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REIDEMEISTER TORSION OF A SYMPLECTIC COMPLEX

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Abstract

We consider a claim mentioned in [33] p.187 about the relation between a symplectic chain complex with ω-compatible bases and Reidemeister Torsion of it. This is an explanation of it.

Introduction

Even though we approach Reidemeister torsion as a linear algebraic object, it actually is a combinatorial invariant for the space of representations of a compact surface into a fixed gauge group [33] [22].

More precisely, let S be a compact surface with genus at least 2 and without boundary, G be a gauge group with its (semi-simple) Lie algebra $\mathfrak{g}$. Then, for a representation $\rho\colon \pi_1(S) \to G$, we can associate the corresponding adjoint bundle $\left(\begin{smallmatrix} \tilde{S} \times_\rho \mathfrak{g} \\ \downarrow \\ S \end{smallmatrix}\right)$ over S, i.e. $\tilde{S} \times_\rho \mathfrak{g} = \tilde{S} \times \mathfrak{g}/\sim$, where (x, t) is identified with all the elements in its orbit i.e. $(\gamma \bullet x, \gamma \bullet t)$ for all $\gamma \in \pi_1(S)$, and where in the first component the element $\gamma \in \pi_1(S)$ of the fundamental group of S acts as a deck transformation, and in the second component by conjugation by $\rho(\gamma)$.

Suppose K is a cell-decomposition of S so that the adjoint bundle $\tilde{S} \times_\rho \mathfrak{g}$ on S is trivial over each cell. Let $\tilde{K}$ be the lift of K to the universal covering $\tilde{S}$ of S. With the action of $\pi_1(S)$ on $\tilde{S}$ as deck transformation, $C_*(\tilde{K};\mathbb{Z})$ can be considered a left-$\mathbb{Z}[\pi_1(S)]$ module and with the action of $\pi_1(S)$ on $\mathfrak{g}$ by adjoint representation, $\mathfrak{g}$ can be considered as a left-$\mathbb{Z}[\pi_1(S)]$ module, where $\mathbb{Z}[\pi_1(S)]$ is the integral group ring $\left\{\sum_{i=1}^{p} m_i\gamma_i;\ m_i \in \mathbb{Z},\ \gamma_i \in \pi_1(S),\ p \in \mathbb{N}\right\}$.

More explicitly, if $\sum_{i=1}^{p} m_i\gamma_i$ is in $\mathbb{Z}[\pi_1(S)]$, t is in $\mathfrak{g}$, and $\sum_{j=1}^{q} n_j\sigma_j \in C_*(\tilde{S};\mathbb{Z})$, then $\left(\sum_{i=1}^{p} m_i\gamma_i\right) \bullet \left(\sum_{j=1}^{q} n_j\sigma_j\right) \overset{\text{defn}}{=} \sum_{i,j} n_j m_i(\gamma_i \bullet \sigma_j)$, where γ_i acts on $\sigma_j \subset \tilde{S}$ by deck transformation, and $\left(\sum_{j=1}^{q} m_j\gamma_j\right) \bullet t \overset{\text{defn}}{=} \sum_{j=1}^{q} m_j(\gamma_j \bullet t)$, where $\gamma_j \bullet t = \mathrm{Ad}_{\rho(\gamma_j)}(t) = \rho(\gamma_j)t\rho(\gamma_j)^{-1}$.

To talk about the tensor product $C_*(\tilde{K};\mathbb{Z}) \otimes \mathfrak{g}$, we should consider the left $\mathbb{Z}[\pi_1(S)]$-module $C_*(\tilde{K};\mathbb{Z})$ as a right $\mathbb{Z}[\pi_1(S)]$-module as $\sigma \bullet \gamma \overset{\text{defn}}{=} \gamma^{-1} \bullet \sigma$, where the action of

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γ^{-1} is as a deck transformation. Note that the relation $\sigma \bullet \gamma \otimes t = \sigma \otimes \gamma \bullet t$ becomes $\gamma^{-1} \bullet \sigma \otimes t = \sigma \otimes \gamma \bullet t$, equivalently $\sigma' \otimes t = \gamma \bullet \sigma' \otimes \gamma \bullet t$, where σ' is $\gamma^{-1} \bullet \sigma$. We may conclude that tensoring with $\mathbb{Z}[\pi_1(S)]$ has the same effect as factoring with $\pi_1(S)$. Thus, $C_*(K; \mathrm{Ad}_\rho) \overset{\text{defn}}{=} C_*(\tilde{K}; \mathbb{Z}) \otimes_\rho \mathfrak{g}$ is defined as the quotient $C_*(\tilde{K}; \mathbb{Z}) \otimes \mathfrak{g}/\sim$, where the elements of the orbit $\{\gamma \bullet \sigma \otimes \gamma \bullet t;$ for all $\gamma \in \pi_1(S)\}$ of $\sigma \otimes t$ are identified.

In this way, we obtain the following complex:

$$0 \to C_2(K; \mathrm{Ad}_\rho) \xrightarrow{\partial_2 \otimes \mathrm{id}} C_1(K; \mathrm{Ad}_\rho) \xrightarrow{\partial_1 \otimes \mathrm{id}} C_0(K; \mathrm{Ad}_\rho) \to 0,$$

where ∂_i is the usual boundary operator. For this complex, we can associate the homologies $H_*(K; \mathrm{Ad}_\rho)$. Similarly, the twisted cochains $C^*(K; \mathrm{Ad}_\rho)$ will result the cohomologies $H^*(K; \mathrm{Ad}_\rho)$, where $C^*(K; \mathrm{Ad}_\rho) \overset{\text{defn}}{=} \mathrm{Hom}_{\mathbb{Z}[\pi_1(S)]}(C_*(\tilde{K}; \mathbb{Z}), \mathfrak{g})$ is the set of $\mathbb{Z}[\pi_1(S)]$-module homomorphisms from $C_*(\tilde{K}; \mathbb{Z})$ into $\mathfrak{g}$. For more information, we refer [22] [26] [33].

If $\rho, \rho' \colon \pi_1(S) \to G$ are conjugate, i.e. $\rho'(\,\cdot\,) = A\rho(\,\cdot\,)A^{-1}$ for some $A \in G$, then $C_*(K; \mathrm{Ad}_\rho)$ and $C_*(K; \mathrm{Ad}_{\rho'})$ are isomorphic. Similarly, the twisted cochains $C^*(K; \mathrm{Ad}_\rho)$ and $C^*(K; \mathrm{Ad}_{\rho'})$ are isomorphic. Moreover, the homologies $H_*(K; \mathrm{Ad}_\rho)$ are independent of the cell-decomposition. For details, see [26] [33] [22].

If $\{e^i_1, \ldots, e^i_{m_i}\}$ is a basis for the $C_i(K; \mathbb{Z})$, then $c_i := \{\tilde{e}^i_1, \ldots, \tilde{e}^i_{m_i}\}$ will be a $\mathbb{Z}[\pi_1(S)]$-basis for $C_i(\tilde{K}; \mathbb{Z})$, where $\tilde{e}^i_j$ is a lift of e^i_j. If we choose a basis $\mathcal{A}$ of $\mathfrak{g}$, then $c_i \otimes_\rho \mathcal{A}$ will be a $\mathbb{C}$-basis for $C_i(K; \mathrm{Ad}_\rho)$, called a *geometric* basis for $C_i(K; \mathrm{Ad}_\rho)$. Recall that $C_i(K; \mathrm{Ad}_\rho) = C_i(\tilde{K}; \mathbb{Z}) \otimes_\rho \mathfrak{g}$, is defined as the quotient $C_i(\tilde{K}; \mathbb{Z}) \otimes \mathfrak{g}/\sim$, where we identify the orbit $\{\gamma \bullet \sigma \otimes \gamma \bullet t;\ \gamma \in \pi_1(S)\}$ of $\sigma \otimes t$, and where the action of the fundamental group in the first slot by deck transformations, and in the second slot by the conjugation with $\rho(\,\cdot\,)$.

In this set-up, one can also define $\mathrm{Tor}(C_*(K; \mathrm{Ad}_\rho), \{c_i \otimes_\rho \mathcal{A}\}^2_{i=0}, \{\mathfrak{h}_i\}^2_{i=0})$ the *Reidemeister torsion* of the triple K, Ad_ρ, and $\{\mathfrak{h}_i\}^2_{i=0}$, where $\mathfrak{h}_i$ is a $\mathbb{C}$-basis for $H_i(K; \mathrm{Ad}_\rho)$. Moreover, one can easily prove that this definition does not depend on the lifts $\tilde{e}^i_j$, conjugacy class of ρ, and cell-decomposition K of the surface S. Details can be found in [26] [22] [33].

Let K, K' be dual cell-decompositions of S so that $\sigma \in K$, $\sigma' \in K'$ meet at most once and moreover the diameter of each cell has diameter less than, say, half of the injectivity radius of S. If we denote $C_* = C_*(K; \mathrm{Ad}_\rho)$, $C'_* = C_*(K'; \mathrm{Ad}_\rho)$, then by the invariance of torsion under subdivision, $\mathrm{Tor}(C_*) = \mathrm{Tor}(C'_*)$. Let D_* denote the complex $C_* \oplus C'_*$. Then, easily we have the short-exact sequence

$$0 \to C_* \to D_* = C_* \oplus C'_* \to C'_* \to 0.$$

The complex $D_* = C_* \oplus C'_*$ can also be considered as a symplectic complex. Moreover, in the case of irreducible representation $\rho \colon \pi_1(S) \to G$, torsion $\mathrm{Tor}(C_*)$ gives a two-form on $H^1(S; \mathrm{Ad}_\rho)$. See [33] [26].

In this article, we will consider Reidemeister torsion as a linear algebraic object and try to rephrase a statement mentioned in [33].

The main result of the article is as stated in [33] p.187 “the torsion of a symplectic complex (C_*, ω) computed using a compatible set of measures is ‘trivial’ in the sense that”

Theorem 0.0.1. *For a general symplectic complex C_*, if $\mathfrak{c}_p$, $\mathfrak{h}_p$ are bases for C_p, H_p, respectively, then*

$$\mathrm{Tor}(C_*, \{\mathfrak{c}_p\}_{p=0}^n, \{\mathfrak{h}_p\}_{p=0}^n) = \left(\prod_{p=0}^{(n/2)-1} (\det[\omega_{p,n-p}])^{(-1)^p} \right) \cdot \left(\sqrt{\det[\omega_{n/2,n/2}]}\right)^{(-1)^{n/2}},$$

where $\det[\omega_{p,n-p}]$ *is the determinant of the matrix of the non-degenerate pairing* $[\omega_{p,n-p}]\colon H_p(C) \times H_{n-p}(C) \to \mathbb{R}$ *in bases* $\mathfrak{h}_p$, $\mathfrak{h}_{n-p}$.

For topological application of this, we refer [26] [33]. For the sake of clarity, the application in [26] will also be explained in §3.

Our main interest started with the observation [27] that Teichmüller space $\mathfrak{Teich}(S)$ of compact hyperbolic surface S with Weil-Petersson form is symplectically the same as the vector space $\mathcal{H}(\lambda; \mathbb{R})$ of transverse cocycles associated to a fixed maximal geodesic lamination λ on S, where we consider the Thurston symplectic form.

The Teichmüller space $\mathfrak{Teich}(S)$ of a fixed compact surface S with negative Euler characteristic (i.e. with genus at least 2) is the space of deformation classes of complex structures on S. By the Uniformization Theorem, it can also be interpreted as the space of isotopy classes of hyperbolic metrics on S (i.e. Riemannian metrics with constant -1 curvature), or as the space of conjugacy classes of all discrete faithful homomorphisms from the fundamental group $\pi_1(S)$ to the group $\mathrm{Isom}^+(\mathbb{H}^2) \cong \mathrm{PSL}_2(\mathbb{R})$ of orientation-preserving isometries of upper-half lane $\mathbb{H}^2 \subset \mathbb{C}$.

$\mathfrak{Teich}(S)$ is a differentiable manifold, diffeomorphic to an open convex cell whose dimension is determined by the topology of the surface S. From its origins in complex geometry, it carries two structures. Namely, it is a complex manifold and admits a naturally defined Hermitian form, called Weil-Petersson Hermitian form [1], [29].

$$\langle\ ,\ \rangle_{\mathrm{WP}}\colon \mathrm{T}_\rho\mathfrak{Teich}(S) \times \mathrm{T}_\rho\mathfrak{Teich}(S) \to \mathbb{C}.$$

The real and imaginary parts of this pairing respectively define on $\mathfrak{Teich}(S)$ a Riemannian metric g_{WP} called *Weil-Petersson Riemannian metric*, and a (real) 2-form ω_{WP} called the *Weil-Petersson* 2*-form*.

In [14], W.M. Goldman proved that the Weil-Petersson 2-form has a very nice topological interpretation and can be described as a cup-product in this context. Namely, he introduced a closed non-degenerate 2-form (or a symplectic form) $\omega_{\mathrm{Goldman}}\colon H^1(S; \mathrm{Ad}_\rho) \times H^1(S; \mathrm{Ad}_\rho) \to \mathbb{R}$, where $H^1(S; \mathrm{Ad}_\rho)$ is the first cohomology space of S with coefficients in the adjoint bundle and also proved that this symplectic form and Weil-Petersson 2-form differ only by a constant multiple.

