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UNKNOTTING SINGULAR SURFACE BRAIDS BY CROSSING CHANGES

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Abstract

C.A. Giller defined a crossing change for surfaces in 4-space, and proved an unknotting theorem. In this paper, we present such an unknotting theorem for singular surface braids, extending S. Kamada's result for those without branch points. As a consequence, we recover Giller's unknotting theorem. We also study finite type invariants for singular surface braids associated with the crossing changes.

1. Introduction

A *surface braid* was introduced by O. Viro [12] and has extensively studied by S. Kamada [9]. L. Rudolph introduced a similar notion earlier in [11]. By a *singular surface braid*, we mean an immersed (closed) surface braid (see §2). We call a transverse double point of a singular surface braid a *crossing point*. By a *crossing change*, we mean an operation for a singular surface braid S inserting a pair of positive and negative crossing points along a chord that is a straight segment connecting adjacent sheets of S (cf. [4, 6]). In this paper, we present an unknotting theorem (Theorem 4.1 in §4) for singular surface braids, which was proved by Kamada [4] for those without branch points. C.A. Giller [2, 8] proved that such an unknotting theorem for surfaces in Euclidean 4-space. We recover Giller's unknotting theorem in Corollary 4.2. In §5, we also study finite type invariants for singular surface braids associated with crossing changes. These invariants are completely determined by the number of sheets, the Euler characteristic and the numbers of (signed) crossing points for each component (Theorem 5.2).

2. Singular surface braids, chart descriptions and C-moves

Let D_1 be an oriented 2-disk and let X_m be a fixed set of m interior points of D_1. Let U_0 be the standard 2-sphere $U_0 = \{(x, y, z, w) \in \mathbf{R}^4 \mid x^2 + y^2 + z^2 = 1,\ w = 0\}$ in $\mathbf{R}^4$ with a base point q_0. We denote by $\mathrm{pr}\colon D_1 \times U_0 \to U_0$ the second factor projection. Let S be a compact oriented immersed surface in $D_1 \times U_0$. Then S is called a (*closed*) *singular surface m-braid* if the following conditions are satisfied: (i) Singularities of S are crossing points, (ii) for an immersion $f\colon F \to D_1 \times U_0$ associated with S, the composition $\mathrm{pr} \circ f$

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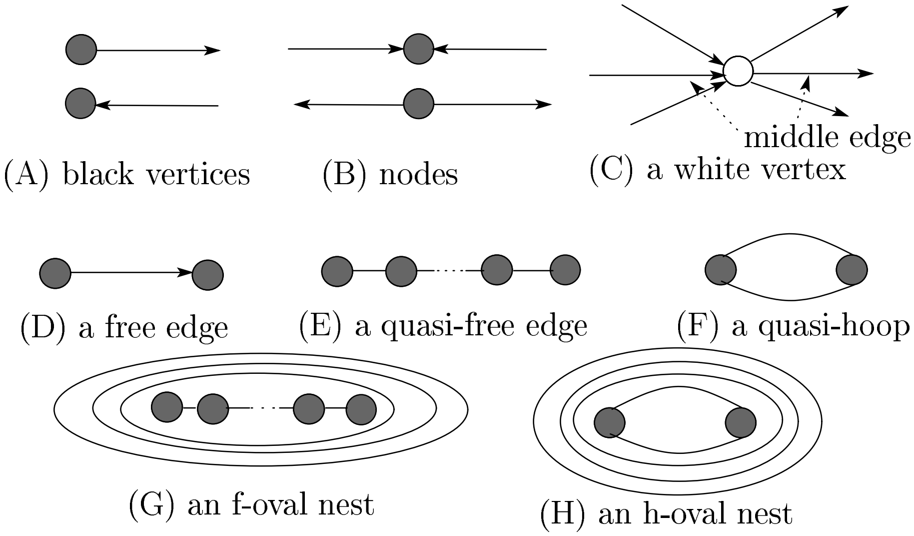


Fig. 1.

is a simple m-fold branched covering map, i.e. for each $y \in U_0$, $\#\{\mathrm{pr}^{-1}(y) \cap S\} = m - 1$ or m. We may assume that $\mathrm{pr}^{-1}(q_0) \cap S = X_m$. Two singular surface m-braids S and S' are *equivalent* if they are ambiently isotopic by a fiber-preserving isotopy $\{h_u\}_{0 \leq u \leq 1}$ of $D_1 \times U_0$, as a D_1-bundle over U_0. A singular surface m-braid is *trivial* if it is equivalent to $X_m \times U_0$. Let $S^1, \dots, S^l$ be components of a singular surface braid S, that is, each S^i is the image $f(F^i)$ of a connected component F^i of F. A (k_1, k_2)-*crossing point* is a crossing point of S^{k_1} and S^{k_2}. In particular, a (k, k)-crossing point is a self-crossing point of S^k.

An m-*chart* Γ is a (possibly empty) finite immersed graph in an oriented 2-sphere U_0 with a base point q_0, which may have *hoops* (that are closed edges without vertices), satisfying the following conditions:

(i) Every vertex has degree one, two or six.
(ii) Every edge is directed, and labeled by an integer in $\{1, 2, \dots, m - 1\}$.
(iii) For each vertex of degree six, three consecutive edges are directed inward and the other three are directed outward; these six edges are labeled by i and $i + 1$ alternately for some i.
(iv) For each vertex of degree two, the two edges are labeled by the same integer and oppositely directed.
(v) Each singularity of Γ is a transverse double point of two edges whose difference in labels is more than one.
(vi) $\Gamma \cap \{q_0\} = \emptyset$.

A vertex of degree one, two or six is called a *black vertex*, a *node* or *a white vertex*, respectively (Fig. 1 (A)–(C)). In [6], Kamada gave a method to present a singular surface m-braid by an m-chart Γ. Black vertices, nodes or white vertices in a chart Γ represent branch points, crossing points or triple points in a diagram of a singular surface m-braid. A node whose adjacent edges are directed inward (or outward) is called a *positive* (or *negative*) *node*. The set of black vertices in Γ is denoted by B_Γ. An edge attached to a white vertex is called a *middle edge* if it is the middle of the three consecutive edges which are oriented in the same directions; otherwise a *non-middle edge*. A *free edge* is an edge both endpoints of which are black vertices (Fig. 1 (D)). A *quasi-free edge* is a smooth arc in a chart whose endpoints are black vertices and the other vertices on it are nodes (Fig. 1 (E)). A quasi-free edge is called *positive* (or *negative*) if the number of positive (or negative) nodes is larger than that of negative (or positive) nodes. A *quasi-hoop* is a simple loop in a chart with two nodes and no other vertices (Fig. 1 (F)). We regard a free edge as a quasi-free edge, but do not regard a hoop as a quasi-hoop. An *f-oval nest* (or *h-oval nest*) is a quasi-free edge (or quasi-hoop) together with some concentric hoops (Fig. 1 (G)–(H)). We always assume that the base point q_0 is outside f-oval nests and h-oval nests.

Let S_1 and S_2 be singular surface m-braids presented by m-charts $\Gamma_1 \subset U_0$ and $\Gamma_2 \subset U_0'$ with base points q_0 and q_0', respectively. The *product* of Γ_1 and Γ_2, denoted by $\Gamma_1 \bullet \Gamma_2$, is an m-chart obtained by identifying the boundaries of $U_0 \setminus \operatorname{Int} N(q_0)$ and $U_0' \setminus \operatorname{Int} N(q_0')$ for neiborhoods $N(q_0) \subset U_0$ and $N(q_0') \subset U_0'$ in such a way that $\overline{N(q_0)} \cap \Gamma_1 = \emptyset$ and $\overline{N(q_0')} \cap \Gamma_2 = \emptyset$. Then, we set a base point of $\Gamma_1 \bullet \Gamma_2$ on the identified boundaries. The *product* of S_1 and S_2, denoted by $S_1 \bullet S_2$, is a singular surface m-braid presented by an m-chart $\Gamma_1 \bullet \Gamma_2$.

REMARK 2.1. A crossing change of a singular surface braid corresponds to insertion of a quasi-hoop in a chart. See [6]. Thus, in this paper, a crossing change also means insertion of a quasi-hoop in a chart.

Operations listed below (and their inverses) are called a C_{I}-, C_{II}-, C_{III}-, C_{IV}- and C_{V}-*move*, respectively. These moves are called C-*moves*. Two m-charts are C-*move equivalent* if they are related by a finite sequence of such C-moves and ambient isotopies.

(C_{I}) For a 2-disk E on U_0 such that $\Gamma \cap E$ has neither black vertices nor nodes, replace $\Gamma \cap E$ with an arbitrary chart that has neither black vertices nor nodes.

(C_{II}) Suppose that an edge α is attached to a black vertex B and intersects another edge β near B. Shorten α to remove the intersection and move B across β.

(C_{III}) Let a black vertex B and a white vertex W be connected by a non-middle edge α of W. Remove α and W, attach B to the edge of W opposite to α, and connect other four edges in a natural way.

(C_{IV}) Let N be a node attached by edges α_1, α_2 and suppose that α_1 intersects an edge β near N. Move N across β.

(C_{V}) Let a node N and a white vertex W be connected by a non-middle edge of W. Move N across W.

We illustrate examples of C_{I}-moves in Fig. 2 and C_{II}–C_{V}-moves in Fig. 3.

