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CASSON-WALKER INVARIANTS OF CYCLIC COVERS BRANCHED ALONG SATELLITE KNOTS

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Abstract

We express the Casson-Walker invariants for the cyclic covering spaces of the three-dimensional sphere branched along satellite knots in terms of companions, patterns, and winding numbers.

1. Introduction

Let C be a knot in S^3, and K a knot in a tubular neighborhood $N(C)$ of C. Let γ be a generator of the kernel of $i_*\colon H_1(\partial N(C);\mathbb{Z}) \to H_1(S^3 - \mathring{N}(C);\mathbb{Z})$, where i is the inclusion. We regard γ as a simple closed curve on $\partial N(C)$. Then, there is a unique embedding $f\colon N(C) \to S^3$, up to isotopy, such that the exterior $E(f(C)) = S^3 - f(\mathring{N}(C))$ is the solid torus and $f(\gamma)$ bounds a disk in $E(f(C))$. Denote by C_0 the core circle of $E(f(C))$. The knot $P = f(K)$ is called a *pattern knot* for K associated to the companion C. The 2-component link $P \cup C_0$ in S^3 is called a pattern link, and $w = |\mathrm{lk}(P, C_0)|$ is called the *winding number* (cf. Fig. 1). We denote by Σ^r_K the r-fold cyclic covering space over S^3 branched along a knot K. In this paper, we present the Casson-Walker invariant of the r-fold cyclic covering space Σ^r_K of S^3 branched along a satellite knot K in terms of patterns, companions, and the winding numbers.

A Laurent polynomial $\Lambda(t) \in \mathbb{Z}[t, t^{-1}]$ is called a *knot-Alexander polynomial* provided that $\Lambda(t^{-1}) = \Lambda(t)$ and $\Lambda(1) = 1$. Namely, a knot-Alexander polynomial can be written as a finite sum $\Lambda(t) = c_0 + \sum_{i>0} c_i(t^i + t^{-i})$, where $c_i \in \mathbb{Z}$ and $c_0 = 1 - 2\sum c_i$. The Alexander polynomial $\Delta_K(t)$ of a knot K in a homology sphere is a knot-Alexander polynomial. Conversely, given a knot Alexander polynomial $\Lambda(t)$ there is a knot K with $\Delta_K(t) = \Lambda(t)$. We also use Conway's version of the Alexander polynomial $\nabla_K(z) = 1 + \sum_{i>0} a_{2i} z^{2i}$, where we denote by $a_{2i}(K)$ the $2i$-th coefficient of the Conway polynomial $\nabla_K(z)$, which is equivalent to $\Delta_K(t)$ via $z = t^{-1/2} - t^{1/2}$. Note that $a_2(K) = 1/2\Delta''_K(1)$, where $\Delta''_K(t)$ is the second derivative of $\Delta_K(t)$. Note also that when $|H_1(\Sigma^r_K;\mathbb{Z})|$ is finite, the order is determined by the Alexander polynomial as $|H_1(\Sigma^r_K;\mathbb{Z})| = \left|\prod_{i=0}^{r-1} \Delta_K(\xi^i)\right|$, where ξ is the rth primitive root of unity. Recall

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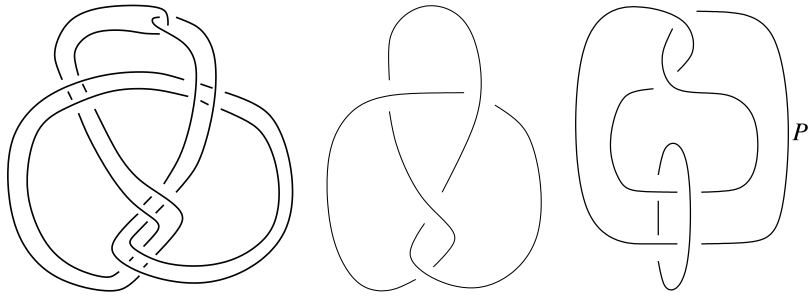


Fig. 1.

that H. Seifert [19] showed the equation $\Delta_K(t) = \Delta_P(t)\Delta_C(t^w)$ between the Alexander polynomials for a knot K with a companion C and a pattern P.