F. Bonahon parametrized the Teichmüller space of S by using a maximal geodesic lamination λ on S [3] [28]. Geodesic laminations are generalizations of deformation classes of simple closed curves on S. More precisely, a geodesic lamination λ on the surface S is by definition a closed subset of S which can be decomposed into family of disjoint simple geodesics, possibly infinite, called its *leaves*. The geodesic lamination is *maximal* if it is maximal with respect to inclusion; this is equivalent to the property that the complement $S - \lambda$ is union of finitely many triangles with vertices at infinity.

The real-analytical parametrization given in [3] identifies $\mathfrak{Teich}(S)$ to an open convex cone in the vector space $\mathcal{H}(\lambda, \mathbb{R})$ of all *transverse cocycles* for λ. In particular, at each $\rho \in \mathfrak{Teich}(S)$, the tangent space $\mathrm{T}_\rho\mathfrak{Teich}(S)$ is now identified with $\mathcal{H}(\lambda, \mathbb{R})$, which is a real vector space of dimension $3|\chi(S)|$, where $\chi(S)$ is the Euler characteristic of S. Transverse cocycles are signed transverse measures (valued in $\mathbb{R}$) associated the maximal geodesic lamination λ on S. The space $\mathcal{H}(\lambda, \mathbb{R})$ has also anti-symmetric bilinear form, namely the Thurston symplectic form ω_{Thurston}, which has also a homological interpretation as an algebraic intersection number. It was proved that up to a multiplicative constant, ω_{Thurston} is the same as ω_{Goldman} [27], and hence is in the same equivalence class of ω_{WP}. More precisely,

Theorem 0.0.2 ([27]). *Let S be a closed oriented surface with negative Euler charactersistic (i.e. of genus at least two), and let λ be a (fixed) maximal geodesic lamination on the surface S. For the identification $\mathrm{T}_\rho\mathfrak{Teich}(S) \cong \mathcal{H}(\lambda;\mathbb{R})$, we have the following commutative diagram $H^1(S;\mathrm{Ad}_\rho) \times H^1(S;\mathrm{Ad}_\rho)$*

$$\begin{tikzcd}
H^1(S;\mathrm{Ad}_\rho) \times H^1(S;\mathrm{Ad}_\rho) \arrow[r, "\smile_B"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & H^2(S;\mathbb{R}) \arrow[d] \\
\mathcal{H}(\lambda;\mathbb{R}) \times \mathcal{H}(\lambda;\mathbb{R}) \arrow[r, "2\tau"] & \mathbb{R}.
\end{tikzcd}$$

Let S be a compact surface with negative Euler characteristic, K be a cell-decomposotion of the surface S. For $p = 0, 1, 2$, let $\mathfrak{c}_p$ be the corresponding geometric bases for $C_p(K;\mathcal{A}d_\rho)$, and let $\mathfrak{h}^1$ be a basis for $H^1(S;\mathcal{A}d_\rho)$.

In [26], we provided the proof of the following theorem; however, for the sake of completeness, we will also explain in §3.

Theorem 0.0.3 ([26]).

$$\mathrm{Tor}(C_*, \{\mathfrak{c}_p\}_{p=0}^2, \{0, \mathfrak{h}_p^1, 0\}) = \frac{6g-6}{\|H\|^2}\mathrm{Pfaff}(\omega_H),$$

where Pfaff(ω_H) *is the* Pfaffian *of the matrix* $H = [\omega_{\text{Goldman}}(\mathfrak{h}_i^1, \mathfrak{h}_j^1)]$, $\|H\|^2 =$

$\mathrm{Trace}(HH^{\mathrm{transpose}})$, *and* $\omega_{\mathrm{Goldman}}\colon H^1(S;\mathcal{A}d_\rho)\times H^1(S;\mathcal{A}d_\rho)\to\mathbb{R}$ *is the Goldman symplectic form.*

Let λ be a maximal geodesic lamination on the surface S. Considering the K_λ triangulation of the surface by using the maximal geodesic lamination (see [27] for details), and by Theorem 3.1.3, we proved the following:

Theorem 0.0.4 ([26]). *Let S be a compact hyperbolic surface, λ be a fixed maximal geodesic lamination on S, and let K_λ be the corresponding triangulation of the surface obtained from λ. For $p = 0, 1, 2,$ let $\mathfrak{c}_p$ be the corresponding geometric bases for $C_p(K_\lambda;\mathcal{A}d_\rho)$, and let $\mathfrak{h}$ be a basis for $\mathcal{H}(\lambda;\mathbb{R})$.*

$$\mathrm{Tor}(C_*, \{\mathfrak{c}_p\}_{p=0}^2, \{0, \mathfrak{h}, 0\}) = \frac{(6g-6)\cdot\sqrt{2^{6g-6}}}{4\cdot\|T\|^2}\mathrm{Pfaff}(\tau),$$

where $\mathrm{Pfaff}(\tau)$ *is the* Pfaffian *of the matrix* $T = [\tau(\mathfrak{h}_i, \mathfrak{h}_j)]$, $\|T\|^2 = \mathrm{Trace}(TT^{\mathrm{transpose}})$, *and* $\tau\colon \mathcal{H}(\lambda;\mathbb{R})\times\mathcal{H}(\lambda;\mathbb{R})\to\mathbb{R}$ *is the Thurston symplectic form.*

For example, when $\lambda = \lambda_{\mathcal{P}}$ is the maximal geodesic lamination obtained from a pant-decomposition $\mathcal{P}$ of the surface S, then since the non-zero transverse-weights $\mathcal{H}(\lambda;\mathbb{R})$ associated to the leaves of λ are nothing but the weights associated to the separating closed curves $\{c_1,\dots,c_{3g-3}\}$ leaves of λ coming from the pant-decomposition $\mathcal{P}$. The cell-decomposition K_λ can be obtained as follows. The 2-cells are the pair-of-pants $\{P_1,\dots,P_{4g-4}\}$, 1-cells are the separating curves $\{c_1,\dots,c_{3g-3}\}$ and 0-cells are obtained by choosing two distinct points on each separating curve.

The plan of paper is as follows. In §1, we will give the definition of Reidemeister torsion for a general complex C_* and recall some properties. See [19] [22] for more information. In §2, we will explain torsion using Witten's notation [33]. Then, symplectic complex will be explained and also the proof of main result Theorem 0.0.1. In §3, we will also provide the proof of the application in [26].

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1. Reidemeister torsion

In this section, we will provide the basic definitions and facts about the Reidemeister torsion. For more information about the subject, we refer the reader to [22] [33].

1.1. Reidemeister torsion of a chain complex of vector spaces. Throughout this section, $\mathbb{F}$ denotes the field $\mathbb{R}$ or $\mathbb{C}$. Let $C_* = \big(C_n \xrightarrow{\partial_n} C_{n-1}\to\cdots\to C_1\xrightarrow{\partial_1} C_0\to 0\big)$ be a chain complex of a finite dimensional vector spaces over $\mathbb{F}$. Let $H_p =$

Z_p/B_p denote the homologies of the complex, where $B_p = \operatorname{Im}\{\partial_{p+1}\colon C_{p+1} \to C_p\}$, $Z_p = \ker\{\partial_p\colon C_p \to C_{p-1}\}$, respectively.

If we start with bases $\mathfrak{b}_p = \{b_p^1, \ldots, b_p^{m_p}\}$ for B_p, and $\mathfrak{h}_p = \{h_p^1, \ldots, h_p^{n_p}\}$ for H_p, a new basis for C_p can be obtained by considering the following short-exact sequences:

$$0 \to Z_p \hookrightarrow C_p \twoheadrightarrow B_{p-1} \to 0, \tag{1.1.1}$$

$$0 \to B_p \hookrightarrow Z_p \twoheadrightarrow H_p \to 0, \tag{1.1.2}$$

where the first row is a result of the 1st-isomorphism theorem and the second follows simply from the definition of H_p.

Starting with (1.1.2) and a section $l_p\colon H_p \to Z_p$, then Z_p will have a basis $\mathfrak{b}_p \oplus l_p(\mathfrak{h}_p)$. Using (1.1.1) and a section $s_p\colon B_{p-1} \to C_p$, C_p will have a basis $\mathfrak{b}_p \oplus l_p(\mathfrak{h}_p) \oplus s_p(\mathfrak{b}_{p-1})$.

If V is a vector space with bases $\mathfrak{e}$ and $\mathfrak{f}$, then we will denote $[\mathfrak{f}, \mathfrak{e}]$ for the determinant of the change-base-matrix $T_{\mathfrak{e}}^{\mathfrak{f}}$ from $\mathfrak{e}$ to $\mathfrak{f}$.

DEFINITION 1.1.1. For $p = 0, \ldots, n$, let $\mathfrak{c}_p$, $\mathfrak{b}_p$, and $\mathfrak{h}_p$ be bases for C_p, B_p and H_p, respectively. $\operatorname{Tor}(C_*, \{\mathfrak{c}_p\}_{p=0}^n, \{\mathfrak{h}_p\}_{p=0}^n) = \prod_{p=0}^n [\mathfrak{b}_p \oplus l_p(\mathfrak{h}_p) \oplus s_p(\mathfrak{b}_{p-1}), \mathfrak{c}_p]^{(-1)^{(p+1)}}$ is called the *torsion of the complex* C_* with respect to bases $\{\mathfrak{c}_p\}_{p=0}^n$, $\{\mathfrak{h}_p\}_{p=0}^n$.

Milnor [19] showed that torsion does not depend on neither the bases $\mathfrak{b}_p$, nor the sections s_p, l_p. In other words, it is well-defined.

REMARK 1.1.2. If we choose another bases $\mathfrak{c}'_p$, $\mathfrak{h}'_p$ respectively for C_p and H_p, then an easy computation shows that

$$\operatorname{Tor}(C_*, \{\mathfrak{c}'_p\}_{p=0}^n, \{\mathfrak{h}'_p\}_{p=0}^n) = \prod_{p=0}^n \left(\frac{[\mathfrak{c}'_p, \mathfrak{c}_p]}{[\mathfrak{h}'_p, \mathfrak{h}_p]}\right)^{(-1)^p} \cdot \operatorname{Tor}(C_*, \{\mathfrak{c}_p\}_{p=0}^n, \{\mathfrak{h}_p\}_{p=0}^n).$$

This follows easily from the fact that torsion is independent of $\mathfrak{b}_p$ and sections s_p, l_p. For example, if $[\mathfrak{c}'_p, \mathfrak{c}_p] = 1$, and $[\mathfrak{h}'_p, \mathfrak{h}_p] = 1$, then they produce the same torsion.

If we have a short-exact sequence of chain complexes $0 \to A_* \overset{\iota}{\hookrightarrow} B_* \overset{\pi}{\twoheadrightarrow} D_* \to 0$, then we also have a long-exact sequence of vector space C_*

$$\cdots \to H_p(A) \xrightarrow{\iota_*} H_p(B) \xrightarrow{\pi_*} H_p(D) \xrightarrow{\triangle} H_{p-1}(A) \to \cdots$$

i.e. an acyclic (or exact) complex C_* of length $3n+2$ with $C_{3p} = H_p(D_*)$, $C_{3p+1} = H_p(A_*)$ and $C_{3p+2} = H_p(B_*)$. In particular, the bases $\mathfrak{h}_p(D_*)$, $\mathfrak{h}_p(A_*)$, and $\mathfrak{h}_p(B_*)$ will serve as bases for C_{3p}, C_{3p+1}, and C_{3p+2}, respectively.

Theorem 1.1.3 (Milnor [19]). *Using the above setup, let $\mathfrak{c}_p^A$, $\mathfrak{c}_p^B$, $\mathfrak{c}_p^D$ be bases for A_p, B_p, D_p, respectively, and let $\mathfrak{h}_p^A$, $\mathfrak{h}_p^B$, $\mathfrak{h}_p^D$ be bases for the corresponding homologies $H_p(A)$, $H_p(B)$, and $H_p(D)$. If, moreover, the bases $\mathfrak{c}_p^A$, $\mathfrak{c}_p^B$, $\mathfrak{c}_p^D$ are compatible in the sense that $[\mathfrak{c}_p^B, \mathfrak{c}_p^A \oplus \tilde{\mathfrak{c}}_p^D] = \pm 1$ where $\pi(\tilde{\mathfrak{c}}_p^D) = \mathfrak{c}_p^D$, then* $\mathrm{Tor}(B_*, \{\mathfrak{c}_p^B\}_{p=0}^n, \{\mathfrak{h}_p^B\}_{p=0}^n) = \mathrm{Tor}(A_*, \{\mathfrak{c}_p^A\}_{p=0}^n, \{\mathfrak{h}_p^A\}_{p=0}^n) \cdot \mathrm{Tor}(D_*, \{\mathfrak{c}_p^D\}_{p=0}^n, \{\mathfrak{h}_p^D\}_{p=0}^n) \cdot \mathrm{Tor}(C_*, \{\mathfrak{c}_{3p}\}_{p=0}^{3n+2}, \{0\}_{p=0}^{3n+2})$.