Lemma 2.2 (cf. [5, 6, 9]). *Two m-charts are C-move equivalent if and only if their presenting singular surface m-braids are equivalent.*

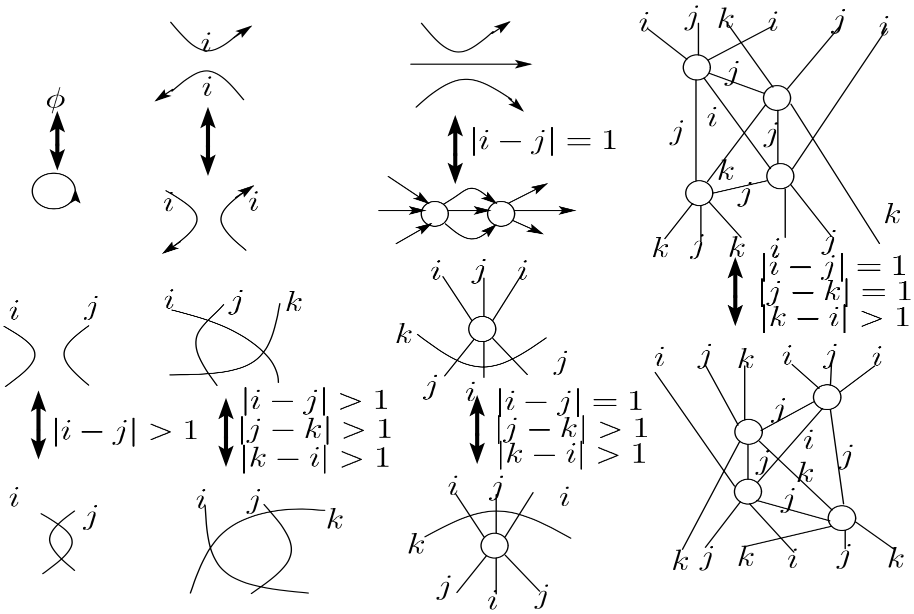


Fig. 2. some C_{I} moves

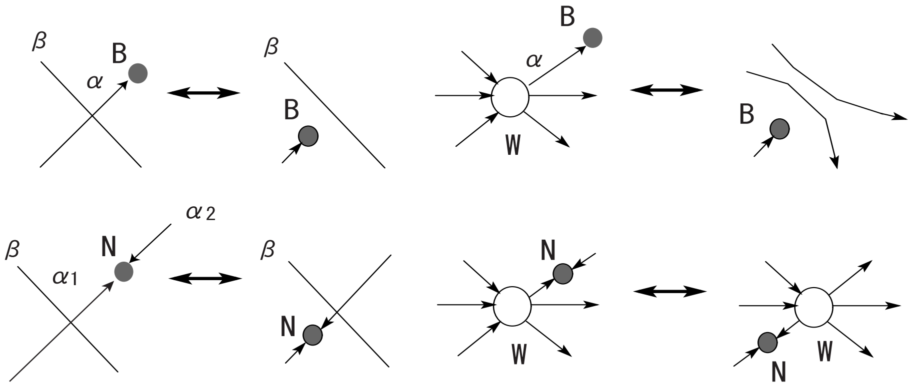


Fig. 3. C_{II}–C_{V} moves

3. Factorization graph of singular surface braids

Here, we introduce the notation of a *factorization graph* [1] to see which sheets of a singular surface braid are connected. It will be useful in proving Theorem 4.1.

Let S be a singular surface m-braid presented by an m-chart Γ and U_0, q_0 and B_Γ be as in §2. A *Hurwitz arc system* of Γ is $H = (a_1, \ldots, a_n)$ with $n = |B_\Gamma|$ such that for $1 \leq i \leq n$, a_i is a simple path intersecting Γ transversely (missing all the vertices except at the initial points) such that

(i) q_0 is the terminal point of a_i for each i,
(ii) the intersection of the images of a_i and a_j is q_0 for $i \neq j$,
(iii) the images of $a_1, \ldots, a_n$ appear in this order around the point q_0,
(iv) the initial points are in B_Γ.

For each i, consider a loop η_i in $U_0 \setminus B_\Gamma$ with base point q_0 such that it goes along a_i, turns around the initial point of a_i in positive direction and comes back along a_i. For loops η_i with $1 \leq i \leq n$, assign to each intersection point with Γ a letter σ_j if its intersecting edge of Γ is labeled j and directed from left to right with respect to η_i; otherwise a letter σ_j^{-1}, where σ_j and σ_j^{-1} are standard generators of the m-string braid group B_m and their inverse. We obtain a word $w_\Gamma(\eta_i)$ of B_m on these standard generators by reading off the letters along η_i. For $w_\Gamma(\eta_i)$ with $1 \leq i \leq n$, let $w_i = \pi(w_\Gamma(\eta_i))$ where $\pi\colon B_m \to \Sigma_m$ is natural homomorphism to symmetry group Σ_m. Then, by the definition of η_i, we see that w_i is a transposition.

DEFINITION 3.1 ([1]). (i) Let H be a Hurwitz arc system of a chart Γ. The *factorization graph* $G = (V, E)$ of Γ associated with H is the graph where $V = \{1, \ldots, m\}$ is the set of vertices of G and $E = \{(x, y) \mid \exists i \text{ s.t. } w_i = (x, y)\}$ is the set of edges of G. And we define the weight $W((x, y))$ of an edge of $(x, y) \in E$ as the number of elements i s.t. $w_i = (x, y)$.
(ii) For a given graph G, we denote the graphs of its connected components as $G^1, \ldots, G^l$ where l is the number of the connected components of the graph. For each connected component, let $G^k = (V^k, E^k)$, where V^k are the vertices of G^k and E^k are the edges.

In Fig. 4, two examples of factorization graphs with $m = 6$ are given.

REMARK 3.2. Two Hurwitz arc system of a chart are related by some *slide actions*. See [9]. It is easy to see that slide actions do not change V^k. Thus, we also denote V^k for a Hurwitz arc system of a chart Γ by $V^k(\Gamma)$.

Let n_k be the maximal number among the elements of V^k. A factorization graph is *good* if it is satisfied that $V^k = \{x \mid n_{k-1} + 1 \leq x \leq n_k\}$ for each $1 \leq k \leq l$. (For example, see Fig. 4 (ii).) By Remark 3.2, the property of being good is independent of the choice of Hurwitz arc systems.

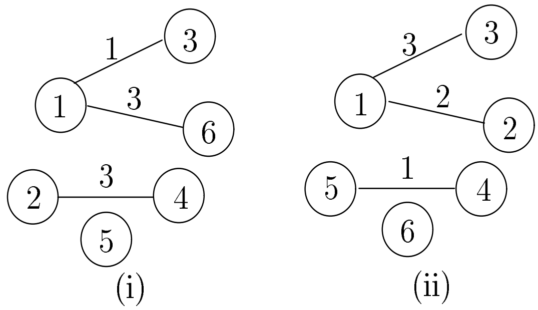


Fig. 4.

Lemma 3.3. *By C-moves, any chart Γ can be transformed to another whose factorization graph G is good.*

Proof. Interchanging sheets in neighborhood of $\mathrm{pr}^{-1}(q_0) \subset S$ leads to exchanging of the vertices of G. This is done by insertion of some concentric hoops around q_0, which is a C_{I}-move. □

4. Unknotting theorem

An m-chart is *unknotted* if it consists of some quasi-free edges or if it is empty (cf. [9]). A singular surface m-braid is *unknotted* if it can be presented by an unknotted m-chart. In this section, we will prove the following theorem.

Theorem 4.1. *Any singular surface braid can be transformed to an unknotted one by crossing changes and its inverse operations.*

A crossing change of a surface in 4-space is inserting a pair of positive and negative crossing points in the sense of [2].

Corollary 4.2 ([2, 6]). *Any surface in* 4*-space can be transformed to an unknotted one by crossing changes and its inverse operations.*

Proof. A surface in 4-space can be represented by a (singular) surface braid S (cf. [9]). By Theorem 4.1, S can be transformed to an unknotted singular surface braid U by crossing changes and its inverse operations. Since an unknotted singular surface braid is an unknotted surface in 4-space (cf. [7, 9]), we have this corollary. □

In order to prove Theorem 4.1, we prepare Proposition 4.3 and Lemmas 4.4–4.8.

Proposition 4.3 ([4]). *Any chart without black vertices can be transformed to the empty chart by the some number of insertion and deletion quasi-hoops and C-moves.*

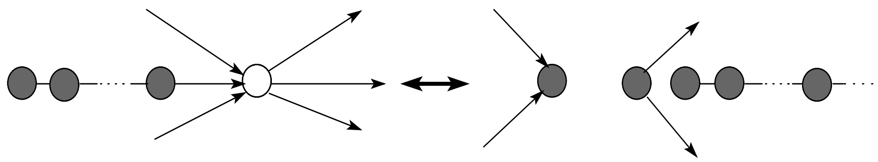


Fig. 5.

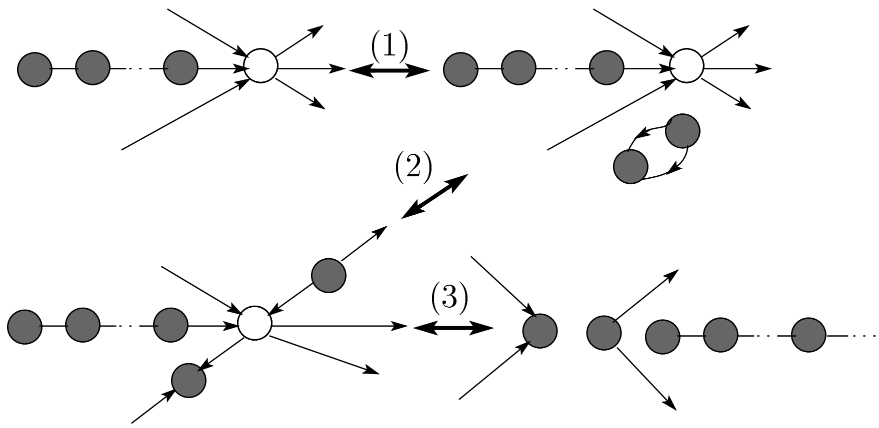


Fig. 6.