A compact orientable 3-manifold M is called a *rational homology circle* if the homology group $H_*(M; \mathbb{Q})$ is isomorphic to that of the standard solid torus $D^2 \times S^1$. Note that each rational homology circle is homeomorphic to the exterior $E(K, H)$ of a knot K in a rational homology sphere H and the symmetrized Alexander polynomial $\Delta_K(t) \in \mathbb{Q}[t, t^{-1}]$ is naturally defined so that $\Delta_K(1) = 1$ (cf. [23]) and $a_2(K) = 1/2\Delta_K''(1)$.

In 1985, A. Casson introduced an integer valued invariant λ for integral homology spheres that counts the number of the conjugacy classes of irreducible $SU(2)$-representations of the fundamental group in some sense (cf. [1], [18]). For two homology spheres H and H', if H' is obtained from H by the $(1/n)$-surgery along a knot K in H, then $\lambda(H') = \lambda(H) + n/2\Delta_K''(1)$. The Casson invariant λ was extended to rational homology spheres by K. Walker [23], who provided a method of defining this invariant in a combinatorial way. Casson's surgery formula is generalized as $\lambda(M') = \lambda(M) + q/pa_2(K \subset M) + \lambda(L(p, q))$, where M' is obtained from M by the q/p-surgery on a knot K in M, and the value of the Casson-Walker invariant of the Lens space $L(p, q)$ is written as the Dedekind sum. C. Lescop [13] extended the Casson-Walker invariant to all closed 3-manifolds.

To state our theorem, we need the following notation: For a 2-component link $l \cup k$, we define $\alpha_l^r(k)$ as follows: Let k^+ denote the preferred longitude for k. Let $\psi : \Sigma_l^r \to S^3$ be the r-fold cyclic cover branched along l. Let k' denote a component of $\psi^{-1}(k)$ and k'^+ the component of $\psi^{-1}(k^+)$ corresponding to k^+. Then, when Σ_l^r is a rational homology sphere, $\mathrm{lk}(k'^+, k)$ is an integer and we put $\alpha_l^r(k) = \mathrm{lk}(k'^+, k)$. It is easy to see that $|\alpha_l^{r>1}(K)| = 2$ for the Whitehead link $W = k \cup l$. Then we show that $\lambda(\Sigma_K^r)$ is written in terms of P and C as follows:

Theorem 1.1. *For a satellite knot K with a pattern P associated to a companion C whose winding number is w, where $\mu = \gcd(r, w)$, $\lambda(\Sigma_K^r) = \lambda(\Sigma_P^r) + \mu\lambda(\Sigma_C^{r/\mu}) - \mu\alpha_P^r(C_0)a_2(\tilde{C} \subset \Sigma_C^{r/\mu})$ when Σ_K^r is a rational homology sphere.*

We notice that Σ^r_K is a rational homology sphere if and only if each of Σ^r_P and $\Sigma^{\mu/r}_C$ is a rational homology sphere. As a special case, we have the following corollary:

Corollary 1.2. *For a satellite knot K with a pattern P associated to a companion C whose winding number is zero, $\lambda(\Sigma^r_K) = \lambda(\Sigma^r_P) - r\alpha^r_P(C_0)a_2(C)$ when Σ^r_P is a rational homology sphere.*

Several authors investigate the Casson-Walker invariant of Σ^r_K for some satellite knots. A satellite knot is called an *untwisted doubled knot* if $P \cup C_0$ forms the Whitehead link. J. Hoste proved that for untwisted doubled knots $\mathcal{D}P$ about P in S^3, $\lambda(\Sigma^r_{\mathcal{D}P}) = 2a_2(P)r$ [9, Theorem 3.2]. In [21], Yasuyoshi Tsutsumi generalized this result for satellite knots whose patterns are two-bridge knots and companions are based on certain Conway's rational tangles. See Ishibe [10] for general doubled knots. In [6], Fujita gave a formula for the Casson-Walker invariant of the manifold obtained from two rational homology circles by gluing their boundaries. See [17] for a simple proof. See also [7] and [3].