1.2. Complex $C_*(S, \mathrm{Ad}_\rho)$ for a homomorphism $\rho\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{F})$. Let S be a compact surface with genus at least 2 (without boundary). For a representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{F})$, we can associate the corresponding adjoint bundle $\begin{pmatrix} \tilde{S} \times_\rho \mathfrak{sl}_2(\mathbb{F}) \\ \downarrow \\ S \end{pmatrix}$ over S, i.e. $\tilde{S} \times_\rho \mathfrak{sl}_2(\mathbb{F}) = \tilde{S} \times \mathfrak{sl}_2(\mathbb{F})/\sim$, where (x,t) is identified with all the elements in its orbit $\{(\gamma \bullet x, \gamma \bullet t);$ for all $\gamma \in \pi_1(S)\}$, and where in the first component γ acts as a deck transformation, and in the second component by the adjoint action i.e. conjugation by $\rho(\gamma)$.

Let K be a fine cell-decomposition of S so that the adjoint bundle $\tilde{S} \times_\rho \mathfrak{sl}_2(\mathbb{F})$ on S is trivial over each cell. If $\tilde{K}$ is the lift of K to the universal covering $\tilde{S}$ of S, then with the action of $\pi_1(S)$ on $\tilde{S}$ as deck transformation, $C_*(\tilde{K};\mathbb{Z})$ will be a left $\mathbb{Z}[\pi_1(S)]$-module and with the action of $\pi_1(S)$ on $\mathfrak{sl}_2(\mathbb{F})$ by adjoint action, $\mathfrak{sl}_2(\mathbb{F})$ will be considered as a left-$\mathbb{Z}[\pi_1(S)]$ module, where $\mathbb{Z}[\pi_1(S)]$ denotes the integral group ring.

Namely, if $\sum_{i=1}^p m_i\gamma_i$ is in $\mathbb{Z}[\pi_1(S)]$, t is a trace zero matrix, and $\sum_{j=1}^q n_j\sigma_j \in C_*(\tilde{S};\mathbb{Z})$, then $\left(\sum_{i=1}^p m_i\gamma_i\right) \bullet \left(\sum_{j=1}^q n_j\sigma_j\right) = \sum_{i,j} n_j m_i(\gamma_i \bullet \sigma_j)$, where γ_i acts on $\sigma_j \subset \tilde{S}$ by deck transformations, and $\left(\sum_{j=1}^q n_j\sigma_j\right) \bullet t \overset{\text{defn}}{=} \sum_{j=1}^q n_j(\sigma_j \bullet t)$, where $\sigma_j \bullet t = \mathrm{Ad}_{\rho(\gamma_j)}(t) = \rho(\gamma_j)t\rho(\gamma_j)^{-1}$.

$C_*(\tilde{K};\mathbb{Z})$ can also be considered as a right $\mathbb{Z}[\pi_1(S)]$-module by $\sigma \bullet \gamma \overset{\text{defn}}{=} \gamma^{-1} \bullet \sigma$, where the action of γ^{-1} is as a deck transformation. Note that the relation $\sigma \bullet \gamma \otimes t = \sigma \otimes \gamma \bullet t$ becomes $\gamma^{-1} \bullet \sigma \otimes t = \sigma \otimes \gamma \bullet t$, equivalently $\sigma' \otimes t = \gamma \bullet \sigma' \otimes \gamma \bullet t$, where σ' is $\gamma^{-1} \bullet \sigma$. Hence, $C_*(K;\mathrm{Ad}_\rho) \overset{\text{defn}}{=} C_*(\tilde{K};\mathbb{Z}) \otimes_\rho \mathfrak{sl}_2(\mathbb{F})$ is defined as the quotient $C_*(\tilde{K};\mathbb{Z}) \otimes \mathfrak{sl}_2(\mathbb{F})/\sim$, where the elements of the orbit $\{\gamma \bullet \sigma \otimes \gamma \bullet t;$ for all $\gamma \in \pi_1(S)\}$ of $\sigma \otimes t$ are identified.

As a result, we have the following complex:

$$0 \to C_2(K;\mathrm{Ad}_\rho) \xrightarrow{\partial_2 \otimes \mathrm{id}} C_1(K;\mathrm{Ad}_\rho) \xrightarrow{\partial_1 \otimes \mathrm{id}} C_0(K;\mathrm{Ad}_\rho) \to 0,$$

where ∂_i is the usual boundary operator. For this complex, one can also associate the twisted homologies $H_*(K;\mathrm{Ad}_\rho)$. Similarly, the cochains $C^*(K;\mathrm{Ad}_\rho)$ will result the cohomologies $H^*(K;\mathrm{Ad}_\rho)$, where $C^*(K;\mathrm{Ad}_\rho) \overset{\text{defn}}{=} \mathrm{Hom}_{\mathbb{Z}[\pi_1(S)]}(C_*(\tilde{K};\mathbb{Z}), \mathfrak{sl}_2(\mathbb{F}))$ is the set of $\mathbb{Z}[\pi_1(S)]$-module homomorphisms from $C_*(\tilde{K};\mathbb{Z})$ into $\mathfrak{sl}_2(\mathbb{F})$.

We will end this section by a list of properties of $C_*(K;\mathrm{Ad}_\rho)$, $C^*(K;\mathrm{Ad}_\rho)$, and for the sake of completeness, we will recall the proofs.

Lemma 1.2.1. (1) *If $\rho, \rho' \colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{F})$ are conjugate, i.e. $\rho'(\cdot) = A\rho(\cdot)A^{-1}$ for some $A \in \mathrm{PSL}_2(\mathbb{F})$, then $C_*(K;\mathrm{Ad}_\rho)$ and $C_*(K;\mathrm{Ad}_{\rho'})$ are isomorphic. Similarly, the twisted cochains $C^*(K;\mathrm{Ad}_\rho)$ and $C^*(K;\mathrm{Ad}_{\rho'})$ are isomorphic.*
(2) *The homologies $H_*(K;\mathrm{Ad}_\rho)$ are independent of the cell-decomposition.*

Proof. (1) Recall that using the homorphisms Ad_ρ, $\mathrm{Ad}_{\rho'} \colon \mathfrak{sl}_2(\mathbb{F}) \to \mathfrak{sl}_2(\mathbb{F})$, $\mathfrak{sl}_2(\mathbb{F})$ becomes a left $\mathbb{Z}[\pi_1(S)]$-module. Since $\mathrm{Ad}_A \colon \mathfrak{sl}_2(\mathbb{F}) \to \mathfrak{sl}_2(\mathbb{F})$ is a homomorphism and the representations $\rho, \rho' \colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{F})$ are conjugate by A, the map $\phi_A \colon \mathfrak{sl}_2(\mathbb{F}) \to \mathfrak{sl}_2(\mathbb{F})$ defined by $\phi_A(t) = \mathrm{Ad}_A(t)$ is actually a $\mathbb{Z}[\pi_1(S)]$-module homomorphism, where in the domain we consider the action by Ad_ρ and in the range by $\mathrm{Ad}_{\rho'}$. By the fact that $\otimes$ is middle-linear and ϕ_A is homomorphism, $\mathrm{id}\otimes\phi_A \colon C_*(\tilde{K};\mathbb{Z}) \times \mathfrak{sl}_2(\mathbb{F}) \to C_*(\tilde{K};\mathbb{Z}) \otimes_{\rho'} \mathfrak{sl}_2(\mathbb{F})$ is also middle linear, i.e. linear in the first component, linear in the second component and $\mathrm{id}\otimes\phi_A(\sigma \bullet \gamma, t) = \mathrm{id}\otimes\phi_A(\sigma, \gamma \bullet t)$. Therefore, there exists a unique homomorphism $\Phi_A \colon C_*(\tilde{K};\mathbb{Z}) \otimes_\rho \mathfrak{sl}_2(\mathbb{F}) \to C_*(\tilde{K};\mathbb{Z}) \otimes_{\rho'} \mathfrak{sl}_2(\mathbb{F})$ such that $\Phi_A(\sigma \otimes t) = \sigma \otimes \phi_A(t)$. Similarly, using the inverse of ϕ_A, i.e. $\phi_{A^{-1}}$, we can obtain the unique homomorphism $\Phi_{A^{-1}}(\sigma \otimes t) = \sigma \otimes \phi_{A^{-1}}(t)$. Moreover, Φ_A and $\Phi_{A^{-1}}$ are inverses of each other, and thus Φ_A is an isomorphism.

(2) This follows from the invariance under subdivision. To define $H_*(K, \mathrm{Ad}_\rho)$, we started with a fine cell-decomposition K of S so that over each cell in K the adjoint bundle is trivial.

Let $\hat{K}$ be the refinement of K obtained by introducing extra cells as follows. For example, if $w \in K$ is a 2-cell (say, n-gon, put a point p, say in the barycenter of w, and adding n new one-cells $y_1, \ldots, y_n$, we also obtain n new two-cells: $w_1, \ldots, w_n$. The refinement $\hat{K}$ gives a chain complex $\hat{C} = C_* \oplus C'_*$, where $C'_* := \hat{C}_*/C_*$ is the chain complex obtained from the added cells. The boundary of w_i consists of two new cells y_i, y_{i+1} and one of the boundary cell of w, thus $\partial'_2[w_i] = [y_{i+1}] - [y_i]$. Similarly, since boundary of y_i is the point p and one of the zero dimensional cell of w, hence $\partial'_1[y_i] = [p]$. Finally, we identify $[y_{i+n}] = [y_i]$ for all i.

Clearly, we have a short-exact sequence of chain complexes

$$0 \to C_* \overset{i}{\hookrightarrow} \hat{C}_* \overset{\pi}{\twoheadrightarrow} C'_* \to 0,$$

which will result the long-exact sequence $0 \to H_2(C_*) \overset{i_*}{\hookrightarrow} H_2(\hat{C}_*) \xrightarrow{\pi_*} H_2(C'_*) \to H_1(C_*) \overset{i_*}{\hookrightarrow} H_1(\hat{C}_*) \xrightarrow{\pi_*} H_0(C'_*) \to H_0(C_*) \overset{i_*}{\hookrightarrow} H_0(\hat{C}_*) \xrightarrow{\pi_*} H_0(C'_*) \to 0$.

We will show that the chain complex C'_* is exact i.e. $H_p(C'_*)$'s are all zero, and thus will conclude that $H_p(C_*) \cong H_p(\hat{C}_*)$.

Lemma 1.2.2. *The chain complex $0 \to C'_2 \xrightarrow{\partial'_2} C'_1 \xrightarrow{\partial'_1} C'_0 \xrightarrow{\partial'_0} 0$ is exact.*

Proof. Recall that the chain complex $C'_* := \hat{C}_*/C_*$ is obtained from the added cells. If w (n-gon) is in C_2, we put a point p inside w, add n new 1-cells $y_1, \ldots, y_n$, and obtain n-new two-cells $w_1, \dots, w_n$ so that $w = w_1 \cup \dots \cup w_n$. Thus $[p]$ is a generator for C_0', $[y_1], \dots, [y_n]$ are in the generating set of C_1', and $[w_1], \dots, [w_n]$ are in the generating set for C_2' with one relation $[w_1] + \dots + [w_n] = 0$. The last is result of $w_1 \cup \dots \cup w_n = w \in C_2$. Moreover, the boundary operators satisfy $\partial_2'[w_i] = [y_{i+1}] - [y_i]$, $\partial_1'[y_i] = [p]$. We also identify $[y_{i+n}] = [y_i]$ for all i.

Clearly, $B_2' = 0$. Let $z_2 = \sum_{i=1}^n \alpha_i[w_i]$ be in $\ker\{\partial_2' \colon C_2' \to C_1'\}$. Since $[w_1] + \dots + [w_n] = 0$, we can assume $z_2 = \sum_{i=1}^{n-1} \beta_i[w_i]$, for some β_i. Then, $\partial_2' z_2$ is equal to $\sum_{i=1}^{n-1} \beta_i([y_{i+1}] - [y_i]) = -\beta_1[y_1] + \sum_{i=1}^{n-2}(\beta_i - \beta_{i+1})[y_{i+1}] + \beta_{n-1}[y_n]$. The linear independence of $[y_1], \dots, [y_n]$ will result that the coefficients are zero, in particular $z_2 = 0$. Thus, we have the exactness at C_2'.

Note that $\mathrm{Im}\{\partial_2' : C_2' \to C_1'\}$ is generated by $\{[y_2] - [y_1], \dots, [y_n] - [y_{n-1}]\}$. Let $z_1 = \sum_{i=1}^n \alpha_i[y_i]$ be in $\ker\{\partial_1' \colon C_1' \to C_0'\}$. Then, since $\mathrm{Im}\{\partial_1' \colon C_1' \to C_0'\}$ is generated by $[p]$, $\sum_{i=1}^n \alpha_i = 0$. Hence z_1 is equal to $\alpha_1([y_1] - [y_2]) + (\alpha_1 + \alpha_2)([y_2] - [y_1]) + \dots + (\alpha_1 + \dots + \alpha_{n-1})([y_{n-1}] - [y_n]) + (\alpha_1 + \dots + \alpha_n)([y_n] - [y_{n+1}])$, or $z_1 \in \mathrm{Im}\{\partial_2' \colon C_2' \to C_1'\}$. Thus, we have the exactness at C_1'.