Lemma 4.4. *Any chart* Γ *can be transformed to a chart consisting of f-oval nests by insertion and deletion of quasi-hoops and* C*-moves.*

Proof. A move illustrated in Fig. 5 is realized by C-moves, insertion and deletion of quasi-hoops. See Fig. 6: (1) Insertion of a quasi-hoop (2) a C_{I}-move and a C_{V}-move (3) C_{V}-moves and a C_{IV}-move. By such moves and C-moves, each black vertex can be an end of a quasi-free edge. Applying the procedure as in Fig. 29.2 of [9], we take all quasi-free edges near the base point q_0 of U_0. Then, we have a chart Γ' such that $\Gamma' \cap E$ includes no black vertices for a 2-disk E in U_0. By Proposition 4.3, $\Gamma' \cap E$ becomes empty by inserting and deleting quasi-hoops and C-moves. This completes the proof. □

DEFINITION 4.5. An f-oval nest (or h-oval nest) is *simple* if the label of the quasi-free edge (or quasi-hoop) is i and the labels of the concentric hoops in the order from inside to outside are $i+1, i+2, \dots, i+k$ (for some k) and orientations of the hoops are induced from that of U_0. See Fig. 7. We consider that a quasi-free edge (or quasi-hoop) is a simple f-oval nest (or simple h-oval nest) with empty hoops.

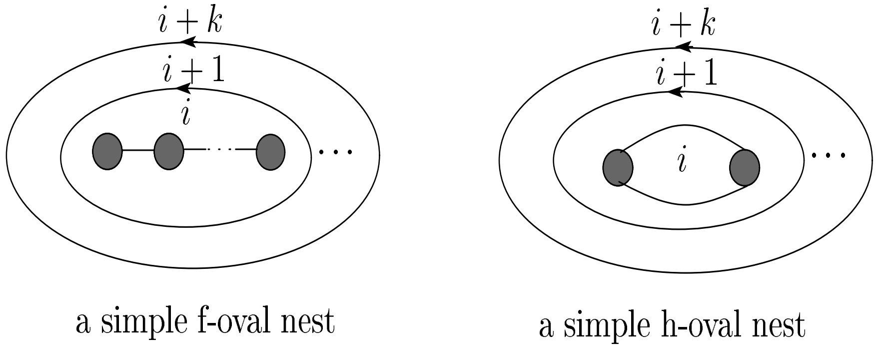


Fig. 7.

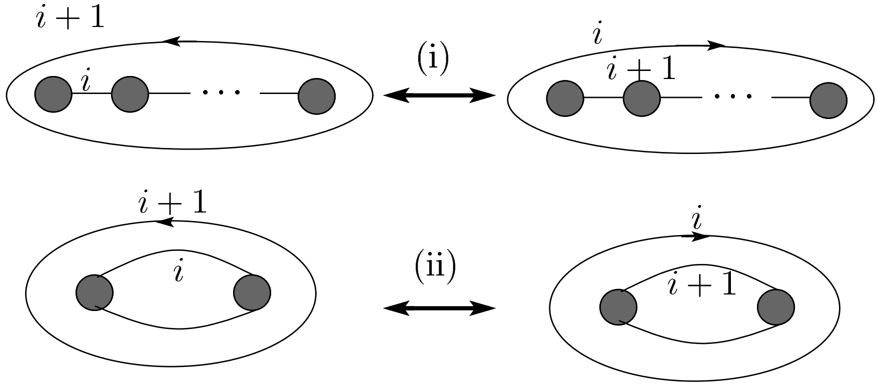


Fig. 8.

Lemma 4.6. *The replacement illustrated in* (i) *and* (ii) *of* Fig. 8 *are realized by C-moves.*

Proof. (i) is given in [3, 5] and (ii) follows from Fig. 9. □

We remark that a given orientation of each hoop in a chart can be reserved by insertion and deletion of quasi-hoops, and we call it an *ID-move*. See Fig. 10.

Lemma 4.7. (i) *Any f-oval nest in a chart is transformed to a simple one by some insertion and deletion of quasi-hoops and C-moves.*
(ii) *Any h-oval nest in a chart is transformed to a simple one by some insertion and deletion of quasi-hoops and C-moves.*

Proof. In this proof, for each f-oval nest f, we denote the number of hoops of f by $n(f)$. We prove the following assertion for any $n(f)$. (i) is a consequence of it.

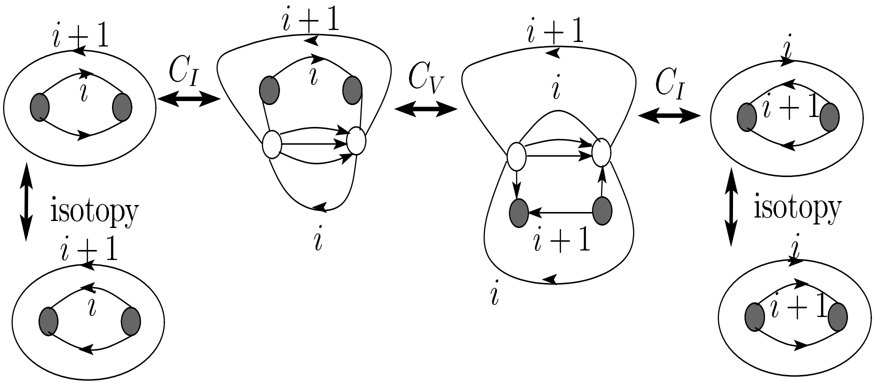


Fig. 9.

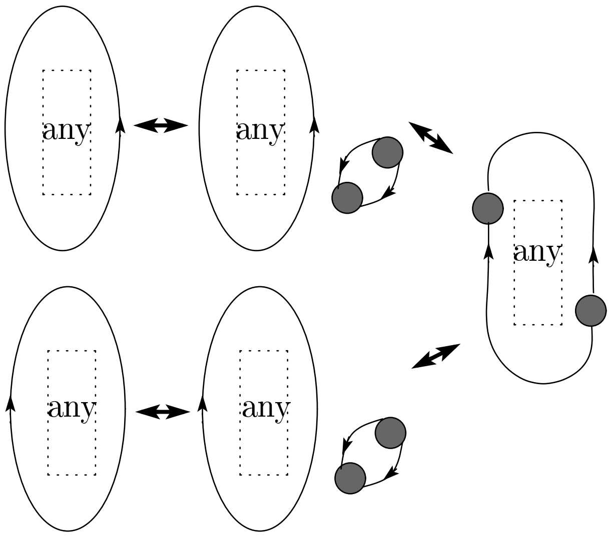


Fig. 10. *ID*-move

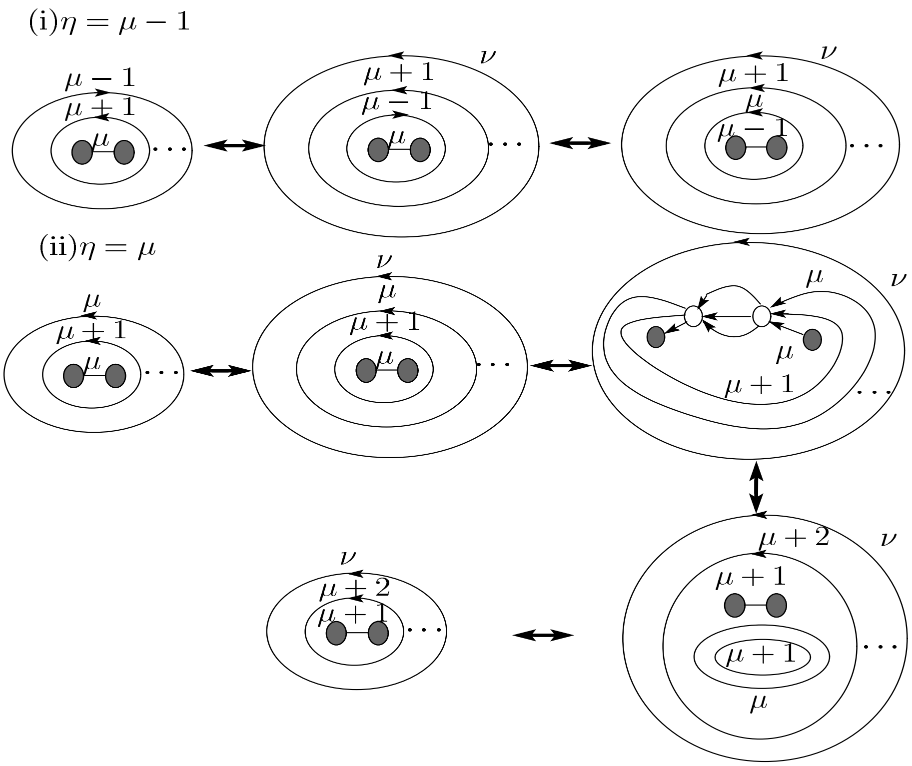


Fig. 11.