D. Mullins [14] gave a relation among the first derivative of the Jones polynomial at -1 $J'_K(-1)$, the Casson-Walker invariant of the double branched cover $\lambda(\Sigma^2_K)$ and the ordinal signature $\sigma(K)$. Shinohara [20] gave a relation between the signatures of satellite knots and the companion, patterns. It is easy to see that when the winding number $w = 0$, K and the pattern have the same signature. Combining Corollary 1.2 with Mullins' result, we have the following:

Corollary 1.3. $J'_K(-1) = J'_P(-1) - 6r\alpha^2_P(C_0)a_2(C)$ *if the winding number is zero.*

There are several relation between $\alpha^r_l(k)$ and some other invariants. In fact, $\alpha^2_l(k)$ is related to Cochran's beta-invariant [4] of the 2-component link and the derived links [2]. One also notices that $\alpha^r_l(k)$ can be derived from Kojima-Yamasaki's function [12] and equivariant linking numbers. In §3, we show the following:

Proposition 1.4. *Let $N \geq 2$ be a natural number greater than one. Given N integers $\lambda_2, \lambda_3, \ldots, \lambda_N, \lambda$, there are knots K in S^3 such that $\Delta_K(t) = 1$ and*

$$\lambda(\Sigma^r_K) = \begin{cases} 2r\lambda_i & \text{if } 2 \leq r \leq N \\ 2r\lambda & \text{if } N < r \end{cases}.$$

We note that Collin and Saveliev [5, Theorem 1] showed that the equivariant Casson invariant $\lambda^{\mathbb{Z}/n}$ of Σ^n_K is determined by a Tristram-Levine equivariant signature of K, that is, the signature of Hermitian form $(1 - e^{2\pi ik/n})S + (1 - e^{-2\pi ik/n})S^{\mathrm{T}}$ for $k = 0, 1, \ldots, n-1$, where S is a Seifert form of K. This implies that $\lambda^{\mathbb{Z}/n}(\Sigma^n_K) = 0$ if $\Delta_K(t) = 1$ since any knots with trivial Alexander polynomials are S-equivalent to the unknot. They also showed that $\lambda^{\mathbb{Z}/n}(\Sigma^n_K) = \lambda(\Sigma^n_K)$ for a graph knot K [5, Theorem 3].

Hence, most knots constructed in the proof of Proposition 1.4 cannot be a graph knot, and the difference between $\lambda^{\mathbb{Z}/n}(\Sigma^n_K)$ and $\lambda(\Sigma^n_K)$ is due to the variation of α^n_* for certain links. According to Garoufalidis and Kricker [8], this difference can be written in terms of residues of the 2-loop polynomials related to the Kontsevich integral of the knot.

2. Proof of Theorem 1.1

Let V be a rational homology circle. Then, ∂V is a single torus and a generator of the kernel of $i_*\colon H_1(\partial V;\mathbb{Z}) \to H_1(V;\mathbb{Z})$ is called a *longitude for* V, where $i\colon \partial V \to V$ is the inclusion. There are a primitive element $\rho \in H_1(\partial V;\mathbb{Z})$ and a positive integer $\delta(V)$ such that $\delta(V)\rho \in H_1(\partial V;\mathbb{Z})$ is the longitude for V. Then, there is a properly embedded orientable surface S in V such that $[\partial S] = \delta(V)\rho$ in $H_1(\partial V;\mathbb{Z})$ and ∂S consists of $\delta(V)$ components. It is well-known that $\delta(V)$ is unique and ∂S is also unique up to isotopy. Such an S is called a *characteristic surface for* V.