Finally, we have the exactness at C_0', because $\mathrm{Im}\{\partial_1' \colon C_1' \to C_0'\}$ has the basis $[p]$, which also generates the $\ker\{\partial_0' \colon C_0' \to 0\}$.

This concludes the Lemma 1.2.2. □

If K_1, K_2 are two such fine cell-decomposition, considering the overlaps, and refining further, we can find a common refinement $\hat{K}$ of both K_1 and K_2 such that the homologies $H_*(\hat{K}; \mathrm{Ad}_\rho)$ isomorphic to $H_*(K_1; \mathrm{Ad}_\rho)$ and $H_*(K_2; \mathrm{Ad}_\rho)$.

This will finish the proof of Lemma 1.2.1. □

Before defining the torsion corresponding to a representation $\rho \colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{F})$, we would like to recall the relation between $H_*(S; \mathrm{Ad}_\rho)$ and $H^*(S; \mathrm{Ad}_\rho)$. Next section will be about this. After that, we will continue with the torsion corresponding to a representation.

1.3. Poincaré duality isomorphisms.

Kronecker dual pairing. Let S be a compact hyperbolic surface with surface (i.e. genus at least 2). Recall that if K is a cell-decomposition of S, and $\rho \colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{F})$ is a representation, we associated the twisted chains $C_*(K; \mathrm{Ad}_\rho)$ and cochains $C^*(K; \mathrm{Ad}_\rho) = \mathrm{Hom}_{\mathbb{Z}[\pi_1(S)]}(C_*(\tilde{K}; \mathbb{Z}), \mathfrak{sl}_2(\mathbb{F}))$, where $\tilde{K}$ is the lift of K to the universal covering $\tilde{S}$ of S.

DEFINITION 1.3.1. For $i = 0, 1, 2$, the *Kronecker pairing* $\langle \cdot, \cdot \rangle \colon C^i(K; \mathrm{Ad}_\rho) \times C_i(K; \mathrm{Ad}_\rho) \to \mathbb{F}$ is defined by associating to $\theta \in C^i(K; \mathrm{Ad}_\rho)$ and $\sigma \otimes_\rho t \in C_i(K; \mathrm{Ad}_\rho)$, the number $B(t, \theta(\sigma))$, where $B(t_1, t_2) = 4\,\mathrm{Trace}(t_1 t_2)$ is the Cartan-Killing form.

The well-definiteness of Kronecker pairing can be verified as follows: Recall that $\sigma \otimes_\rho t \in C_i(K; \mathrm{Ad}_\rho)$ denotes the orbit $\{\gamma \bullet \sigma \otimes \gamma \bullet t;$ for all $\gamma \in \pi_1(S)\}$ of $\sigma \otimes t$, where the action of the fundamental group in the first component is by deck transformations and in the second one by adjoint action. Since trace is invariant under conjugation, and $\theta \in C^i(K; \mathrm{Ad}_\rho)$, we have $B(\gamma \bullet t, \theta(\gamma \bullet \sigma)) = B(t, \theta(\sigma))$ for all $\gamma \in \pi_1(S)$.

Naturally, the pairing can be extended to $\langle \cdot , \cdot \rangle \colon H^i(S; \mathrm{Ad}_\rho) \times H_i(S; \mathrm{Ad}_\rho) \to \mathbb{F}$. More explicitly, let $\theta' = \theta + \delta\theta''$, where θ'' is in C^{i-1} and δ denotes the coboundary operator, let $\sigma' = \sigma + d\sigma''$, for some $\sigma'' \in C_{i+1}$. Then, $B(t, \theta'(\sigma'))$ equals to $B(t, \theta(\sigma)) + B(t, \theta(d\sigma'')) + B(t, (\delta\theta'')(\sigma)) + B(t, (\delta\theta'')(d\sigma''))$. By the relation between d and δ and the choice of θ'', σ'', the last three terms vanish. Finally, since B is non-degenerate and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ is a field, $\langle \cdot , \cdot \rangle \colon H^i(S; \mathrm{Ad}_\rho) \times H_i(S, \mathrm{Ad}_\rho) \to \mathbb{F}$ is a pairing, too.

Cup product $\smile_B$. Here, we will explain the cup product

$$\smile_B \colon H^p(S; \mathrm{Ad}_\rho) \times H^q(S; \mathrm{Ad}_\rho) \to H^{p+q}(S; \mathbb{F}),$$

induced by the Cartan-Killing form B.

Let K be a cell-decomposition of the compact hyperbolic surface S without boundary. Consider the cup product

$$\tilde{\cup} \colon C^p(K; \mathrm{Ad}_\rho) \times C^q(K; \mathrm{Ad}_\rho) \to C^{p+q}(\tilde{S}; \mathfrak{sl}_2(\mathbb{F}) \otimes \mathfrak{sl}_2(\mathbb{F}))$$

defined by $(\theta_p \tilde{\cup} \theta_q)(\sigma_{p+q}) = \theta_p((\sigma_{p+q})_{\mathrm{front}}) \otimes \theta_q((\sigma_{p+q})_{\mathrm{back}})$, where σ_{p+q} is in $C_{p+q}(\tilde{K}; \mathbb{Z})$. Since $\theta_p \colon C_p(\tilde{K}; \mathbb{Z}) \to \mathfrak{sl}_2(\mathbb{F})$ and $\theta_q \colon C_q(\tilde{K}; \mathbb{Z}) \to \mathfrak{sl}_2(\mathbb{F})$ are $\mathbb{Z}[\pi_1(S)]$-module homomorphisms and $B \colon \mathfrak{sl}_2(\mathbb{F}) \times \mathfrak{sl}_2(\mathbb{F}) \to \mathbb{F}$ is non-degenerate, we can also define

$$\cup' \colon C^p(K; \mathrm{Ad}_\rho) \times C^q(K; \mathrm{Ad}_\rho) \to C^{p+q}(\tilde{S}; \mathbb{F})$$

by $(\theta_p \cup' \theta_q)(\sigma_{p+q}) = B(\theta_p((\sigma_{p+q})_{\mathrm{front}}), \theta_q((\sigma_{p+q})_{\mathrm{back}}))$. By the fact that B is invariant under adjoint action, $\theta_p \cup' \theta_q$ is invariant under the action of $\pi_1(S)$. Therefore, we have the cup product

$$\smile_B \colon C^p(K; \mathrm{Ad}_\rho) \times C^q(K; \mathrm{Ad}_\rho) \to C^{p+q}(K; \mathbb{F}).$$

We can naturally extend $\smile_B$ to twisted cohomologies. Like twisted homologies, twisted cohomologies are also independent of the cell-decomposition. Thus, we have

$$\begin{aligned} \smile_B \colon H^p(S; \mathrm{Ad}_\rho) \times H^q(S; \mathrm{Ad}_\rho) &\to H^{p+q}(S; \mathbb{F}) \\ [\theta_p], [\theta_q] &\mapsto [\theta_p \smile_B \theta_q]. \end{aligned}$$

Actually, considering the trivializations, one may also think $\theta_p = \omega_p \otimes t_1$ and $\theta_q = \omega_q \otimes t_2$ for some $\omega_p \in H^p(S), \omega_q \in H^q(S)$, and $t_1, t_2 \in \mathfrak{sl}_2(\mathbb{F})$. As a result, $\theta_p \smile_B \theta_q = \omega_p \wedge \omega_q B(t_1, t_2)$.

Intersection Form. Let S be a compact hyperbolic surface without boundary, let K, K^* be dual triangulation of S. Recall that if $\sigma \in K$ is a 2-simplex, $\sigma^* \in K^*$ is a vertex in σ. If $\sigma_1, \sigma_2 \in K$ are two 2-simplexes meeting along a 1-simplex α, then $\alpha^* \in K^*$ is the 1-simplex with end points $\sigma_1^*, \sigma_2^* \in K^*$ and transversely meeting with α.

If $\tilde{K}, \tilde{K}^*$ are the lifts of K, K^*, respectively, then they will also be dual in the universal covering $\tilde{S}$ of S. Let α, β be in $C_i(\tilde{K};\mathbb{Z})$, $C_{2-i}(\tilde{K}^*;\mathbb{Z})$, respectively. If $\alpha \cap \beta = \emptyset$, then the intersection number $\alpha\,.\,\beta$ is 0. If $\alpha \cap \beta = \{x\}$, then it is respectively $1, -1$, when the orientation of $T_x\alpha \oplus T_x\beta$ coincides with that of $T_x\tilde{S}$, and when the orientation of $T_x\alpha \oplus T_x\beta$ does not coincide with that of $T_x\tilde{S}$.

Using the Cartan-Killing form B of $\mathfrak{sl}_2(\mathbb{F})$, we can define an intersection form on the twisted chains as follows

$$(\,\cdot\,,\ \cdot\,)\colon C_i(K;\mathrm{Ad}_\rho) \times C_{2-i}(K^*;\mathrm{Ad}_\rho) \to \mathbb{F}$$

$(\sigma_1 \otimes t_1, \sigma_2 \otimes t_2) = \sum_{\gamma\in\pi_1(S)} \sigma_1\,.\,(\gamma\bullet\sigma_2)B(t_1, \gamma\bullet t_2)$, where the action of γ on t_2 by $\mathrm{Ad}_{\rho(\gamma)}$, and on σ_2 as deck transformation, and “ . ” denotes the above intersection number.

Note that the set $\{\gamma \in \pi_1(S);\ \sigma_1 \cap \gamma\bullet\sigma_2\}$ is finite, because the action of $\pi_1(S)$ on $\tilde{S}$ properly, discontinuously, and freely, and σ_1, σ_2 are compact. Note also that since intersection number is anti-symmetric and B is invariant under adjoint action, $(\,\cdot\,,\ \cdot\,)$ is anti-symmetric, too.

We can naturally extend the intersection form to twisted homologies $(\cdot,\cdot)\colon H_i(K;\mathrm{Ad}_\rho) \times H_{2-i}(K^*;\mathrm{Ad}_\rho) \to \mathbb{F}$. Recall that twisted homologies do not depend on the cell-decomposition. Thus, we have a non-degenerate anti-symmetric form

$$(\,\cdot\,,\ \cdot\,)\colon H_i(S;\mathrm{Ad}_\rho) \times H_{2-i}(S;\mathrm{Ad}_\rho) \to \mathbb{F}.$$

Finally, the isomorphisms induced by the Kronecker pairing and the intersection form will give us the Poincare duality isomorphisms. Namely,

$$\mathrm{PD}\colon H_i(S;\mathrm{Ad}_\rho) \overset{\text{intersection form}}{\cong} H_{2-i}(S;\mathrm{Ad}_\rho)^* \overset{\text{Kronecker pairing}}{\cong} H^{2-i}(S;\mathrm{Ad}_\rho).$$

Therefore, for $i = 0, 1, 2$, we have the following commutative diagram

$$\begin{tikzcd}
H^{2-i}(S;\mathrm{Ad}_\rho) \times H^{i}(S;\mathrm{Ad}_\rho) \arrow[r, "\smile_B"] & H^2(S;\mathbb{F}) \\
H_i(S;\mathrm{Ad}_\rho) \times H_{2-i}(S;\mathrm{Ad}_\rho) \arrow[u, "\mathrm{PD}\ \ \mathrm{PD}"] \arrow[r, "(\,,\,)"'] \arrow[ur, phantom, "\circlearrowleft"] & \mathbb{F}, \arrow[u]
\end{tikzcd}$$

where $\mathbb{F} \to H^2(S;\mathbb{F})$ is the isomorphism sending $1 \in \mathbb{F}$ to the fundamental class of $H^2(S;\mathbb{F})$.

If $\rho\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{F})$ is irreducible (e.g. when ρ is discrete, faithful), then $H_0(S;\mathrm{Ad}_\rho)$, $H_2(S;\mathrm{Ad}_\rho)$, $H^0(S;\mathrm{Ad}_\rho)$, and $H^2(S;\mathrm{Ad}_\rho)$ are all 0. Hence, we only have the commutative diagram

$$\begin{tikzcd} H^1(S;\mathrm{Ad}_\rho) \times H^1(S;\mathrm{Ad}_\rho) \arrow[r, "\smile_B"] & H^2(S;\mathbb{F}) \\ H_1(S;\mathrm{Ad}_\rho) \times H_1(S;\mathrm{Ad}_\rho) \arrow[u, "\mathrm{PD}"] \arrow[r, "(\,,\,)"] \arrow[ur, phantom, "\circlearrowleft"] & \mathbb{F}. \arrow[u] \end{tikzcd}$$

Finally, for future reference, we would like to mention the fact that $H^1(S;\mathrm{Ad}_\rho)$, $H_1(S;\mathrm{Ad}_\rho)$ are isomorphic respectively to the tangent space $T_\rho\mathfrak{Teich}(S)$ and of the Teichmüller space of S and to the cotangent space $T^*_\rho\mathfrak{Teich}(S)$ and of the Teichmüller space of S, when the field $\mathbb{F}$ is $\mathbb{R}$.