Assertion. *Let f be an f-oval nest in a chart. Then, f can be transformed to a simple one f' with $n(f') \leq n(f)$ by ID-moves and C-moves.*

We prove this assertion by induction on $n(f)$. If $n(f) = 0$, it is obvious. Supposed that $n(f) = 1$. In the case where the difference of labels of the quasi-free edge and the hoop of f is 1, by Lemma 4.6 (i) and an *ID*-move (if necessary), f can be transformed to a simple f-oval nest f'. In the other cases, the hoop is removed by C-moves (C_{II}-moves and C_{IV}-moves). Thus, we proved the assertion if $n(f) = 1$. If $n(f) \geq 2$, we consider sub-f-oval nest $\tilde{f}$ of f consisting of the quasi-free edge and hoops except outermost hoop l of f. By induction, we transform $\tilde{f}$ to a simple f-oval nest $\hat{f}$ with $n(\hat{f}) \leq n(\tilde{f})$. Then, $\hat{f} \cup l$ is an f-oval nest with $n(\hat{f}) \leq n(f) - 1$. If $n(\hat{f}) < n(f) - 1$, by induction hypothesis, we can transform $\hat{f} \cup l$ to a simple f-oval nest f'. It is supposed that $n(\hat{f}) = n(f) - 1$. Let μ, ν and η be the labels of the quasi-free edge of $\hat{f}$, the outermost hoop of $\hat{f}$ and l, respectively. In the case where $\eta < \mu - 1$ or $\eta > \nu + 1$, we remove l by C-moves, so we have $f' = \hat{f}$. In the cases where $\eta = \nu$, $\nu + 1$, by an *ID*-move and C-moves (if necessary), we transform $\hat{f} \cup l$ to a simple f-oval nest f'. In the other case, see Fig. 11 and Fig. 12. (If the orientation of l is reverse in Fig. 11

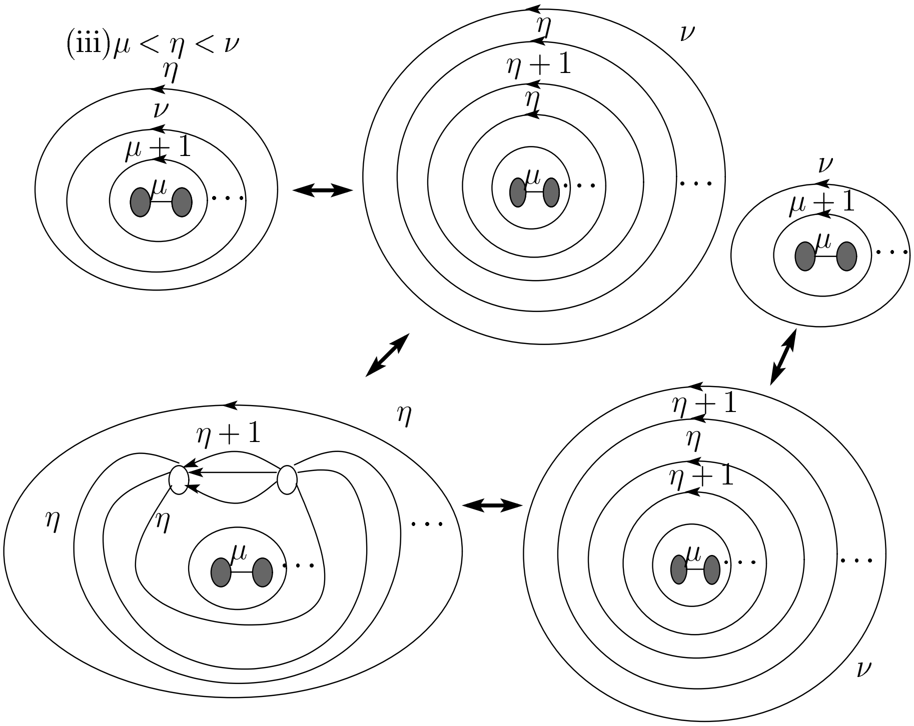


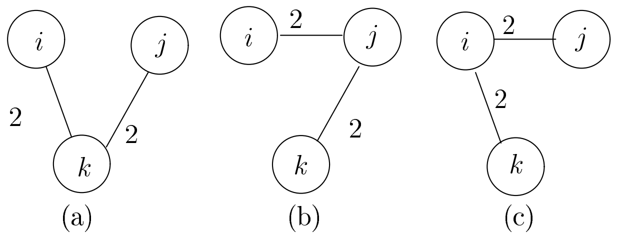
Fig. 12.

and Fig. 12, we should change it by an *ID*-move.) Thus, the proof of the assertion is complete.

(ii) is proved by a similar method. (Use Lemma 4.6 (ii) instead of Lemma 4.6 (i).) This completes the proof. □

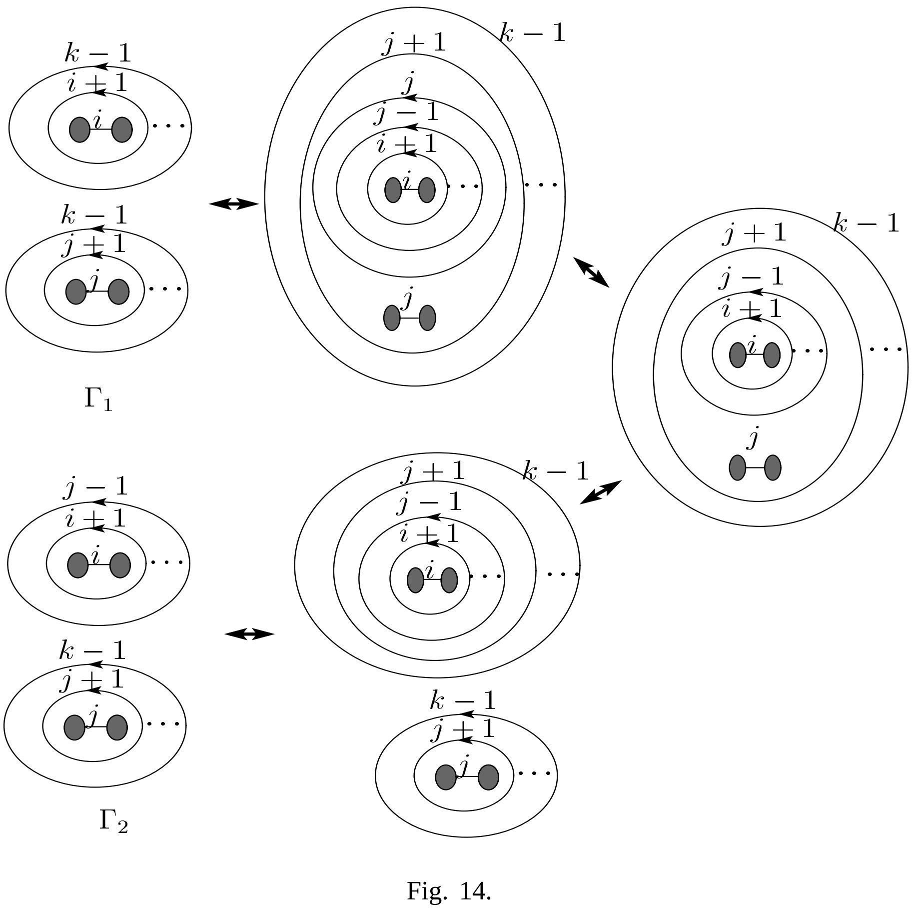
For a chart Γ consisting of simple f-oval nests $f_1, \ldots, f_k$, we consider a Hurwitz arc system $H = (a_1, \ldots, a_n)$ such that for any $1 \leq i \leq n$, $a_i \cap \Gamma \subset f_j$. Such a Hurwitz arc system is called *simple*. We remark that the factorization graph G is independent of the choice of a simple Hurwitz arc system H.

Lemma 4.8. *Let G_1, G_2 and G_3 be the factorization graphs of charts Γ_1, Γ_2 and Γ_3 consisting of simple f-oval nests, respectively, associated with simple Hurwitz arc systems. Suoopse that G_1, G_2 and G_3 are locally different each other as in* Fig. 13 (a), (b) *and* (c) *and the remainding parts are the same. Then, Γ_1, Γ_2 and Γ_3 can be changed each other by C-moves.*



i
j
2
2
k
(a)
i
2
j
2
k
(b)
i
2
j
2
k
(c)

Fig. 13.



$k-1$
$i+1$
i
$k-1$
$j+1$
j
Γ_1
$j+1$
$k-1$
j
$j-1$
$i+1$
i
j
$j+1$
$k-1$
$j-1$
$i+1$
i
j
$j-1$
$i+1$
i
$k-1$
$j+1$
j
Γ_2
$j+1$
$k-1$
$j-1$
$i+1$
i
$k-1$
$j+1$
j

Fig. 14.

Proof. In the case $i < j < k$, Γ_1 and Γ_2 are locally different each other as in the first stage and the last stage of Fig. 14. Each steps in Fig. 14 is done by C-moves. Thus Γ_1 and Γ_2 are changed each other by C-moves. And see Fig. 15 for Γ_2 and Γ_3. The other cases are treated similary. $\square$

Proof of Theorem 4.1. Let S be a singular surface braid presented by a chart Γ. By Lemma 4.4, Γ is transformed to a chart Γ' consisting of some f-oval nests. We may assumed that a factorization graph of Γ' is good by Lemma 3.3. By Lemma 4.7 (i), Γ' is transformed to a chart Γ'' consisting of some simple f-oval nests. By Lemma 4.8, Γ'' is transformed to a chart Γ''' of which factorization graph associated with simple Hurwitz arc system is illustrated in Fig. 16. Γ''' is unknotted, so we have Theorem 4.1. $\square$

Corollary 4.9. *Any singular surface braid can be transformed to be a product $U \bullet O^1 \bullet \cdots \bullet O^s$ by crossing changes* (*and no inverse operations*), *where U is an unknotted singular surface braid and O^t is a singular surface braid presented by a chart consisting a simple h-oval nest for each $1 \leq t \leq s$.*

Proof. Instead of deleting quasi-hoops on proof of Theorem 4.1, we take a quasi-hoop near the base point q_0 of U_0 by applying the procedure as in Fig. 29.2 of [9]. Then, the resulting chart is the product of an unknotted chart and some h-oval nests. Thus, by Lemma 4.7 (ii), we have this corollary. $\square$

5. Finite type invariants

S. Kamada introduced finite type invariants of surfaces in 4-space associated with crossing changes (finger moves) in [8] and 1-handle surgeries in [10]. The author [3] defined finite type invariants of surface braids associated with simple 1-handle surgeries. We consider similar invariants of singular surface braids, which are finite type invariants associated with crossing changes.