A *meridian* γ for a rational homology circle V is a non-trivial simple closed curve (or its isotopy class) on ∂V which intersects the longitude ρ transversely in $\delta(V)$ points. A rational homology circle V equipped with a pair (ρ, γ) on ∂V is called a *framed homology circle*, where the pair (ρ, γ) is also regarded as a pair of homology classes in $H_1(\partial V;\mathbb{Z})$. We denote by $\hat{V}$ the rational homology sphere obtained from V by attaching a solid torus $D^2 \times S^1$ so that the boundary of a meridian disk $\partial D^2 \times \{*\}$ is identified with γ. The linking number $\mathrm{lk}(K_1, K_2)$ for a link $K_1 \cup K_2$ is defined as (the number of algebraic intersections of K_1 with a characteristic surface of $E(K_2)$)$/\delta(E(K_2))$. Then, the linking number $\mathrm{lk}(\,\cdot\,,\,\cdot\,)$ for two disjoint oriented knots in V and the Alexander-Conway polynomials of knots in V are well-defined as in $\hat{V}$. And, we put $\lambda(V) = \lambda(\hat{V})$. We note that, if $M = V \cup V'$ is a rational homology sphere obtained from V by gluing a rational homology circle V' so that the meridian of V' is identified with the longitude of V, then the linking numbers of links in $V \subset M$ and the Alexander polynomials of knots in $V \subset M$ coincide with that of $\hat{V}$.

Let K be an oriented knot in V, and K^+ the preferred longitude for K. That is, K^+ is an oriented knot in $\partial N(K)$ with $\mathrm{lk}(K, K^+) = 0$. We denote the *winding number* by $w = w(K)$ which is defined as the absolute value of the algebraic intersection number of K with a characteristic surface S, which is independent of the choice of S. For a framed rational homology circle C in a rational homology sphere (or in a framed rational homology circle), we denote by $\mathrm{fr}(C)$ the framing number of C, namely the linking number of the framing γ and the knot γ^+ parallel to γ in ∂C. Notice that $\mathrm{fr}(C)$ is an integer. The following lemma can be shown by Fujita's formula [6] and Walker's surgery formula [23].

Lemma 2.1. *Let H be a rational homology sphere, and C a framed rational homology circle with $\delta(C) = 1$ embedded in H. Let V be a framed rational homology circle with $\delta(V) = 1$. Denote by M the manifold obtained from H by replacing C with*

V by a meridian-longitude preserving map. Then, $\lambda(M) = \lambda(H) + \lambda(V) - \mathrm{fr}(C)a_2(V)$.

Proof. Let M' denote the homology sphere obtained from H by replacing C with V by a longitude preserving map such that the image of the meridian of V is preferred in M'. Denote by (γ, ρ) the meridian-longitude pair of V in M'. Notice that $M = \chi(M'; (\gamma, -1/\mathrm{fr}(C)))$. By Walker's surgery formula, we have that $\lambda(M) = \lambda(M') - \mathrm{fr}(C)a_2(\gamma)$. Since γ bounds a characteristic surface as a Seifert surface in M', we see that $\Delta_{\gamma \subset M'}(t) = \Delta_V(t)$ and thus $a_2(\gamma \subset M') = a_2(V)$. On the other hand, it follows from Fujita's splicing formula [6] that $\lambda(M') = \lambda(H) + \lambda(V)$. Now we have that $\lambda(M) = \lambda(H) + \lambda(V) - \mathrm{fr}(C)a_2(V)$. This completes the proof. □

Let V be a rational homology circle with $\delta(V) = 1$. Then, the r-fold cyclic covering space V^r of V is obtained from r copies of $V - S$ by gluing up cyclically, where S is a characteristic surface for V with a single boundary. Denote by $\varphi_r : V^r \to V$ the covering projection. Let ρ_r denote a component of $\varphi_r^{-1}(\rho)$ for the longitude ρ of V. Notice that $\gamma_r = \varphi_r^{-1}(\gamma)$ is connected and intersects ρ_r transversely in a single point. Then, the pair (γ_r, ρ_r) is regarded as a meridian-longitude pair on ∂V^r. Hereafter we assume that V^r is a homology circle. Let K be a knot in V. It is elementary to see that $\varphi_r^{-1}(K)$ consists of $\mu_r = \gcd(w, r)$ components $K_{r,1}, \ldots, K_{r,\mu_r}$. We number them so that $K_{r,i}$ is mapped to $K_{r,i+1}$ by a natural generator of the group of covering translations $V^r \to V^r$, that is, $K_{r,i+1}$ is next to $K_{r,i}$. Then, $\varphi_r|_{K_{r,i}} : K_{r,i} \to K$ is an r/μ-fold cyclic cover. For the preferred longitude K^+ for K, we denote by $K^+_{r,i}$ the component of $\varphi_r^{-1}(K^+)$ that corresponds to $K_{r,i}$. Here $\gcd(0, r) = r$ and 0-fold cyclic covers mean the infinite cyclic covers. Put $\alpha_V^r(K) = \mathrm{lk}(K_{r,i}, K^+_{r,i})$, which is independent of the choice of i.