1.4. Torsion corresponding to a representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{F})$. In the previous section, for a fixed compact hyperbolic surface S without boundary and a representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{F})$, we associated the twisted chain complex $0 \to C_2(K;\mathrm{Ad}_\rho) \to C_1(K;\mathrm{Ad}_\rho) \to C_0(K;\mathrm{Ad}_\rho)$. Recall that $C_i(K;\mathrm{Ad}_\rho) = C_i(\tilde{K};\mathbb{Z}) \otimes_\rho \mathfrak{sl}_2(\mathbb{F})$ is defined as the quotient $C_i(\tilde{K};\mathbb{Z}) \otimes \mathfrak{sl}_2(\mathbb{F})/\!\sim$, where we identify the orbit $\{\gamma \bullet \sigma \otimes \gamma \bullet t;\ \gamma \in \pi_1(S)\}$ of $\sigma \otimes t$, and where the action of the fundamental group in the first slot by deck transformations, and in the second slot by the conjugation with $\rho(\,\cdot\,)$.

We will now explain the torsion of the twisted chain complex, and will follow the notations of [22]. If $\{e^i_1, \dots, e^i_{m_i}\}$ is a basis for the $C_i(K;\mathbb{Z})$, then $c_i := \{\tilde{e}^i_1, \dots, \tilde{e}^i_{m_i}\}$ is a $\mathbb{Z}[\pi_1(S)]$-basis for $C_i(\tilde{K};\mathbb{Z})$, where $\tilde{e}^i_j$ is a lift of e^i_j. If we choose a $\mathbb{F}$-basis $\mathcal{A} = \{\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3\}$ of $\mathfrak{sl}_2(\mathbb{F})$, then $c_i \otimes_\rho \mathcal{A}$ will be an $\mathbb{F}$-basis for $C_i(K,\mathrm{Ad}_\rho)$, called a *geometric* for $C_i(K;\mathrm{Ad}_\rho)$.

DEFINITION 1.4.1. If S is a compact hyperbolic surface without boundary, $\rho\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{F})$ is a representation, and K is a cell-decomposition of S, then $\mathrm{Tor}(C_*(K;\mathrm{Ad}_\rho), \{c_p \otimes_\rho \mathcal{A}\}^2_{p=0}, \{\mathfrak{h}_p\}^2_{p=0})$ is the *Reidemeister torsion* of the triple K, Ad_ρ, and $\{\mathfrak{h}_p\}^2_{p=0}$, where $\mathfrak{h}_p$ is a $\mathbb{F}$-basis for $H_p(K;\mathrm{Ad}_\rho)$.

In the next lemma, we will see that the definition does not depend on $\mathcal{A}$, lifts $\tilde{e}^i_j$, and conjugacy class of ρ. In later sections, we will also conclude that torsion is independent of the cell-decomposition.

Lemma 1.4.2. $\mathrm{Tor}(C_*(K;\mathrm{Ad}_\rho), \{c_p \otimes_\rho \mathcal{A}\}^2_{p=0}, \{\mathfrak{h}_p\}^2_{p=0})$ *is independent of $\mathcal{A}$, lifts $\tilde{e}^i_j$, and conjugacy class of ρ.*

Proof. *Independence of $\mathcal{A}$*: Let $\mathcal{A}'$ be another $\mathbb{F}$-basis for $\mathfrak{sl}_2(\mathbb{F})$ and let T be the change-base-matrix from $\mathcal{A}'$ to $\mathcal{A}$. Using the techniques presented in §1,

$\mathrm{Tor}(C_*(K;\mathrm{Ad}_\rho), \{c_p \otimes_\rho \mathcal{A}'\}_{p=0}^2, \{\mathfrak{h}_p\}_{p=0}^2)$ is $\prod_{p=0}^2 [\mathfrak{b}_p \oplus \tilde{\mathfrak{h}}_p \oplus \tilde{\mathfrak{b}}_{p-1}, \mathfrak{c}_p \otimes \mathcal{A}']^{(-1)^{p+1}}$. By the change-base-formula Remark 1.1.2, $\mathrm{Tor}(C_*(K;\mathrm{Ad}_\rho), \{c_p \otimes_\rho \mathcal{A}'\}_{p=0}^2, \{\mathfrak{h}_p\}_{p=0}^2)$ equals to the product of $\mathrm{Tor}(C_*(K;\mathrm{Ad}_\rho), \{c_p \otimes_\rho \mathcal{A}\}_{p=0}^2, \{\mathfrak{h}_p\}_{p=0}^2)$ and $\det(T)^{-\chi(S)}$, where the last term is by the fact that $[\mathfrak{b}_i \oplus \tilde{\mathfrak{h}}_i \oplus \tilde{\mathfrak{b}}_{i-1}, \mathcal{A}' \otimes c_i] = [\mathfrak{b}_i \oplus \tilde{\mathfrak{h}}_i \oplus \tilde{\mathfrak{b}}_{i-1}, \mathcal{A} \otimes c_i] \cdot \det(T)^{\#c_i}$, and $\#X$ denotes the cardinality of the set X, and $[\mathfrak{a}, \mathfrak{b}]$ is the determinant of the base-change-matrix from basis $\mathfrak{b}$ to $\mathfrak{a}$.

If, for example, $\det T = \pm 1$, then $\mathcal{A}$ and $\mathcal{A}'$ will produce the same torsion, because the Euler-characteristic $\chi(S)$ is even. Or, if $\mathbb{F} = \mathbb{C}$ and $\mathcal{A}$, $\mathcal{A}'$ are two B-orthonormal bases, where B is the Cartan-Killing form of $\mathfrak{sl}_2(\mathbb{C})$, then T is in $O(3,\mathbb{C})$. Again since the Euler-characteristic $\chi(S)$ is even, the corresponding torsions will be the same.

Independence of lifts: Let $c_i' = \{\tilde{e}_1^i \bullet \gamma, \ldots, \tilde{e}_{m_i}^i\}$ be another lift of $\{e_1^i, \ldots, e_{m_i}^i\}$, where we take another lift of e_1^i, and leave the others the same. Recall that $\tilde{e}_1^i \bullet \gamma \otimes t = \tilde{e}_1^i \otimes \gamma \bullet t$, where the action in the second slot is by $\mathrm{Ad}_{\rho(\gamma)}$. Then, $\mathfrak{c}_i' \otimes \mathcal{A} = \mathfrak{c}_i \otimes \mathrm{Ad}_{\rho(\gamma)}(\mathcal{A})$ and $\mathrm{Tor}(C_*(K;\mathrm{Ad}_\rho), \{c_p' \otimes_\rho \mathcal{A}\}_{p=0}^2, \{\mathfrak{h}_p\}_{p=0}^2)$ is equal to $\mathrm{Tor}(C_*(K;\mathrm{Ad}_\rho), \{c_p \otimes_\rho \mathcal{A}\}_{p=0}^2, \{\mathfrak{h}_p\}_{p=0}^2) \cdot \det(T)^{-\chi(S)}$, where T is the matrix of $\mathrm{Ad}_{\rho(\gamma)} \colon \mathfrak{sl}_2(\mathbb{F}) \to \mathfrak{sl}_2(\mathbb{F})$ with respect to basis $\mathcal{A}$.

For instance, if $\det T = \pm 1$, then we have the same torsion. Or, if $\mathbb{F} = \mathbb{C}$ and $\mathcal{A}$ is B-orthonormal, then T will be in $SO(3, \mathbb{C})$. The latter can be verified as follows: Recall that the adjoint representation $\mathrm{Ad}\colon \mathrm{PSL}_2(\mathbb{C}) \to \mathcal{E}nd(\mathfrak{sl}_2(\mathbb{C}))$ assigns to each $x \in \mathrm{PSL}_2(\mathbb{C})$ the conjugation endomorphism $\mathrm{Ad}_x \colon \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{sl}_2(\mathbb{C})$ by x. Since Ad_x has the inverse $\mathrm{Ad}_{x^{-1}}$, the adjoint representation maps $\mathrm{PSL}_2(\mathbb{C})$ into $\mathrm{GL}(\mathfrak{sl}_2(\mathbb{C}))$.

Let $\mathcal{A} = \{a_1, a_2, a_3\}$ be a B-orhonormal basis of $\mathfrak{sl}_2(\mathbb{C})$ i.e. the matrix of B in this basis is the 3×3 identity matrix. Note that since trace is invariant under conjugation, Ad_x also preserves B. Therefore, the matrix T of Ad_x in this basis is an orthogonal 3×3 matrix with complex entries, because $\mathrm{Id}_{3\times 3} = T\,\mathrm{Id}_{3\times 3}\,T^{\mathrm{trans}}$. This gives that $\det T = \pm 1$ and finalizes the proof since the Euler characteristic of S is even.

Actually, if the matrix $x \in \mathrm{PSL}_2(\mathbb{C})$ is a hyperbolic (e.g. if x is in $\rho(\pi_1(S))$), then Ad_x is in $SO(3, \mathbb{C})$. This is because of the following: determinant of the matrix of $\mathrm{Ad}_{\rho(\gamma)}$ is independent of basis, so consider $\mathcal{A}' = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$, which is not B-orthonormal. Since the surface S is compact hyperbolic (without boundary), $\pi_1(S)$ consists of only hyperbolic elements. Thus, $\rho(\gamma) \in \mathrm{PSL}_2(\mathbb{C})$ is also hyperbolic i.e. let λ, λ^{-1} be the eigenvalues of $\rho(\gamma)$, then $Q\rho(\gamma)Q^{-1} = D$ for some $Q \in \mathrm{PSL}_2(\mathbb{C})$, where $D = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$. Hence, if we use the basis $\mathcal{A}'$, then it is enough to find the determinant of the matrix of Ad_D in the basis $\mathcal{A}'$. An easy computation will result that the matrix of Ad_D in the basis $\mathcal{A}'$ is simply $\mathrm{Diagonal}(\lambda^2, \lambda^{-2}, 1)$. This verifies that $\mathrm{Ad}_x \in SO(3, \mathbb{C})$ and will also conclude the proof of the independence of lifts.

Independence of conjugacy class of ρ: This follows from the fact that if ρ, ρ' are conjugate representation, then the corresponding twisted chains and cochains are isomorphic. $\square$

2. Reidemeister torsion using Witten's notations

Let V be a vector space of dimension k over $\mathbb{R}$. Let $\det(V)$ denote the top exterior power $\bigwedge^k V$ of V. A *measure* on V is a non-zero functional $\alpha\colon \det(V) \to \mathbb{R}$ on $\det(V)$, i.e. $\alpha \in \det(V)^{-1}$, where -1 denotes the dual space.

Recall that the isomorphism between $\det(V)^{-1}$ and $\det(V^*)$ is given by the pairing $\langle\,\cdot\,,\,\cdot\,\rangle\colon \det(V^*) \times \det(V) \to \mathbb{R}$, defined by

$$\langle f_1^* \wedge \cdots \wedge f_k^*, e_1 \wedge \cdots \wedge e_k\rangle = \det[f_i^*(e_j)],$$

i.e. the determinant $[\mathfrak{f}, \mathfrak{e}]$ of the change-base-matrix from basis $\mathfrak{e} = \{e_1, \ldots, e_k\}$ to $\mathfrak{f} = \{f_1, \ldots, f_k\}$, where f_i^* is the dual element corresponding to f_i, namely, $f_i^*(f_j) = \delta_{ij}$. Below $(v_1 \wedge \cdots \wedge v_k)^{-1}$ will denote $(v_1)^* \wedge \cdots \wedge (v_k)^*$

Note also that $\langle f_1^* \wedge \cdots \wedge f_k^*, e_1 \wedge \cdots e_k\rangle = \langle e_1^* \wedge \cdots \wedge e_k^*, f_1 \wedge \cdots f_k\rangle^{-1}$, i.e. $[\mathfrak{f}, \mathfrak{e}] = [\mathfrak{e}, \mathfrak{f}]^{-1}$. So, using the pairing, $[\mathfrak{f}, \bullet]$ can be considered a linear functional on $\det(V)$ and $[\bullet, \mathfrak{e}]$ can be considered a linear functional on $\det(V^*)$.

Let $C_*\colon 0 \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \to 0$ be a chain complex of finite dimensional vector spaces with *volumes* $\alpha_p \in \det(C_p)^{-1}$, i.e. $\alpha_p = (c_1^p)^* \wedge \cdots \wedge (c_{m_p}^p)^*$ for some basis $\{c_1^p, \ldots, c_{m_p}^p\}$ for C_p. If, moreover, we assume that C_* is exact (or acyclic), then $H_p(C_*) = 0$ for all p. In particular, we have the short exact sequence

$$0 \to \underbrace{\operatorname{Im}\{\partial_{p+1}\colon C_{p+1} \to C_p\}}_{B_p} \overset{i_p}{\hookrightarrow} C_p \overset{\partial_p}{\twoheadrightarrow} \underbrace{\operatorname{Im}\{\partial_p\colon C_p \to C_{p-1}\}}_{B_{p-1}} \to 0.$$

Let $\{b_1^p, \ldots, b_{k_p}^p\}$, $\{b_1^{p-1}, \ldots, b_{k_{p-1}}^{p-1}\}$ be bases for B_p, B_{p-1}, respectively. Then, $\{b_1^p, \ldots, b_{k_p}^p, \tilde{b}_1^{p-1}, \ldots, \tilde{b}_{k_{p-1}}^{p-1}\}$ is a basis for C_p, where $\partial_p(\tilde{b}_{p-1}^i) = b_{p-1}^i$ and thus $b_1^p \wedge \cdots \wedge b_{k_p}^p \wedge \tilde{b}_1^{p-1} \wedge \cdots \wedge \tilde{b}_{k_{p-1}}^{p-1}$ is a basis for $\det(C_p)$.