Let L^m be the family of equivalence classes of singular surface m-braids. We consider a pair $\mathfrak{S} = \{S, \{c_1, c_2, \ldots, c_n\}\}$ where S is a singular surface m-braid and $c_1, c_2, \ldots, c_n$ are mutually disjoint chords that are straight segment connecting adjacent sheets of S. (See [8] for a precise definition of a "*chord*".) For each n-tuple of signs $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, we denote by

$$\mathfrak{S}_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n} \tag{5.1}$$

the singular surface m-braid obtained from S by a crossing change about c_i (that is a finger move along c_i) for every i $(1 \leq i \leq n)$ with $\varepsilon_i = +1$. In a chart description, a crossing change is presented by the insertion of a quasi-hoop. A map $v\colon L^m \to A$ (A is an abelian group) is called an *order k invariant* if, for any pair $\mathfrak{S} = \{S, \{c_1, c_2, \ldots, c_{k+1}\}\}$,

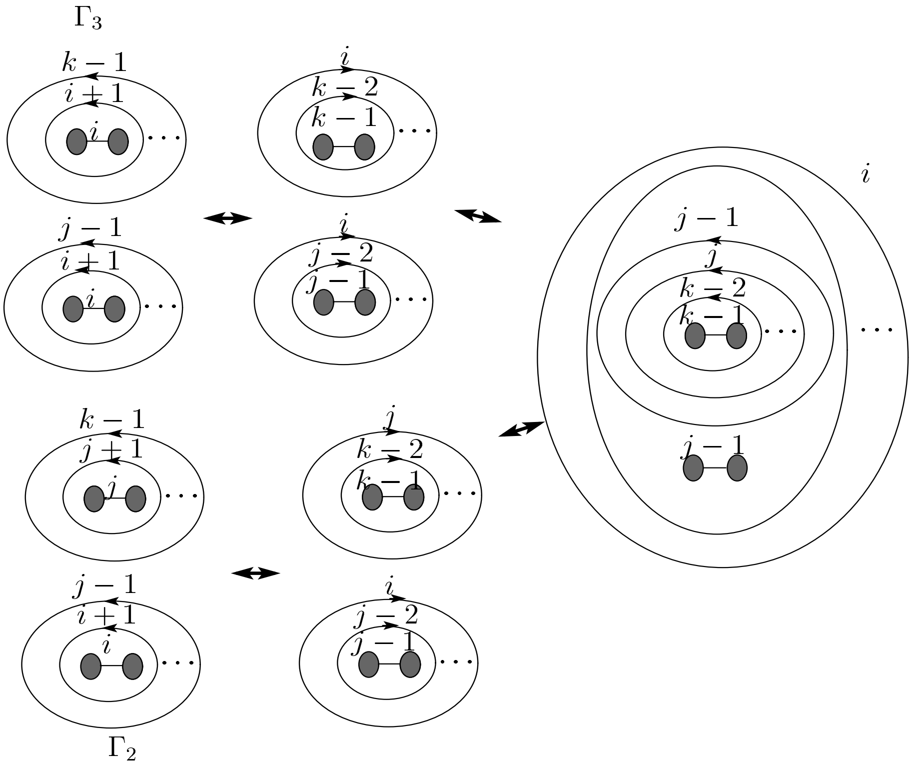

Γ_3
$k-1$
$i+1$
i
i
$k-2$
$k-1$
i
$j-1$
$i+1$
i
i
$j-2$
$j-1$
$j-1$
j
$k-2$
$k-1$
$j-1$
$k-1$
$j+1$
j
j
$k-2$
$k-1$
$j-1$
$i+1$
i
i
$j-2$
$j-1$
Γ_2

Fig. 15.

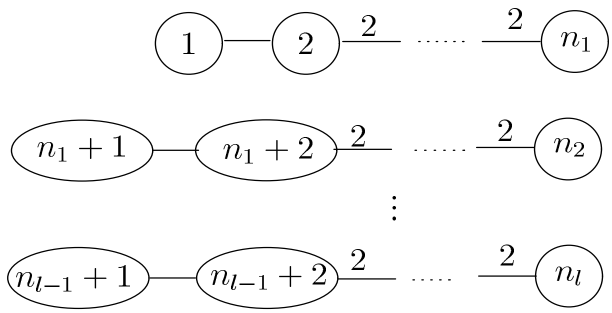

1
2
2
2
n_1
n_1+1
n_1+2
2
2
n_2
$n_{l-1}+1$
$n_{l-1}+2$
2
2
n_l

Fig. 16.

the following equation holds:

$$\sum_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{k+1})} \varepsilon_1\varepsilon_2\cdots\varepsilon_{k+1}\ v([\mathfrak{S}_{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_{k+1}}]) = 0.$$

A map $v\colon L^m \to A$ is called a *finite type invariant* if v is an order k invariant for some k.

EXAMPLE 5.1. Let S be a singular surface m-braid with l components $S^1, \ldots, S^l$. For each k $(1 \le k \le l)$, let F^k be the component of the source F of the immersion f associated with S such that $f(F^k) = S^k$. (See §2.) We define maps from L^m to $\mathbf{Z}$ as follows;

$$\begin{aligned}
\alpha_k([S]) &= (\text{the number of sheets of } S^k),\\
\chi_k([S]) &= (\text{the Euler characteristic of } F^k),\\
d^+_{k_1,k_2}([S]) &= \#(\text{positive } (k_1, k_2)\text{-crossing points}),\\
d^-_{k_1,k_2}([S]) &= \#(\text{negative } (k_1, k_2)\text{-crossing points}),\\
d_{k_1,k_2}([S]) &= \#((k_1, k_2)\text{-crossing points}),\\
e_k([S]) &= d^+_{k,k}([S]) - d^-_{k,k}([S])
\end{aligned}$$

for $1 \le k, k_1, k_2 \le l$. Then, we define the following two invariants;

$$\begin{aligned}
d([S]) &= \sum_{k_1=1}^{l} \sum_{k_1 \le k_2}^{l} d_{k_1,k_2}([S]),\\
e([S]) &= \sum_{k=1}^{l} e_k([S]).
\end{aligned}$$

If S_2 is obtained from S_1 by a crossing change, then

$$d([S_1]) = d([S_2]) + 2, \quad e([S_1]) = e([S_2]), \quad \chi([S_1]) = \chi([S_2]).$$

Therefore, e and χ are order zero invariants and d is an order one invariant.

Theorem 5.2. *Let $v\colon L^m \to A$ be a finite type invariant. Let S_1 and S_2 be singular surface m-braids with l components. If $\alpha_k([S_1]) = \alpha_{\tau(k)}([S_2])$, $d([S_1]) = d([S_2])$, $e_k([S_1]) = e_{\tau(k)}([S_2])$ and $\chi_k([S_1]) = \chi_{\tau(k)}([S_2])$ for any $1 \le k \le l$ and some $\tau \in \Sigma_l$, then $v([S_1]) = v([S_2])$.*

This theorem is a consequence of Theorem 5.3.

For any pair $\mathfrak{S} = \{S, \{c_1, c_2, \dots, c_n\}\}$ of a singular surface m-braid S and a set of n mutually distinct chords $c_1, c_2, \dots, c_n$ each of which is a straight segment connecting adjacent sheets of S, we have an element

$$\sum_{(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n)} \varepsilon_1\varepsilon_2 \cdots \varepsilon_n [\mathfrak{S}_{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n}]$$

of the free $\mathbf{Z}$-module $\mathbf{Z}L^m$ generated by L^m. Denote by $\mathfrak{L}_n^m$ the submodule of $\mathbf{Z}L^m$ spanned by all elements as above. Evidently, we have $\mathfrak{L}_1^m \supset \mathfrak{L}_2^m \supset \cdots$.

Theorem 5.3. *Let S_1 and S_2 be singular surface* m*-braids with l components. If $\alpha_k([S_1]) = \alpha_{\tau(k)}([S_2])$, $d([S_1]) = d([S_2])$, $e_k([S_1]) = e_{\tau(k)}([S_2])$ and $\chi_k([S_1]) = \chi_{\tau(k)}([S_2])$ for any $1 \le k \le l$ and some $\tau \in \Sigma_l$, then $[S_1] - [S_2] \in \mathfrak{L}_n^m$ for any n.*

From now on, we may assume that $\alpha_k([S]) = \alpha_k([S'])$, $\chi_k([S]) = \chi_k([S'])$, $e_k([S]) = e_k([S'])$, $d_{k,k'}([S]) = d_{k,k'}([S'])$ and $d_{t,t'}([S])+2 = d_{t,t'}([S'])$ for $1 \le k, k' \le l$ and $\{k, k'\} \neq \{t, t'\}$ where S' is a singular surface braid obtained from a singular surface braid S with l components by a crossing change such that inserting crossing points are (t, t')-crossing points. This is possible by a suitable choice of indices of components of S'.

In order to prove Theorem 5.3, we use the following lemmas and Proposition 5.9.