For a 2-component link $K \cup k$, we regard the exterior $E(k, S^3)$ as a framed solid torus, K as a knot in $E(k, S^3)$. Then we put $\alpha_k^r(K) = \alpha^r_{E(k,S^3)}(K)$.

Notice that Σ^r_K is a rational homology sphere if and only if each of $\Sigma_C^{r/\mu}$, Σ^r_P is a rational homology sphere. Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We denote by $\psi_K : \Sigma^r_K \to S^3$, $\psi_P : \Sigma^r_P \to S^3$ the covering projections. Note that C_0 is regarded as a knot in the homology circle $E(P, S^3)$ with winding number w, and note also that $E(\psi_P^{-1}(P), \Sigma^r_P)$ is the r-fold cyclic covering space of $E(P, S^3)$. Hence, $\psi_P^{-1}(N(C_0, S^3))$ is the union of $\mu = \gcd(w, r)$ solid tori $U_1, \ldots, U_\mu$ and $\psi_P|_{U_i} : U_i \to N(C_0, S^3)$ is the r/μ-fold cyclic cover. Note that $E(K, S^3)$ is obtained from $E(P, S^3)$ by replacing the solid torus $N(C_0, S^3)$ with the homology circle $E(C, S^3)$ via a meridian-longitude preserving map. Then, Σ^r_K is obtained from Σ^r_P by replacing each solid torus U_i with a copy U'_j of the r/μ-fold cyclic covering space $E(\tilde{C}, \Sigma_C^{r/\mu})$ of $E(C, S^3)$ via a meridian-longitude preserving map, where $\psi_C : \Sigma_C^{r/\mu} \to S^3$ is the r/μ cyclic branched cover along C and $\tilde{C} = \psi_C^{-1}(C)$. Recall that $\mathrm{fr}(U_i) = \alpha_P^{r/\mu}(C_0)$. By applications of Lemma 2.1, we have that $\lambda(\Sigma^r_K) =$

$\lambda(\Sigma_P^r) + \mu\lambda(\Sigma_C^{r/\mu}) - \mu\alpha_P^{r/\mu}(C_0)a_2(\tilde{C})$. This completes the proof. □

3. Proof of Proposition 1.4

Let $l \cup k(m, n)$ be the link illustrated in Fig. 2. Notice that $k(m, n)$ is trivial in S^3 and $\mathrm{lk}(l, k(m, n)) = 0$. Then, it is direct to see that

$$\alpha_l^r(k(m, n)) = \begin{cases} 0 & (r \mid n) \\ -2m & (r \nmid n) \end{cases}.$$

In particular, $\alpha_l^n(k(m, n)) = 0$ and $\alpha_l^{r>n}(k(m, n)) = -2m$.

Let $l \cup K(m, n)$ denote the link obtained from $l \cup k(m, n)$ by replacing $k(m, n)$ with the untwisted Whitehead double about $k(m, n)$. See Fig. 3-(1) for $l \cup K(-1, 1)$. Note that $K(m, n)$ is also a trivial knot in S^3. Note also that $l \cup K(m, n)$ is a boundary link. Let $\bar{K}^r(m, n)$ denote a component of $\psi_r^{-1}(K(m, n))$, where $\psi_r : S^3 \to S^3$ is the r-fold cyclic cover branched along l. Then,

$$\Delta_{\bar{K}^r(m,n)}(t) = \begin{cases} 1 & (r \mid n) \\ 1 + 2m(t^{1/2} - t^{-1/2})^2 & (r \nmid n) \end{cases}.$$

Put $K^*(m, n) = \chi(l;(K(m, n), -1)) \sharp \chi(l;(K(-1, 1), 1))$. We see that $\Delta_{K^*(m,n)}(t) = 1$ since each factor is obtained from a trivial knot l and $l \cup K(m, n)$ is a boundary link. Further we have that $\Sigma^r_{K^*(m,n)} = \Sigma^r_{\chi(l;(K(m,n),-1))} \sharp \Sigma^r_{\chi(l;(K(-1,1),1))}$ and

$$\lambda(\Sigma^r_{K^*(m,n)}) = \begin{cases} 2mr & (r \mid n) \\ 0 & (r \nmid n) \end{cases}$$

by the additivity of the Casson invariant.