If u denotes $\bigotimes_{p=0}^n \bigl(b_1^p \wedge \cdots \wedge b_{k_p}^p \wedge \tilde{b}_1^{p-1} \wedge \cdots \wedge \tilde{b}_{k_{p-1}}^{p-1}\bigr)^{(-1)^p}$, then u is an element of $\bigotimes_{p=0}^n (\det(C_p))^{(-1)^p}$, where the exponent (-1) denotes the dual of the vector space. E. Witten describes the torsion as:

$$\begin{aligned}\operatorname{Tor}(C_*) &= \left\langle u, \bigotimes_{p=0}^n \alpha_p^{(-1)^p}\right\rangle \\ &= \prod_{p=0}^n \bigl\langle b_1^p \wedge \cdots \wedge b_{k_p}^p \wedge \tilde{b}_1^{p-1} \wedge \cdots \wedge \tilde{b}_{k_{p-1}}^{p-1}, (c_1^p)^* \wedge \cdots \wedge \bigl(c_{m_p}^p\bigr)^*\bigr\rangle^{(-1)^p},\end{aligned}$$

which is nothing but $\prod_{p=0}^n \bigl[\{c_1^p, \ldots, c_{m_p}^p\}, \{b_1^p, \ldots, b_{k_p}^p, \tilde{b}_1^{p-1}, \ldots, \tilde{b}_{k_{p-1}}^{p-1}\}\bigr]^{(-1)^p}$ or $\prod_{p=0}^n \bigl([\{b_1^p, \ldots, b_{k_p}^p, \tilde{b}_1^{p-1}, \ldots, \tilde{b}_{k_{p-1}}^{p-1}\}, \{c_1^p, \ldots, c_{m_p}^p\}]^{(-1)}\bigr)^{(-1)^p}$. The last term coincides with the $\operatorname{Tor}(C_*, \{c_p\}_{p=0}^n, \{0\}_{p=0}^n)$ defined in §1.

We will now explain how a general chain complex can be (unnaturally) written as a direct sum of two chain complexes, one of which is exact and the other is ∂-zero.

Theorem 2.0.3. *If $C_*\colon 0 \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \to 0$ is any chain complex, then it can be splitted as $C_* = C'_* \oplus C''_*$, where C'_* is exact, and C''_* is ∂-zero.*

Proof. Consider the short-exact sequences

$$0 \to \ker \partial_p \hookrightarrow C_p \overset{\partial_p}{\twoheadrightarrow} \operatorname{Im} \partial_p \to 0,$$

$$0 \to \operatorname{Im} \partial_{p+1} \hookrightarrow \ker \partial_p \overset{\pi_p}{\twoheadrightarrow} H_p(C) \to 0.$$

If $l_p\colon \operatorname{Im} \partial_p \to C_p$, and $s_p\colon H_p(C) \to \ker \partial_p$ are sections, i.e. $\partial_p \circ l_p = \mathrm{id}_{\operatorname{Im} \partial_p}$, and $\pi_p \circ s_p = \mathrm{id}_{H_p(C)}$, then C_p is equal to $\ker \partial_p \oplus l_p(\operatorname{Im} \partial_p)$ or $\operatorname{Im} \partial_{p+1} \oplus s_p(H_p(C)) \oplus l_p(\operatorname{Im} \partial_p)$. Define $C'_p := \operatorname{Im} \partial_{p+1} \oplus l_p(\operatorname{Im} \partial_p)$ and $C''_p := s_p(H_p(C))$. Restricting $\partial_p\colon C_p \to C_{p-1}$ to these, we obtain two chain complexes $(C'_*, \partial'_*)(C''_*, \partial''_*)$.

As C''_p is a subspace of $\ker \partial_p$, $\partial''_p\colon C''_p \to C''_{p-1}$ is the zero map, i.e. C''_* is ∂-zero chain complex. Note also $\ker\{\partial''_p\colon C''_p \to C''_{p-1}\}$ equals to C''_p and $\operatorname{Im}\{\partial''_{p+1}\colon C''_{p+1} \to C''_p\}$ is $\{0\}$. Then, $H_p(C''_*) = C''_p/\{0\}$ is isomorphic to $H_p(C)$, because $C''_p = s_p(H_p(C))$ is isomorphic to $H_p(C)$.

The exactness of (C'_*, ∂'_*) can be seen as follows: Since $\operatorname{Im} \partial_{p+1}$ is a subspace of $\ker \partial_p$, the image of $\operatorname{Im} \partial_{p+1}$ under ∂'_p is zero. Hence, $\ker\{\partial'_p\colon C'_p \to C'_{p-1}\}$ equals to $\operatorname{Im}\{\partial_{p+1}\colon C_{p+1} \to C_p\}$. Since $\partial_p \circ l_p = \mathrm{id}_{\operatorname{Im} \partial_p}$, and $\partial'_p\colon C'_p \to C'_{p-1}$ is the restriction of $\partial_p\colon C_p \to C_{p-1}$, then $\operatorname{Im}\{\partial'_p\colon C'_p \to C'_{p-1}\}$ equals to $\operatorname{Im}\{\partial_p : C_p \to C_{p-1}\}$. Similarly, $\operatorname{Im}\{\partial'_{p-1}\colon C'_{p-1} \to C'_{p-2}\} = \operatorname{Im}\{\partial_{p-1}\colon C_{p-1} \to C_{p-2}\}$ and $\ker\{\partial'_{p-1}\colon C'_{p-1} \to C'_{p-2}\} = \operatorname{Im}\{\partial_p\colon C_p \to C_{p-1}\}$, because $\operatorname{Im} \partial_p$ is a subspace of $\ker \partial_{p-1}$ and l_{p-1} is a section of $\partial_{p-1}\colon C_{p-1} \to \operatorname{Im} \partial_{p-1}$. Consequently, $\operatorname{Im}\{\partial'_p\colon C'_p \to C'_{p-1}\} = \ker\{\partial'_{p-1}\colon C_{p-1} \to C_{p-2}\} = \operatorname{Im} \partial_p$ and we have the exactness of C'_*.

This concludes Theorem 2.0.3. □

In the next result, we will explain Witten's remark on ([33] p.185) how torsion $\operatorname{Tor}(C_*)$ of a general complex can be interpreted as an element of the dual of the one dimensional vector space $\bigotimes_{p=0}^{n}(\det(H_p(C)))^{(-1)^p}$.

Theorem 2.0.4. $\operatorname{Tor}(C_*)$ *of a general complex is as an element of the dual of the one dimensional vector space* $\bigotimes_{p=0}^{n}(\det(H_p(C)))^{(-1)^p}$.

Proof. Let C_* be a general chain complex of finite dimensional vector spaces with volumes $\alpha_p \in (\det(C_p))^{-1}$, i.e. $\alpha_p = (c_p^1)^* \wedge \cdots \wedge \left(c_p^{i_p}\right)^*$, for some basis $\mathfrak{c}_p = \left\{c_p^1, \ldots, c_p^{i_p}\right\}$ of C_p. Let $C_* = C'_* \oplus C''_*$ be the above (unnatural) splitting of C_* i.e. $C'_p = \operatorname{Im} \partial_{p+1} \oplus l_p(\operatorname{Im} \partial_p)$ and $C''_p = s_p(H_p(C))$, where $l_p\colon \operatorname{Im} \partial_p \to C_p$ is the section of $\partial_p\colon C_p \twoheadrightarrow \operatorname{Im} \partial_p$ and $s_p\colon H_p(C) \to \ker \partial_p$ is the section of $\pi_p\colon \ker \partial_p \twoheadrightarrow H_p(C)$ used in Theorem 2.0.3.

Since $C_p = \operatorname{Im}\partial_{p+1} \oplus s_p(H_p(C)) \oplus l_p(\operatorname{Im}\partial_p)$, we can break the basis $\mathfrak{c}_p$ of C_p into three blocks as $\mathfrak{c}_1^p \sqcup \mathfrak{c}_2^p \sqcup \mathfrak{c}_3^p$, where $\mathfrak{c}_1^p$ generates $\operatorname{Im}\partial_{p+1}$, $\mathfrak{c}_2^p$ is basis for $s_p(H_p(C))$ i.e. $[\mathfrak{c}_2^p] = \pi_p(\mathfrak{c}_2^p)$ generates $H_p(C)$, and $\partial_p(\mathfrak{c}_3^p)$ is a basis for $\operatorname{Im}\partial_p$. As the determinant of change-base-matrix from $\mathfrak{c}_p$ to $\mathfrak{c}_p$ is 1, the bases $\mathfrak{c}_2^p$, $\mathfrak{c}_p = \mathfrak{c}_1^p \sqcup \mathfrak{c}_2^p \sqcup \mathfrak{c}_3^p$, and $\mathfrak{c}_1^p \sqcup \mathfrak{c}_3^p$ for C''_p, C_p, C'_p, will be compatible with the short-exact sequence of complexes

$$0 \to C''_* \hookrightarrow C_* = C''_* \oplus C'_* \twoheadrightarrow C'_* \to 0,$$

where we consider the inclusion as section $C'_p \to C_p$. Note also that $H_p(C'') = C''_p/0$ i.e. $s_p(H_p(C))$ which is isomorphic to $H_p(C)$.

By Milnor's result Theorem 1.1.3, we have $\operatorname{Tor}(C_*, \{\mathfrak{c}_p\}_{p=0}^n, \{\mathfrak{h}_p\}_{p=0}^n)$ is the product of $\operatorname{Tor}(C''_*, \{\mathfrak{c}_p^2\}_{p=0}^n, \{s_p(\mathfrak{h}_p)\}_{p=0}^n)$, $\operatorname{Tor}(C'_*, \{\mathfrak{c}_p^1 \sqcup \mathfrak{c}_p^3\}_{p=0}^n, \{0\}_{p=0}^n)$, and $\operatorname{Tor}(\mathcal{H}_*)$, where $\mathcal{H}_*$ is the long-exact sequence obtained from the above short-exact of chain complexes.

More precisely, $\mathcal{H}_*$: $0 \to H_n(C'') \to H_n(C) \to H_n(C') \to H_{n-1}(C'') \to H_{n-1}(C) \to H_{n-1}(C') \to \cdots \to H_0(C'') \to H_0(C) \to H_0(C') \to 0$. Since C'_* is exact, then $\mathcal{H}_*$ is the long exact-sequence $0 \to H_n(C'') \to H_n(C) \to 0 \to H_{n-1}(C'') \to H_{n-1}(C) \to 0 \to \cdots \to 0 \to H_0(C'') \to H_0(C) \to 0 \to 0$. Using the isomorphism $H_p(C) \to H_p(C'')$, namely s_p as section, we conclude that $\operatorname{Tor}(\mathcal{H}_*, \{s_p(\mathfrak{h}_p), \mathfrak{h}_p, 0\}_{p=0}^n, \{0\}_{p=0}^{3n+2}) = 1$.

Moreover, we can also verify that $\operatorname{Tor}(C'_*, \{\mathfrak{c}_p^1 \sqcup \mathfrak{c}_p^3\}_{p=0}^n, \{0\}_{p=0}^n) = 1$ as follows:

$$0 \to \ker\{\partial'_p : C'_p \to C'_{p-1}\} \hookrightarrow C'_p \overset{\partial'_p \overset{\text{i.e.}}{=} \partial_p}{\twoheadrightarrow} \operatorname{Im}\{\partial'_p : C'_p \to C'_{p-1}\} \to 0,$$

where $\ker\{\partial'_p : C'_p \to C'_{p-1}\}$ is $\operatorname{Im}\{\partial_{p+1} : C_{p+1} \to C_p\}$ and $\operatorname{Im}\{\partial'_p : C'_p \to C'_{p-1}\}$ is $\operatorname{Im}\{\partial_p : C_p \to C_{p-1}\}$. If we consider the section l_p, then we also have $\operatorname{Tor}(C'_*, \{\mathfrak{c}_p^1 \sqcup \mathfrak{c}_p^3\}_{p=0}^n, \{0\}_{p=0}^n) = 1$.

Therefore, $\operatorname{Tor}(C_*, \{\mathfrak{c}_p\}_{p=0}^n, \{\mathfrak{h}_p\}_{p=0}^n)$ is equal to $\operatorname{Tor}(C''_*, \{\mathfrak{c}_p^2\}_{p=0}^n, \{s_p(\mathfrak{h}_p)\}_{p=0}^n)$ i.e. $\prod_{p=0}^n [s_p(\mathfrak{h}_p), \mathfrak{c}_p^2]^{(-1)^{(p+1)}}$, where $[s_p(\mathfrak{h}_p), \mathfrak{c}_p^2]$ is the determinant of the change-base-matrix from $\mathfrak{c}_p^2$ to $s_p(\mathfrak{h}_p)$ of $C''_p = s_p(H_p(C))$. Recall that $s_p : H_p(C) \to \ker\partial_p$ is the section of $\pi_p : \ker\partial_p \twoheadrightarrow H_p(C)$. So, $[\mathfrak{c}_p^2]$, i.e. $\pi_p(\mathfrak{c}_p)$, and $\mathfrak{h}_p = [s_p(\mathfrak{h}_p)]$ are bases for $H_p(C)$. Since s_p is isomorphism onto its image, change-base-matrix from $\mathfrak{c}_p^2$ to $s_p(\mathfrak{h}_p)$ coincides with the one from $[\mathfrak{c}_p^2]$ to $\mathfrak{h}_p$.