Lemma 5.4. *Let S be a singular surface braid with l components. If $k \neq k'$ $(1 \le k, k' \le l)$, then $d^+_{k,k'}([S]) = d^-_{k,k'}([S])$. In particular, $d_{k,k'}([S])$ is even if $k \neq k'$.*

Proof. If S is unknotted, let Γ be an unknotted chart presenting S. Since each node in Γ is on a quasi-free edge, each double point is $(\tilde{k}, \tilde{k})$-double points for some $\tilde{k}$ $(1 \le \tilde{k} \le l)$. Therefore, $d^+_{k,k'}([S]) = d^-_{k,k'}([S]) = 0$. If S is not unknotted, by Theorem 4.1, S can be transformed to an unknotted singular surface braid S' by crossing changes and its inverse operations. Since a crossing change is insertion of a pair of a positive and negative double points and $d^+_{k,k'}([S']) = d^-_{k,k'}([S'])$, we see that $d^+_{k,k'}([S]) = d^-_{k,k'}([S])$. □

Let $V^k(\Gamma)$ be as in Remark 3.2 and we denote the minimal and maximal numbers of $V^k(\Gamma)$ by $\eta_k(\Gamma)$ and $n_k(\Gamma)$, or η_k and n_k for short, respectively.

Lemma 5.5. *Let S_1 and S_2 be singular surface m-braids with l components. If $\alpha_k([S_1]) = \alpha_{\tau(k)}([S_2])$, $d([S_1]) = d([S_2])$, $e_k([S_1]) = e_{\tau(k)}([S_2])$ and $\chi_k([S_1]) = \chi_{\tau(k)}([S_2])$ for any $1 \le k \le l$ and some $\tau \in \Sigma_l$, then S_1 and S_2 can be transformed to $S'_1 = U_1 \bullet O_1^1 \bullet \cdots \bullet O_1^{s_1}$ and $S'_2 = U_2 \bullet O_2^1 \bullet \cdots \bullet O_2^{s_2}$ with $V^k(\Gamma_1) = V^{\tau(k)}(\Gamma_2)$, $d_{k_1,k_2}([S'_1]) = d_{\tau(k_1),\tau(k_2)}([S'_2])$, $e_k([S'_1]) = e_{\tau(k)}([S'_2])$ and $\chi_k([S'_1]) = \chi_{\tau(k)}([S'_2])$ by the same number of crossing changes where U_j is an unknotted singular surface braid, O_j^t is a singular surface braid presented by chart consisting of a simple h-oval nest and Γ_j is a chart presenting S'_j for each $j = 1, 2$ and $1 \le t \le s_j$.*

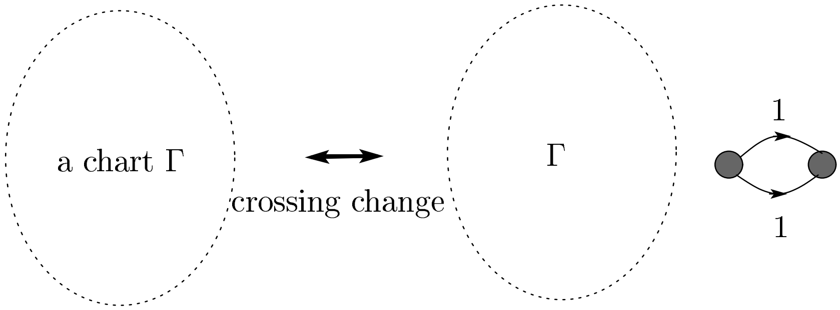


Fig. 17.

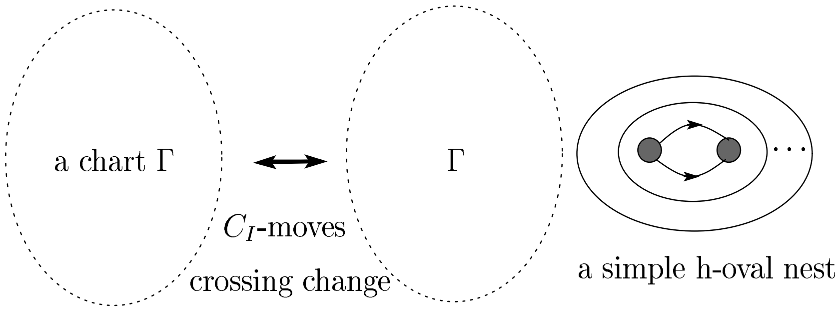


Fig. 18.

Proof. By Corollary 4.9, S_1 and S_2 are transformed to $\hat{S}_1 = \hat{U}_1 \bullet \hat{O}_1^1 \bullet \cdots \bullet \hat{O}_1^{s_1}$ and $\hat{S}_2 = \hat{U}_2 \bullet \hat{O}_2^1 \bullet \cdots \bullet \hat{O}_2^{s_2}$ by some crossing changes, respectively. We may asuume that $V^k(\hat{\Gamma}_1) = V^{\tau(k)}(\hat{\Gamma}_2)$ by insertion of some concentric hoops around q_0 before applying Lemma 4.7 (i) in the proof of Corollary 4.9 (Theorem 4.1) where $\hat{\Gamma}_j$ is a chart presenting $\hat{S}_j$ for $j = 1, 2$. Since $d(S_1) = d(S_2)$, applying crossing changes trivially as in Fig. 17 if necessary, we may also assume that $d(\hat{S}_1) = d(\hat{S}_2)$. It is obvious that crossing changes do not change χ_k and e_k for each k. Thus, it is satisfied that $e_k([\hat{S}_1]) = e_{\tau(k)}([\hat{S}_2])$ and $\chi_k([\hat{S}_1]) = \chi_{\tau(k)}([\hat{S}_2])$. By Lemma 5.4, $d_{k_1,k_2}(\hat{S}_1)$ and $d_{\tau(k_1),\tau(k_2)}(\hat{S}_2)$ are even numbers for $1 \le k_1 \ne k_2 \le l$, so we see that $d_{k_1,k_2}([\hat{S}_1]) - d_{\tau(k_1),\tau(k_2)}([\hat{S}_2])$ is even. Since $e_k([\hat{S}_1]) = e_{\tau(k)}([\hat{S}_2])$, we also see that $d_{k,k}([\hat{S}_1]) - d_{\tau(k),\tau(k)}([\hat{S}_2])$ is even for $1 \le k \le l$. Therefore, applying crossing changes for $\hat{S}_1$ and $\hat{S}_2$ as in Fig. 18 for all pairs (k_1, k_2) $(1 \le k_1, k_2 \le l)$ such that $d_{k_1,k_2}([\hat{S}_1]) \ne d_{\tau(k_1),\tau(k_2)}([\hat{S}_2])$, we can obtain singular surface braids S_1' and S_2' such that $d_{k_1,k_2}([S_1']) = d_{\tau(k_1),\tau(k_2)}([S_2'])$ for any $1 \le k_1, k_2 \le l$. Then, the charts Γ_1 and Γ_2 thus obtained, or the charts corresponding to S_1' and S_2', respectively. Here, we need the same number of crossing changes for $\hat{S}_1$ and $\hat{S}_1$ to have S_1' and S_2' because of $d(\hat{S}_1) = d(\hat{S}_2)$. Since the crossing changes as in Fig. 17 and Fig. 18 do not change V^k for $1 \le k \le l$, the lemma is proved. □

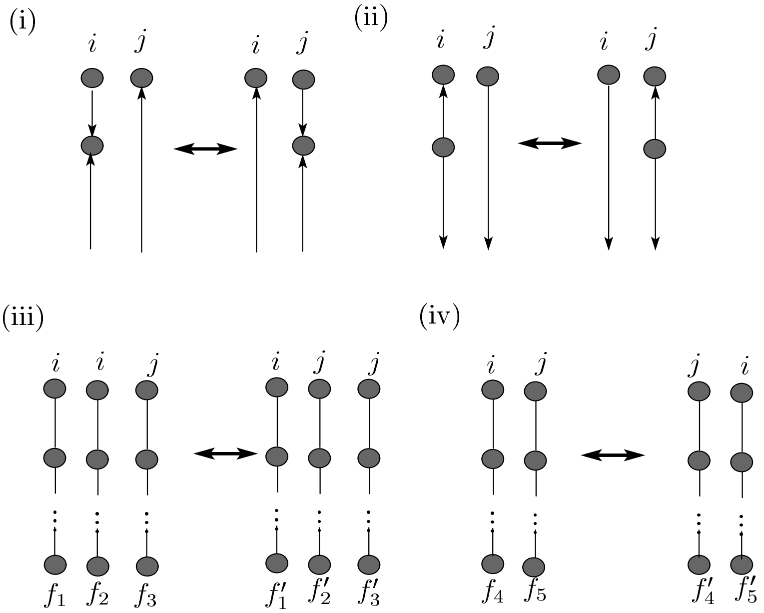


Fig. 19.

By $u(S_1, S_2)$, we denote the minimal number of crossing changes that are needed to satisfy the statement of Lemma 5.5.

Lemma 5.6. *If* $|i - j| = 1$, *then the local operations illustrated in* Fig. 19 (i)–(iv) *are C-move equivalence, where each* f_r *and* f_s' *is a quasi-free edge for* $1 \leq r, s \leq 5$ *such that the number of nodes in* f_r *is equal to the number of nodes in* f_r'.

Proof. See [7] for (i)–(iii). The operation (iv) follows from Fig. 20. □

Lemma 5.7. *Let* U_1 *and* U_2 *be unknotted singular surface braids with* l *components such that* $V^k(\Gamma_1) = V^{\tau(k)}(\Gamma_2)$, $d_{k,k}([U_1]) = d_{\tau(k),\tau(k)}([U_2])$, $e_k([U_1]) = e_{\tau(k)}([U_2])$ *and* $\chi_k([U_1]) = \chi_{\tau(k)}([U_2])$ *where* Γ_j *is a chart presenting* U_j *for* $1 \leq k \leq l$ *and* $j = 1, 2$. *Then,* U_1 *is equivalent to* U_2.

Proof. The k-th component U_j^k of U_j is presented by a chart Γ_j^k consisting of quasi-free edges with label s for each $s \in V^k(\Gamma_j) \setminus \{n_k(\Gamma_j)\}$. Then, U_j is presented by a chart $\Gamma_j^1 \bullet \cdots \bullet \Gamma_j^l$. We prove the following assertion.