Now we have the following claim.

Claim 3.1. *For an integer number $n \geq 2$, there is a knot K_n such that $\Delta_{K_n}(t) = 1$, $\lambda(\Sigma^n_{K_n}) = 2n$, $\lambda(\Sigma^{r>n}_{K_n}) = 0$ and $\lambda(\Sigma^{1<r<n}_{K_n})$ is divisible by $2r$.*

Proof of Proposition 1.4. By taking connected-sums of $K^*(m, n)$'s according to the given integers for $n = N, N - 1, N - 2, \ldots, 2$, we get a desired knot. □

Since the exterior $E(K)$ of a knot K in S^3 is a Haken manifold, $E(K)$ is uniquely decomposed into hyperbolic manifolds and Seifert fibered spaces by characteristic tori. Hence, if we define the volume $\mathrm{vol}(K)$ of K as the sum of volumes of the hyperbolic manifolds in the JSJ-family, it is a topological invariant of a knot. Then, we can define a graph knot as that of whose volume is zero. The construction of knots in the proof of Proposition 1.4 always yields satellite knots. There are several ways to prove the next proposition. Here we refer the reader to Kawauchi's techniques [11] or Myers' gluing lemma [15].

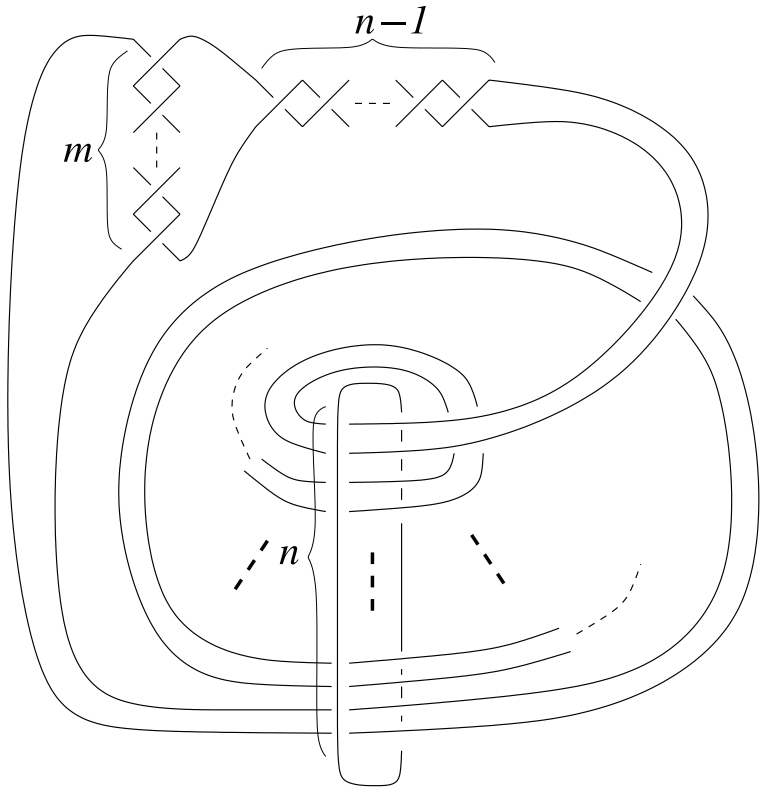
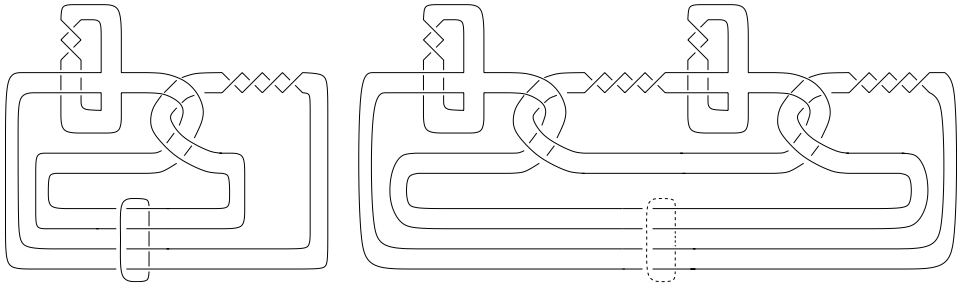


Fig. 2. $l \cup k(m, n)$.