As a result, we obtained that

$$\begin{aligned}\operatorname{Tor}(C_*, \{\mathfrak{c}_p\}_{p=0}^n, \{\mathfrak{h}_p\}_{p=0}^n) &= \prod_{p=0}^n [\mathfrak{h}_p, [\mathfrak{c}_p^2]]^{(-1)^{(p+1)}} \\ &= [\mathfrak{h}_0, [\mathfrak{c}_0^2]]^{-1} \cdot [\mathfrak{h}_1, [\mathfrak{c}_1^2]] \cdots [\mathfrak{h}_n, [\mathfrak{c}_n^2]]^{(-1)^{(n+1)}}.\end{aligned}$$

For p odd, $[\mathfrak{h}_p, [\mathfrak{c}_p^2]]^{(-1)^{(p+1)}}$ is $[\mathfrak{h}_p, [\mathfrak{c}_p^2]]$, and for p even, it is $[\mathfrak{h}_p, [\mathfrak{c}_p^2]]^{-1}$ or $[[\mathfrak{c}_p^2], \mathfrak{h}_p]$.

By the remark at the beginning of §2, for even p's, $[[\mathfrak{c}_p^2], \bullet]$ is linear functional on $\det(H_p(C))$, and for odd p's, $[[\mathfrak{c}_p^2], \bullet]$ is linear functional on $\det(H_p(C)^*) \equiv \det(H_p(C))^{-1}$, where the exponent -1 denotes the dual of the space.

This finishes the proof of Theorem 2.0.4. $\square$

In particular, considering the complex

$$C_*\colon 0 \to C_2(S;\mathrm{Ad}_\rho) \xrightarrow{\partial_2\otimes\mathrm{id}} C_1(S;\mathrm{Ad}_\rho) \xrightarrow{\partial_1\otimes\mathrm{id}} C_0(S;\mathrm{Ad}_\rho) \to 0,$$

where $\rho\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$, we conclude $\mathrm{Tor}(C_*)$ is in

$$(\det(H_2(S;\mathrm{Ad}_\rho)))^{(-1)^{0+1}} \otimes (\det(H_1(S;\mathrm{Ad}_\rho)))^{(-1)^{1+1}} \otimes (\det(H_0(S;\mathrm{Ad}_\rho)))^{(-1)^{2+1}}.$$

If, moreover, the representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$ is irreducible (e.g. when ρ is discrete, faithful), then $H_2(S;\mathrm{Ad}_\rho)$ and $H_0(S;\mathrm{Ad}_\rho)$ both vanish. Therefore, $\mathrm{Tor}(C_*)$ is in $\det(H_1(S;\mathrm{Ad}_\rho)) = \bigwedge^{\dim H_1(S;\mathrm{Ad}_\rho)} H_1(S;\mathrm{Ad}_\rho)$. We should also recall here that when $\rho\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$ is discrete, faithful, then $H_1(S;\mathrm{Ad}_\rho)$, $H^1(S;\mathrm{Ad}_\rho)$ can be identified with the cotangent space $T^*_\rho\mathfrak{Teich}(S)$ and the tangent space $T_\rho\mathfrak{Teich}(S)$ of the *Teichmüller space* of S, respectively.

We will close this section with the fact that torsion $\mathrm{Tor}(C_*(K;\mathrm{Ad}_\rho))$, where K is a cell-decomposition of compact hyperbolic surface S without boundary, $\rho\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$, is independent of the cell-decomposition, too.

Lemma 2.0.5. $\mathrm{Tor}(C_*(K;\mathrm{Ad}_\rho))$ *is independent of the cell-decompostion, it depends only on the representation* ρ.

Proof. Let K be a fine cell-decompositions of S as in the definition. Let $\hat{K}$ be a further refinement of K. As in Lemma 1.2.1, we obtain the chain complexes $\hat{C}_* = C_* \oplus \hat{C}'_*$, where $\hat{C}'_* = \hat{C}_*/\hat{C}_*$ is obtained by the added cells. We have the short-exact sequence of complexes $0 \to C_* \hookrightarrow \hat{C}_* \twoheadrightarrow C'_* := \hat{C}_*/C_* \to 0$, where C_* is obtained by the cell-decomposition K, $\hat{C}_*$ is obtained by the refined cell-decomposition $\hat{K}$, and C'_* is obtained by the added cells. Then, we have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C_2 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \hat{C}_2 \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & C'_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_1 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \hat{C}_1 \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & C'_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_0 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \hat{C}_0 \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & C'_0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Note that each row is exact, and torsion of each row is 1. More precisely, for $p = 0, 1, 2$, we have the exact row $0 \to C_p \hookrightarrow \hat{C}_p \twoheadrightarrow C'_p \to 0$. Considering the inclusion $s_2\colon C'_p \to \hat{C}_p$ as a section, we have torsion of each row is 1. Hence, bases $\mathfrak{c}_p$, $\mathfrak{c}_p \oplus \mathfrak{c}'_p$, $\mathfrak{c}'_p$ of C_p, $\hat{C}_p$, and C'_p are compatible in the sense that determinant of the change-base-matrix corresponding to the bases $\mathfrak{c}_p \oplus s_p(\mathfrak{c}'_p)$ and $\mathfrak{c}_p \oplus \mathfrak{c}'_p$ is (clearly) 1.

The short-exact sequence of complexes $0 \to C_* \hookrightarrow \hat{C}_* \twoheadrightarrow C'_* := \hat{C}_*/C_* \to 0$ also results the long-exact sequence of vector space $\mathcal{H}_*\colon 0 \to H_2(C_*) \to H_2(\hat{C}_*) \to H_2(C'_*) \to H_1(C_*) \to H_1(\hat{C}_*) \to H_1(C'_*) \to H_0(C_*) \to H_0(\hat{C}_*) \to H_2(C'_*) \to 0$. By Lemma 1.2.2, the chain complex C'_* is exact. Then, $H_p(C'_*) = 0$, for $p = 0, 1, 2$, and thus $H_p(C_*) \cong H_p(\hat{C}_*)$. Considering the isomorphism $H_p(\hat{C}_*) \to H_p(C_*)$ as section, we have $\mathrm{Tor}(\mathcal{H}_*) = 1$.

Since the bases of C_*, $\hat{C}_*$, and C'_* are clearly compatible, thus by Milnor's result Lemma 1.1.3, we get $\mathrm{Tor}(\hat{C}_*) = \mathrm{Tor}(C_*) \cdot \mathrm{Tor}(C'_*) \cdot \underbrace{\mathrm{Tor}(\mathcal{H}_*)}_{=1}$.

Lemma 2.0.6. $\mathrm{Tor}(C'_*)$ *is also* 1.

Proof. Recall that the exact complex $0 \to C'_2 \xrightarrow{\partial'_2} C'_1 \xrightarrow{\partial'_1} C_0 \to 0$, where $C'_* := \hat{C}_*/C_*$, is obtained from the added cells. Namely, for n-gon $w \in C_2$, we added a point p inside w, and n new 1-cells $y_1, \ldots, y_n$, so that we obtain n-new two-cells $w_1, \ldots, w_n$ with $w = w_1 \cup \cdots \cup w_n$. So, $\{[p]\}$, $\{[y_1], \ldots, [y_n]\}$, and $\{[w_1], \ldots, [w_n]\}$ are in the generating sets of C'_0, C'_1, and C'_2, respectively. Because the $w \in C_2$ is union of $w_1, \ldots, w_n$, $[w_1] + \cdots + [w_n] = 0$. Recall also that the boundary operators satisfy $\partial'_2[w_i] = [y_{i+1}] - [y_i]$, $\partial'_1[y_i] = [p]$. We also identify $[y_{i+n}] = [y_i]$ for all i.

The exactness of C'_* results $\ker\{\partial'_2\colon C'_2 \to C'_1\} = 0$. Thus, from the short-exact sequence, $0 \to \ker\{\partial'_2\colon C'_2 \to C'_1\} \hookrightarrow C'_2 \twoheadrightarrow \mathrm{Im}\{\partial'_2\colon C'_2 \to C'_1\} \to 0$, we have the isomorphism $C'_2 \cong \mathrm{Im}\{\partial'_2\colon C'_2 \to C'_1\}$. Consider the inverse of $C'_2 \to \mathrm{Im}\{\partial'_2\colon C'_2 \to C'_1\}$ as section $s_2\colon \mathrm{Im}\{\partial'_2\colon C'_2 \to C'_1\} \to C'_2$, namely, $s_2([y_{i+1}] - [y_i]) = [w_i]$. Recall also that $\{[y_2]-[y_1], [y_3]-[y_2], \ldots, [y_n]-[y_{n-1}]\}$ are in the generating set of $\mathrm{Im}\{\partial'_2\colon C'_2 \to C'_1\}$. Clearly, determinant of the change-base-matrix for C'_2 is 1.

For the short-exact sequence $0 \to \ker\{\partial'_1\colon C'_1 \to C'_0\} \hookrightarrow C'_1 \twoheadrightarrow \mathrm{Im}\{\partial'_1\colon C'_1 \to C'_0\} \to 0$, consider the section $s_1\colon \mathrm{Im}\{\partial'_1\colon C'_1 \to C'_0\} \to C'_1$ defined by $s_1([p]) = (-1)^{n-1}[y_n]$. Here, recall that $\{[p]\}$ is in the generating set of $\mathrm{Im}\{\partial'_1\colon C'_1 \to C'_0\}$. Since C'_* is exact complex, hence $\{[y_2]-[y_1], [y_3]-[y_2], \ldots, [y_n]-[y_{n-1}]\}$ also in the generating set of $\ker\{\partial'_1\colon C'_1 \to C'_0\}$. Then, the determinant of change-base-matrix from $\{[y_1], [y_2], \ldots, [y_n]\}$ to $\{[y_2]-[y_1], \ldots, [y_n]-[y_{n-1}], (-1)^{n-1}[y_n]\} = \underbrace{(-1)\cdots(-1)}_{n-1}(-1)^{n-1} = 1$.

Therefore, $\mathrm{Tor}(C'_*) = 1$, which concludes Lemma 2.0.6. □

As a result, we proved that $\mathrm{Tor}(\hat{C}_*) = \mathrm{Tor}(C_*) \cdot \overbrace{\mathrm{Tor}(C'_*)}^{=1} \cdot \overbrace{\mathrm{Tor}(\mathcal{H}_*)}^{=1} = \mathrm{Tor}(C_*)$, i.e. Tor is *invariant under subdivision*. If K_1, K_2 are two fine cell-decompositions, considering the overlaps and refining as before, we get a common refinement $\hat{K}$ for both K_1 and K_2. Hence, the corresponding torsions will be $\mathrm{Tor}(\hat{C}_*)$.

This finishes the proof of Lemma 2.0.5 □

E. Witten describes the fact that *rows of the short-exact sequence* $0 \to C_* \hookrightarrow \hat{C}_* \to C'_* := \hat{C}_*/C_* \to 0$ *has torsion* 1 by saying that the short-exact sequence of complexes is *volume exact*. Hence, Lemma 2.0.5 says that in a short-exact sequence of complexes which is also volume exact, then the alternating product of the torsions is 1 i.e. $\mathrm{Tor}(C_*)\,\mathrm{Tor}(\hat{C}_*)^{-1}\,\mathrm{Tor}(C'_*) = 1$, which is actually $\mathrm{Tor}(\mathcal{H}_*)$.

2.1. Symplectic chain complex.

DEFINITION 2.1.1. $C_*\colon 0 \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_{n/2} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \to 0$ is a *symplectic chain complex*, if

- $n \equiv 2 \pmod 4$ and
- there exist non-degenerate anti-symmetric ∂-compatible bilinear maps i.e. $\omega_{p,n-p}\colon C_p \times C_{n-p} \to \mathbb{R}$ s.t. $\omega_{p,n-p}(a,b) = (-1)^{p(n-p)}\omega_{n-p,p}(b,a)$ and $\omega_{p,n-p}(\partial_{p+1}a, b) = (-1)^{p+1} \times \omega_{p+1,n-(p+1)}(a, \partial_{n-p}b)$.

In the definition, since $n \equiv 2 \pmod 4$ i.e. n is even and $n/2$ is odd, $\omega_{p,n-p}(a,b) = (-1)^p\omega_{n-p,p}(b,a)$.