Assertion. Γ_j^k *can be transformed to a chart* D_k *by C-moves that satisfies the following conditions* (*see* Fig. 21):

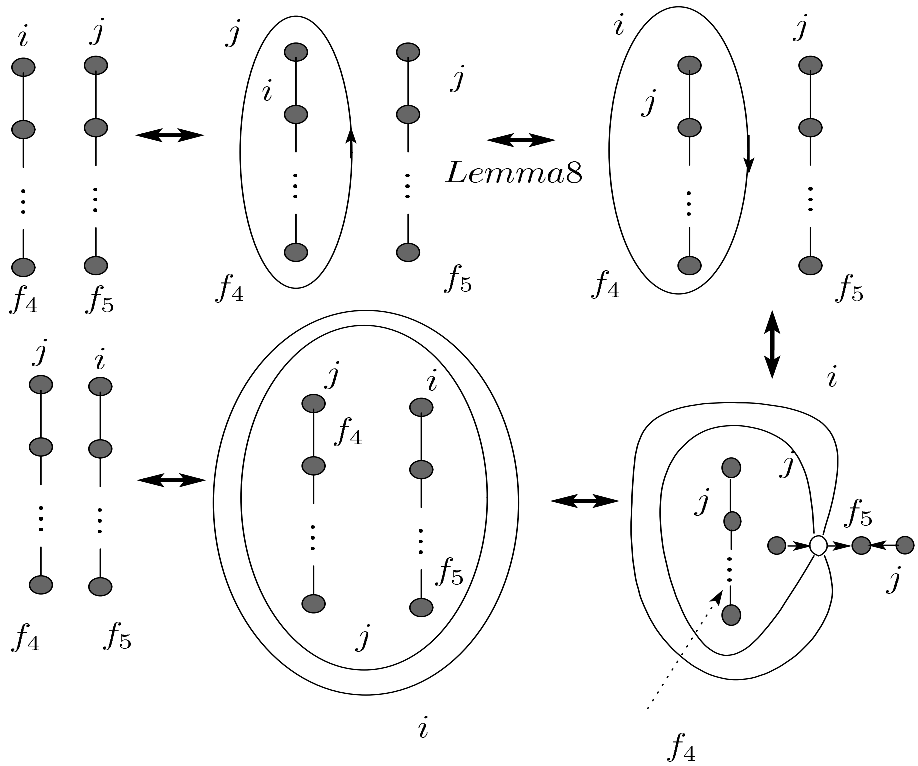


Fig. 20.

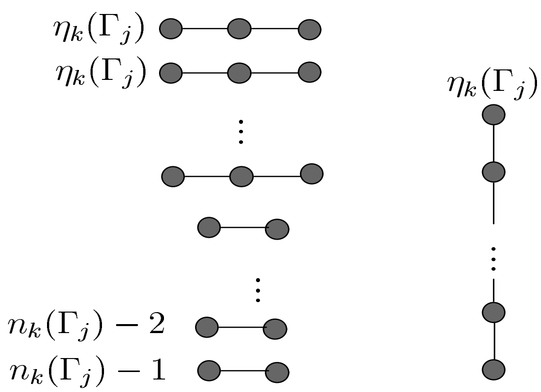


Fig. 21. D_k

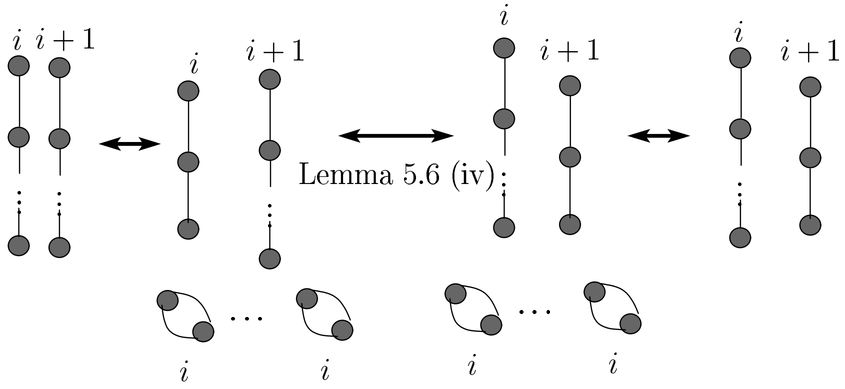


Fig. 22.

(1) *The factorization graph of* D_k *associated with a simple Hurwitz arc system is as in* Fig. 16.
(2) *The number of quasi-free edge with at least one node is* $|e_k(U_j)|$ *if* $e_k(U_j) \neq 0$; *otherwise* 1.
(3) *The number of quasi-free edge with at least two node is* 0 *or* 1.
(4) *The quasi-free edge with label* w *has nodes if* $w \geq n_k(\Gamma_j) - d$ *where* d *is the number of quasi-free edge without nodes*; *otherwise it dose not have node.*
(5) $V(D_k) = V^k(U_j)$.

If the number of quasi-free edges in Γ_j^k is more than $|V(\Gamma_j^k)| - 1$, that is, U_j^k is not a sphere, then it is easy to prove this assertion by Lemma 5.6 (iii). In case the number of quasi-free edges in Γ_j^k is equal to $|V(\Gamma_j^k)| - 1$ and $e(S) > 0$, by Lemma 5.6 (i) and (ii), Γ_j^k is a chart Γ_1 consisting of quasi-free edges that are positive. We may assume that Γ_1 satisfies the condition (4) by Lemma 5.6 (iv). Applying the operation as illustrated in Fig. 22 if necessary, we obtain a chart D_k satisfying the conditions (1)–(5). The other cases are treated similary. This completes the proof of the assertion.

Since $V^k(\Gamma_1) = V^{\tau(k)}(\Gamma_2)$, $d_{k,k}([U_1]) = d_{\tau(k),\tau(k)}([U_2])$, $e_k([U_1]) = e_{\tau(k)}([U_2])$ and $\chi_k([U_1]) = \chi_{\tau(k)}([U_2])$, both U_1^k and U_2^k can be also presented by the same chart D_k for $1 \leq k \leq l$. Therefore, both U_1 and U_2 are presented by the chart $D_1 \bullet \cdots \bullet D_l$, and hence U_1 is equivalent to U_2. □

Lemma 5.8. *Let* S *be a singular surface braid with* l *components such that* $S = U \bullet O$ *where* U *is an unknotted singular surface braid,* O *is a singular surface braid, whose crossing points are two* (k, k')*-crossing points* $(k < k')$*, presented by a chart consisting of a simple h-oval nest. Then,* S *is equivalent to* $U \bullet \hat{O}$ *such that* $\hat{O}$ *is a singular surface braid presented by a chart consisting of a simple h-oval nest of which*

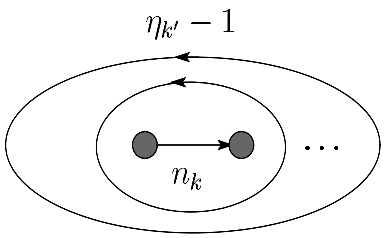


Fig. 23.

the labels of the quasi-free edge and the outermost circle are $n_k(\Gamma^u)$ *and* $\eta_{k'}(\Gamma^u) - 1$, *respectively, where* Γ^u *is a chart presenting* U. (Fig. 23.)

Proof. Let $\tilde{\Gamma}$ be a chart presenting O. ($\tilde{\Gamma}$ is a simple h-oval nest.) It is seen that the labels of the free edge and the outermost circle are in $V^k \setminus \{n_k\}$ and $V^{k'} \setminus \{n_{k'}\}$ where $n_k = n_k(\Gamma^u)$ and $n_{k'} = n_k(\Gamma^u)$. The chart Γ^u has quasi-free edges with label i for any $i \in (V^k \cup V^{k'}) \setminus \{n_k, n_{k'}\}$. Applying the operation in Fig. 24, $\Gamma^u \bullet \tilde{\Gamma}$ is equivalent to $\Gamma^u \bullet \hat{\Gamma}$ where $\hat{\Gamma}$ is as in Fig. 23. □

Proposition 5.9. *Let* S_1 *and* S_2 *be singular surface braids with* l *components such that* $S_1 = U_1 \bullet O_1^1 \bullet \cdots \bullet O_1^{s_1}$ *and* $S_2 = U_2 \bullet O_2^1 \bullet \cdots \bullet O_2^{s_2}$ *with* $V^k(\Gamma_1) = V^{\tau(k)}(\Gamma_2)$, $d_{k_1,k_2}([S_1]) = d_{\tau(k_1),\tau(k_2)}([S_2])$, $e_k([S_1]) = e_{\tau(k)}([S_2])$ *and* $\chi_k([S_1]) = \chi_{\tau(k)}([S_2])$ *where for each* $1 \le t \le s_j$ *and* $j = 1, 2$, U_j *is an unknotted singular surface braid,* O_j^t *is a singular surface braid presented by chart consisting of a simple h-oval nest and* Γ_j *is a chart presenting* S_j. *Then,* S_1 *is equivalent to* S_2.