(1): $l \cup K(-1, 1)$. (2): $\psi_2^{-1}(K(-1, 1))$.

Fig. 3.

Proposition 3.2. *Given a knot K in S^3, there are infinitely many hyperbolic knots K^{**} such that K^{**} have the same Seifert form as K, $\Sigma^r_{K^{**}}$ is hyperbolic, and $\lambda(\Sigma^r_{K^{**}}) = \lambda(\Sigma^r_K)$ for any r.*

Proof of Proposition 3.2. Let S_0 be a Seifert surface for $K_0 = K$ of positive genus. Take two disjoint trivial knots K_1, K_2 in $E(S_0)$ so that K_1 and K_2 bound disjoint disks in $E(S_0)$, and take disjoint genus one Seifert surfaces S_1, S_2 contained in disjoint 3-balls B_1 and B_2 for K_1, K_2. Let Γ_0, Γ_1, Γ_2 be spines of S_0, S_1, S_2. Then, $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2$ forms a 3-component graph embedded in S^3. By Kawauchi [11, Theorem 1.1] there are graphs $\Gamma_0^* \cup \Gamma_1^* \cup \Gamma_2^*$ such that $E(\Gamma_0^* \cup \Gamma_1^* \cup \Gamma_2^*)$ is hyperbolic and $\Gamma_0^* \cup \Gamma_1^* \cup \Gamma_2^* - \Gamma_i^*$ is isotopic to $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2 - \Gamma_i$ for $i = 0, 1, 2$. We construct Seifert surfaces S_0^*, S_1^*, S_2^* along Γ_i^*'s for knots $K_0^* = \partial S_0^*$, $K_1^* = \partial S_1^*$, $K_2^* = \partial S_2^*$ corresponding to S_0, S_1, S_2 respectively such that $K_0^* \cup K_1^* \cup K_2^* - K_i^*$ is equivalent to $K_0 \cup K_1 \cup K_2 - K_i$. Since $K_0^* \cup K_1^* \cup K_2^*$ bounds a disconnected Seifert surface $S_0^* \cup S_1^* \cup S_2^*$ whose exterior is hyperbolic and since each of the components has positive genus, it follows from [22, Proposition 3.1] that $K_0^* \cup K_1^* \cup K_2^*$ is a hyperbolic link in S^3. By Thurston's hyperbolic surgery theorem [16], twistings along K_1^* and K_2^* produce infinitely many hyperbolic knots K^{**} of distinct volumes such that $E(S_0^{**})$ is hyperbolic, where S_0^{**} is the Seifert surface obtained from S_0^* by twisting along K_1^* and K_2^*. Then $\Sigma^r_{K^{**}}$ is hyperbolic by [22, Proposition 3.2]. We shall show that the sequence of the knots K^{**} is a desired one. Let $\psi\colon \Sigma^r_K \to S^3$ be the r-fold cyclic branched cover, and $\phi\colon \Sigma^r_{K^*} \to S^3$ the r-fold cyclic branched cover. Note that $\Sigma^r_{K^{**}}$ is obtained from Σ^r_K by $1/n$-surgeries on all components of $\phi^{-1}(K_1^* \cup K_2^*)$. Since $S_1^* \cup S_2^*$ is disjoint from S_0^*, $\phi^{-1}(S_1^* \cup S_2^*)$ consists of r-copies of $S_1^* \cup S_2^*$. Since S_i^* is contained in a 3-ball which lifts to Σ^r_K, we see that $\phi^{-1}(K_1^* \cup K_2^*)$ is a boundary link such that the Alexander polynomial of each component is trivial. Then we see that $\lambda(\Sigma^r_{K^{**}}) = \lambda(\Sigma^r_K)$. This completes the proof. □

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