}(U)$.

REMARK 1.3. As already mentioned in [5] a Banach space framework of the "div-curl" lemma is given. There, the case of $p_i = p$, $1 \le i \le n$ is covered.

Proof. The proof follows the ideas of the proof of the "div-curl lemma" given in [2].

In a frist step we define functions u_i^k, $1 \le i \le n$ as (unique) solutions of the problem

$$-\Delta u_i^k = w_i^k \quad \text{in} \quad U, \quad u_i^k = 0 \quad \text{on} \quad \partial U. \tag{3}$$

The u_i^k are uniformly bounded in $W^{2,p_i}(U)$ for $1 \le i \le n$.

In the second step we define the functions

$$z^k = -\operatorname{div} u^k, \quad y^k = w^k - \nabla z^k \tag{4}$$

with

$$y_i^k = w_i^k - z_{x_i}^k = \left(u^k_{(j,x_i)} - u^k_{i,x_j}\right)_{x_j} = ((\operatorname{curl} u^k)_{j,i})_{x_j}. \tag{5}$$

Therefore, $\{z^k\}_{k=1}^{\infty}$ is bounded in $W^{1,p_{\min}}(U)$ and compact in $L^r(U)$ with $1/p_{\min} - 1/n < 1/r \le 1$. The sequence $\{y_i^k\}_{k=1}^{\infty}$ is on one hand bounded in $L^{p_{min}}(U)$ (second derivatives of u^k), on the other hand compact in $L^{\min_{1\le j\le n}(s_{j,i})}(U)$. Indeed, according to the assumptions $(\operatorname{curl} w^k)_{j,i}$ is compact in $W^{-1,s_{j,i}}(U)$. Therefore, $(\operatorname{curl} u^k)_{j,i}$ lies compactely in $W^{1,s_{j,i}}(U)$ according to the results in [4] (at this point the smooth boundary (C^{∞}) is required).

Then, the limits z, y, u of z^k, y^k, u^k satisfy $z = -\operatorname{div} u$, $y = w - \nabla z$ and

$$-\Delta u_i = w_i \quad \text{in} \quad U, \quad u_i = 0 \quad \text{on} \quad \partial U. \tag{6}$$

Finally, using a testfunction $\Phi \in C_0^{\infty}(U)$ we obtain for $\min_{1\le j\le n}(s_{j,i}) \ge p_i$, $1 \le i \le n$

$$\int_U v^k \cdot y^k \Phi \, dx \to \int_U v \cdot y \Phi \, dx. \tag{7}$$

Similarly, for $t \geq (p_{\min})' = \max_{1\leq i\leq n}(p_i')$ we have

$$\int_U \operatorname{div} v^k z^k \Phi \, dx \to \int_U \operatorname{div} vz\Phi \, dx. \tag{8}$$

Due to the assumption $1/p_{\min} - 1/n < 1/p_{\max}$ there exist always values of r such that (where r is the parameter used above)

$$\frac{1}{p_{\min}} - \frac{1}{n} < \frac{1}{r} < \frac{1}{p_{\max}}. \tag{9}$$

Then, the second inequality (in (9)) garantuees the convergence

$$\int_U v^k \cdot \nabla\Phi z^k \, dx \to \int_U v \cdot \nabla\Phi z \, dx. \tag{10}$$

Combining (7)–(10) we obtain

$$\begin{aligned} \int_U v^k \cdot w^k \Phi \, dx &= \int_U v^k \cdot y^k \Phi \, dx - \int_U \operatorname{div} v^k z^k \Phi \, dx - \int_U v^k \cdot \nabla\Phi z^k \, dx \\ &\to \int_U v \cdot y\Phi \, dx - \int_U \operatorname{div} vz\Phi \, dx - \int_U v \cdot \nabla\Phi z \, dx \\ &= \int_U v \cdot w\Phi \, dx. \end{aligned} \tag{11}$$

This ends the proof. □

EXAMPLE. Suppose $n = 2$ and $p_1 < 2$ $(p_1' > 2)$, $p_2 > 2$ $(p_2' < 2)$. The curl matrix has two (nonvanishing) elements $(\operatorname{curl} w^k)_{1,2} = -(\operatorname{curl} w^k)_{2,1}$. Then $t > p_1'$ and $s_{1,2} > p_2$ are required in order to apply the theorem. This is an examples where the results in [5] do not apply.

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