Proof. If the crossing points of O_j^i are (k,k)-crossing points, $U_j \bullet O_j^i$ is unknotted. Thus, we may assumed that the crossing points of O_j^i are (k, k')-crossing points with $k < k'$. Now, $V^k(\Gamma_1) = V^{\tau(k)}(\Gamma_2)$, $d_{k,k}([U_1]) = d_{\tau(k),\tau(k)}([U_2])$, $e_k([U_1]) = e_{\tau(k)}([U_2])$ and $\chi_k([U_1]) = \chi_{\tau(k)}([U_2])$ where Γ_j is a chart presenting U_j for $1 \le k \le l$ and $j = 1, 2$. Therefore, by Lemma 5.7, U_1 is equivalent to U_2. By Lemma 5.8, S_j is equivalent to $U_j \bullet \hat{O}_j^1 \bullet \cdots \bullet \hat{O}_j^{s_j}$ where $\hat{O}_j^i$ is a singular surface braid as $\hat{O}$ in Lemma 5.8. Since $d_{k_1,k_2}([S_1]) = d_{\tau(k_1),\tau(k_2)}([S_2])$ for $k_1 \neq k_2$, it is see that $\hat{O}_1^1 \bullet \cdots \bullet \hat{O}_1^{s_1}$ is equivalent to $\hat{O}_2^1 \bullet \cdots \bullet \hat{O}_2^{s_2}$. U_1 is also equivalent to U_2, so $U_1 \bullet \hat{O}_1^1 \bullet \cdots \bullet \hat{O}_1^{s_1}$ is equivalent to $U_2 \bullet \hat{O}_2^1 \bullet \cdots \bullet \hat{O}_2^{s_2}$. Therefore, S_1 is equivalent to S_2. □

Proof of Theorem 5.3. We prove the following assersion for every $n \in \mathbf{N}$.

Assertion. *Let* S *and* S' *be singular surface* m*-braids with* $\alpha_k([S_1]) = \alpha_{\tau(k)}([S_2])$, $d([S_1]) = d([S_2])$, $e_k([S_1]) = e_{\tau(k)}([S_2])$ *and* $\chi_k([S_1]) = \chi_{\tau(k)}([S_2])$ *for any* $1 \le k \le l$ *and some* $\tau \in \Sigma_l$. *If* $u(S_1, S_2) \le r$, *then* $[S_1] - [S_2] \in \mathfrak{L}_n^m$ *for any* $n \ge r + 1$.

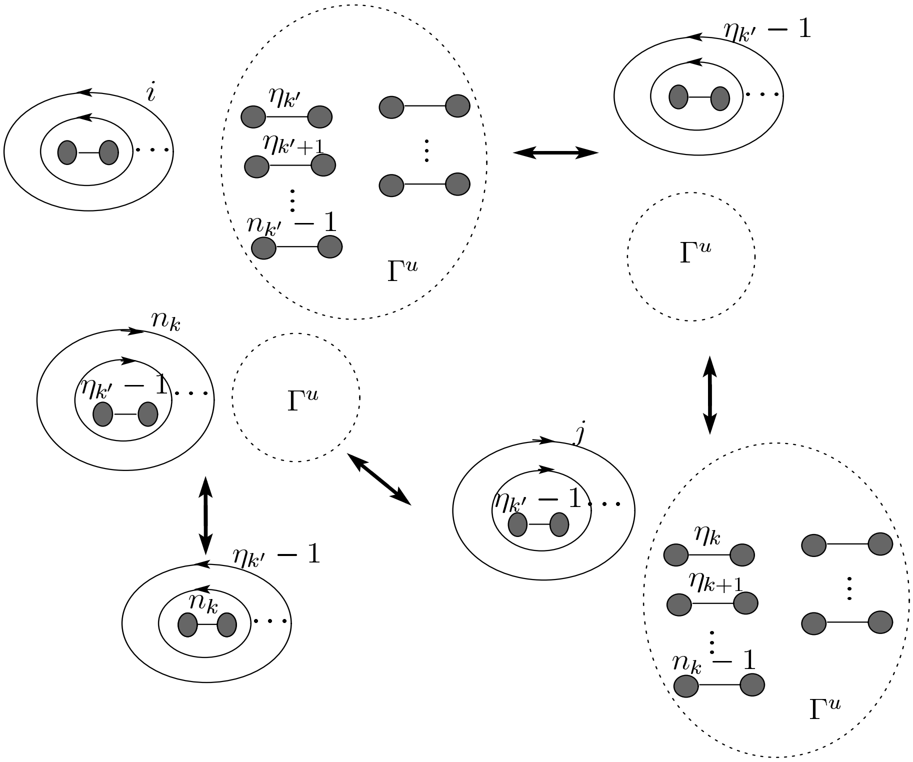


Fig. 24.

Let $\alpha_{k,0} := \alpha_k([S_1]) = \alpha_{\tau(k)}([S_2])$, $d_0 := d([S_1]) = d([S_2])$, $e_{k,0} := e_k([S_1]) = e_{\tau(k)}([S_2])$ and $\chi_{k,0} := \chi_k([S_1]) = \chi_{\tau(k)}([S_2])$ for any $1 \leq k \leq l$ and some $\tau \in \Sigma_l$.

Let n be an integer with $n \geq r+1$. Take r mutually disjoint cord c_i or d_i for $1 \leq i \leq r$ each of which is a straight segment connecting adjacent sheets of S_1 (or S_2) satisfying the following conditions:

(i) The surgery result S_1' (or S_2') along cord $c_1, c_2, \dots, c_r$ (or $d_1, d_2, \dots, d_r$) are as in Lemma 5.5.

(ii) c_i (or d_i) is a parallel copy of c_1 or d_1 for any $r+1 \leq i \leq n$.

For the pair $\mathfrak{S}_1 = \{S_1, \{c_1, \dots, c_n\}\}$ (or $\mathfrak{S}_2 = \{S_2, \{d_1, \dots, d_n\}\}$) and for an n-tuple of signs $(\varepsilon_1, \dots, \varepsilon_n)$, let $(\mathfrak{S}_1)_{\varepsilon_1,\dots,\varepsilon_n}$ (or $(\mathfrak{S}_2)_{\varepsilon_1,\dots,\varepsilon_n}$) be a singular surface m-braid as the formula (5.1). And let $p = p(\varepsilon_1, \dots, \varepsilon_n)$ be the number of positive signs in the n-tuple of signs $(\varepsilon_1, \dots, \varepsilon_n)$. Then $\alpha_k([(\mathfrak{S}_1)_{\varepsilon_1,\dots,\varepsilon_n}]) = \alpha_{\tau(k)}([(\mathfrak{S}_2)_{\varepsilon_1,\dots,\varepsilon_n}]) = \alpha_{k,0}$, $d([(\mathfrak{S}_1)_{\varepsilon_1,\dots,\varepsilon_n}]) = d([(\mathfrak{S}_2)_{\varepsilon_1,\dots,\varepsilon_n}]) = d_0+2p$, $e_k([(\mathfrak{S}_1)_{\varepsilon_1,\dots,\varepsilon_n}]) = e_{\tau(k)}([(\mathfrak{S}_2)_{\varepsilon_1,\dots,\varepsilon_n}]) = e_{k,0}$ and $\chi_k([(\mathfrak{S}_1)_{\varepsilon_1,\dots,\varepsilon_n}]) = \chi_{\tau(k)}([(\mathfrak{S}_2)_{\varepsilon_1,\dots,\varepsilon_n}]) = \chi_{k,0}$. If $p > 0$, we see that $u((\mathfrak{S}_1)_{\varepsilon_1,\dots,\varepsilon_n}, (\mathfrak{S}_2)_{\varepsilon_1,\dots,\varepsilon_n}) \leq r-1$.

We prove the assertion by induction on r. If $r = 1$, then $(\mathfrak{S}_1)_{\varepsilon_1,\dots,\varepsilon_n} \cong (\mathfrak{S}_2)_{\varepsilon_1,\dots,\varepsilon_n}$ for any $(\varepsilon_1, \dots, \varepsilon_n)$ with $p = p(\varepsilon_1, \dots, \varepsilon_l) > 0$ by Lemma 5.9. Thus,

$$\begin{aligned}&\sum_{(\varepsilon_1,\dots,\varepsilon_n),\,p>0} \varepsilon_1 \cdots \varepsilon_n[(\mathfrak{S}_1)_{\varepsilon_1,\dots,\varepsilon_n}] + (-1)^n([S_1])\\ &\equiv \sum_{(\varepsilon_1,\dots,\varepsilon_n),\,p>0} \varepsilon_1 \cdots \varepsilon_n[(\mathfrak{S}_2)_{\varepsilon_1,\dots,\varepsilon_n}] + (-1)^n([S_2]) \pmod{\mathfrak{L}_n^m}.\end{aligned}$$

Therefore, we have $[S] - [S'] \in \mathfrak{L}_n^m$. If $r \geq 2$, then by the induction hypothesis we have $[(\mathfrak{S}_1)_{\varepsilon_1,\dots,\varepsilon_n}] - [(\mathfrak{S}_2)_{\varepsilon_1,\dots,\varepsilon_n}] \in \mathfrak{L}_n^m$ for any $(\varepsilon_1, \dots, \varepsilon_n)$ with $p = p(\varepsilon_1, \dots, \varepsilon_n) > 0$. Thus,

$$\begin{aligned}&\sum_{(\varepsilon_1,\dots,\varepsilon_n),\,p>0} \varepsilon_1 \cdots \varepsilon_n[(\mathfrak{S}_1)_{\varepsilon_1,\dots,\varepsilon_n}] + (-1)^n([S_1])\\ &\equiv \sum_{(\varepsilon_1,\dots,\varepsilon_n),\,p>0} \varepsilon_1 \cdots \varepsilon_n[(\mathfrak{S}_2)_{\varepsilon_1,\dots,\varepsilon_n}] + (-1)^n([S_2]) \pmod{\mathfrak{L}_n^m}.\end{aligned}$$

Therefore, we see that $[(\mathfrak{S}_1)_{\varepsilon_1,\dots,\varepsilon_n}] - [(\mathfrak{S}_2)_{\varepsilon_1,\dots,\varepsilon_n}] \in \mathfrak{L}_n^m$. This completes the proof of assertion. Since $\mathfrak{L}_1^m \supset \mathfrak{L}_2^m \supset \cdots$, we have $[S] - [S'] \in \mathfrak{L}_n^m$ for all n. □

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