Using the ∂-compatibility of the non-degenerate anti-symmetric bilinear maps $\omega_{p,n-p}\colon C_p \times C_{n-p} \to \mathbb{R}$, one can easily extend these to homologies. Namely,

Lemma 2.1.2. *The bilinear map* $[\omega_{p,n-p}]\colon H_p(C) \times H_{n-p}(C) \to \mathbb{R}$ *defined by* $[\omega_{p,n-p}]([x],[y]) = \omega_{p,n-p}(x,y)$ *is anti-symmetric and non-degenerate.*

Proof. For the well-definiteness, let x, x' be in $\ker \partial_p$ with $x - x' = \partial_{p+1}x''$ for some $x'' \in C_{p+1}$ and let y, y' be in $\ker \partial_{n-p}$ with $y - y' = \partial_{n-p+1}y''$ for some $y'' \in C_{n-p+1}$. Then from the bilinearity and ∂-compatibility, $[\omega_{p,n-p}]([x],[y])$ is equal to $\omega_{p,n-p}(x',y') + (-1)^p\omega_{p-1,n-p+1}(\partial_p x', y'') + (-1)^{p+1}\omega_{p+1,n-p-1}(x'', \partial_{n-p}y') + (-1)^{p+1} \times \omega_{p+1,n-p-1}(x'', \partial_{n-p} \circ \partial_{n-p+1}y'') = \omega_{p,n-p}(x',y')$.

Assume for some $[y_0] \in H_{n-p}(C)$, $[\omega_{p,n-p}]([x],[y_0]) = 0$ for all $[x] \in H_p(C)$.

Lemma 2.1.3. y_0 *is in* $\mathrm{Im}\,\partial_{n-p+1}$.

Proof. Let $\varphi\colon C_p/Z_p \to \mathbb{R}$ be defined by $\varphi(x + Z_p) = \omega_{p,n-p}(x, y_0)$. This is a well-defined linear map because if $x - x' \in Z_p$, then $\omega_{p,n-p}(x,y_0) - \omega_{p,n-p}(x',y_0) =$

$[\omega_{p,n-p}]([x-x'],[y_0])$ equals to 0. By the 1st-isomorphism theorem, $C_p/Z_p \overset{\overline{\partial}_p}{\cong} \operatorname{Im}\partial_p = B_{p-1}$, where $\overline{\partial}_p(x+Z_p)$ is $\partial_p(x)$.

Consider the linear functional $\tilde{\varphi} := \varphi\circ(\overline{\partial}_p)^{-1}$ on B_{p-1}, where $(\overline{\partial}_p)^{-1}(\partial_p y) = y+Z_p$. Extend $\tilde{\varphi}$ to $\hat{\varphi}\colon C_{p-1} = B_{p-1}\oplus(C_{p-1}/B_{p-1}) \to \mathbb{R}$ as zero on complement of B_{p-1}. Since $\omega_{p-1,n-p+1}\colon C_{p-1}\times C_{n-p+1}\to\mathbb{R}$ is non-degenerate, it induces an isomorphism between the dual space C^*_{p-1} of C_{p-1} and C_{n-p+1}. Therefore, there exists a unique $u_0\in C_{n-p+1}$ such that $\hat{\varphi}(\,\cdot\,) = \omega_{p-1,n-p+1}(\,\cdot\,,u_0)$.

For $x\in C_p$, $v=\partial_p x$ is in B_{p-1}. Then, on one hand, $\hat{\varphi}(v)$ is $\omega_{p-1,n-p+1}(\partial_p x,u_0)$ or $(-1)^p\omega_{p,n-p}(x,\partial_{n-p+1}u_0)$ by the ∂-compatibility. On the other hand, by the construction of $\hat{\varphi}$, $\hat{\varphi}(v) = \omega_{p,n-p}(x,y_0)$ so $\omega_{p,n-p}(x,y_0)$ is $\omega_{p,n-p}(x,(-1)^p\partial_{n-p+1}u_0)$ for all $x\in C_p$.

The nondegeneracy of $\omega_{p,n-p}$ finishes the proof of Lemma 2.1.3. □

This concludes the proof of Lemma 2.1.2 □

We will define ω-compatibility for bases in a symplectic chain complex.

DEFINITION 2.1.4. Let $C_*\colon 0\to C_n \xrightarrow{\partial_n} C_{n-1}\to\cdots\to C_{n/2}\to\cdots\to C_1\xrightarrow{\partial_1} C_0\to 0$ be a symplectic chain complex. Bases $\mathfrak{c}_p,\mathfrak{c}_{n-p}$ of C_p, C_{n-p} are *ω-compatible* if the matrix of $\omega_{p,n-p}$ in bases $\mathfrak{c}_p,\mathfrak{c}_{n-p}$ is

$$\begin{cases} \mathrm{Id}_{k\times k}; & p\neq \dfrac{n}{2} \\ \begin{bmatrix} O_{m\times m} & \mathrm{Id}_{m\times m} \\ -\mathrm{Id}_{m\times m} & 0_{m\times m}\end{bmatrix}; & p=\dfrac{n}{2}\end{cases}$$

where k is $\dim C_p = \dim C_{n-p}$ and $2m = \dim C_{n/2}$. In the same way, considering $[\omega_{p,n-p}]\colon H_p(C)\times H_{n-p}(C)\to\mathbb{R}$, we can also define $[\omega_{p,n-p}]$-compatibility of bases $\mathfrak{h}_p$, $\mathfrak{h}_{n-p}$ of $H_p(C)$, $H_{n-p}(C)$.

In the next result, we will explain how a general symplectic chain complex C_* can be splitted ω-orthogonally as a direct sum of an exact and ∂-zero symplectic complexes.

Theorem 2.1.5. *Let $C_*\colon 0\to C_n\xrightarrow{\partial_n} C_{n-1}\to\cdots\to C_1\xrightarrow{\partial_1} C_0\to 0$ be a symplectic chain complex. Assume $\mathfrak{c}_p,\mathfrak{c}_{n-p}$ ω-compatible. Then C_* can be splitted as a direct sum of symplctic complexes C'_*, C''_*, where C'_* is exact, C''_* is ∂-zero and C'_* is perpendicular to C''_*.*

Proof. Start with the following short-exact sequence

$$0 \to \ker \partial_p \hookrightarrow C_p \overset{\partial_p}{\twoheadrightarrow} \operatorname{Im} \partial_p \to 0,$$
$$0 \to \operatorname{Im} \partial_{p+1} \hookrightarrow \ker \partial_p \overset{\pi_p}{\twoheadrightarrow} H_p(C) \to 0.$$

Consider the section $l_p\colon \operatorname{Im} \partial_p \to C_p$ defined by $l_p(\partial_p x) = x$ for $\partial_p x \neq 0$, and $s_p\colon H_p(C) \to \ker \partial_p$ by $s_p([x]) = x$.

As C_p disjoint union of $\operatorname{Im} \partial_{p+1}$, $s_p(H_p(C))$, and $l_p(\operatorname{Im} \partial_p)$, the basis $\mathfrak{o}_p$ of C_p has three blocks $\mathfrak{o}_p^1, \mathfrak{o}_p^2, \mathfrak{o}_p^3$, where $\mathfrak{o}_p^1$ is a basis for $\operatorname{Im} \partial_{p+1}$, $\mathfrak{o}_p^2$ generates $s_p(H_p(C))$ the rest of $\ker \partial_p$, i.e. $[\mathfrak{o}_p^2]$ generates $H_p(C)$, and $\partial_p \mathfrak{o}_p^3$ is a basis for $\operatorname{Im} \partial_p$. Similarly, $\mathfrak{o}_{n-p} = \mathfrak{o}_{n-p}^1 \sqcup \mathfrak{o}_{n-p}^2 \sqcup \mathfrak{o}_{n-p}^3$. Because $[\omega]_{p,n-p}\colon H_p(C) \times H_{n-p}(C) \to \mathbb{R}$ defined by $[\omega]_{p,n-p}([a],[b]) = \omega_{p,n-p}(a,b)$ is non-degenerate and bases $\mathfrak{o}_p, \mathfrak{o}_{n-p}$ of C_p, C_{n-p} are ω-compatible, $\omega_{p,n-p}(\,\cdot\,, s_{n-p}(H_{n-p}(C)))\colon C_p \to \mathbb{R}$ vanishes on $\operatorname{Im} \partial_{p+1} \oplus l_p(\operatorname{Im} \partial_p)$. Likewise, $\omega_{p,n-p}(s_p(H_p(C)), \,\cdot\,)\colon C_{n-p} \to \mathbb{R}$ vanishes on $\operatorname{Im} \partial_{n-p+1} \oplus l_{n-p}(\operatorname{Im} \partial_{n-p})$.

Set $C'_p = \operatorname{Im} \partial_{p+1} \oplus l_p(\operatorname{Im} \partial_p)$ and $C''_p = s_p(H_p(C))$. Note that C'_p with basis $\mathfrak{o}_p^1 \sqcup \mathfrak{o}_p^3$ and C''_{n-p} with basis $\mathfrak{o}_{n-p}^2$ are ω-orhogonal to each other. Hence, (C'_*, ∂), (C''_*, ∂) will be the desired splitting, where we consider the corresponding restrictions of $\omega_{p,n-p}\colon C_p \times C_{n-p} \to \mathbb{R}$.

Clearly, (C''_*, ∂) is ∂-zero for C''_p being subspace of $\ker \partial_p$. Since $[\omega_{p,n-p}]\colon H_p(C) \times H_{n-p}(C) \to \mathbb{R}$ is non-degenerate, the restriction $\omega_{p,n-p}\colon C''_p \times C''_{n-p} \to \mathbb{R}$ is also non-degenerate. Being the restriction of $\omega_{p,n-p}$, it is also ∂-compatible. Hence C''_* becomes symplectic chain complex with ∂-zero.

In the sequence $C'_{p+1} \xrightarrow{\partial_{p+1}} C'_p \xrightarrow{\partial_p} C'_{p-1}$, first map ∂_{p+1} sends $\operatorname{Im} \partial_{p+2}$, $l_{p+1}(\operatorname{Im} \partial_{p+1})$ to zero and $\operatorname{Im} \partial_{p+1}$, respectively. Hence, $\ker\{\partial_{p+1}\colon C'_{p+1} \to C'_p\}$ equals to $\operatorname{Im}\{\partial_{p+2}\colon C_{p+2} \to C_{p+1}\}$ and $\operatorname{Im}\{\partial_{p+1}\colon C'_{p+1} \to C'_p\}$ is $\operatorname{Im}\{\partial_{p+1}\colon C_{p+1} \to C_p\}$. Similarly, $\ker\{\partial_p\colon C'_p \to C'_{p-1}\} = \operatorname{Im}\{\partial_{p+1}\colon C_{p+1} \to C_p\}$ and $\operatorname{Im}\{\partial_p\colon C'_p \to C'_{p-1}\}$ is $\operatorname{Im}\{\partial_p\colon C_p \to C_{p-1}\}$. Thus, C'_* is exact.

Moreover, since $\omega_{p,n-p}\colon C_p \times C_{n-p} \to \mathbb{R}$ is non-degenerate, and C'_p, C'_{n-p} are ω-perpendicular to C''_{n-p}, C''_p, respectively, $\omega_{p,n-p}\colon C'_p \times C'_{n-p} \to \mathbb{R}$ is non-degenerate. Also, it is ∂-compatible for being restriction of the ∂-compatible map $\omega_{p,n-p}\colon C_p \times C_{n-p} \to \mathbb{R}$.

This concludes the proof of Theorem 2.1.5 □

Above theorem is a special case of Theorem 2.0.3. The only difference is using ω-compatible bases $\mathfrak{o}_p$ the splitting is ω-orthogonal, too.

We will now explain how the torsion of a symplectic complex with ∂-zero is connected with Pfaffian of the anti-symmetric $[\omega_{n/2,n/2}]\colon H_{n/2}(C) \times H_{n/2}(C) \to \mathbb{R}$. Then, Pfaffian will be defined. After that, we will give the relation for a general symplectic complex.

Theorem 2.1.6. *Let C_* be symplectic chain complex with ∂-zero. Let $\mathfrak{h}_p$ be a basis for H_p. Assume the bases $\mathfrak{o}_p, \mathfrak{o}_{n-p}$ of C_p, C_{n-p} are ω-compatible with the property that the bases $\mathfrak{o}_{n/2}$ and $h_{n/2}$ of $H_{n/2}(C)$ are in the same orientation class. Then,*

$$\operatorname{Tor}(C_*, \{\mathfrak{o}_p\}_{p=0}^n, \{\mathfrak{h}_p\}_{p=0}^n) = \left(\prod_{p=0}^{(n/2)-1} (\det[\omega_{p,n-p}])^{(-1)^p}\right) \cdot \left(\sqrt{\det[\omega_{n/2,n/2}]}\right)^{(-1)^{n/2}},$$

where $\det[\omega_{p,n-p}]$ *is the determinant of the matrix of the non-degenerate pairing* $[\omega_{p,n-p}]\colon H_p(C) \times H_{n-p}(C) \to \mathbb{R}$ *in bases* $\mathfrak{h}_p, \mathfrak{h}_{n-p}$.

Proof. C_* is ∂-zero complex, so all B_p's are zero and $Z_p = C_p$. In particular, $H_p(C)$ is equal to $C_p/\{0\}$ and hence the basis $\mathfrak{h}_p$ of $H_p(C)$ can also be considered as a basis for C_p. Recall $\operatorname{Tor}(C_*, \{\mathfrak{o}_p\}_{p=0}^n, \{\mathfrak{h}_p\}_{p=0}^n)$ is defined as the alternating product

$$\prod_{p=0}^{n} [\mathfrak{o}_p, \mathfrak{h}_p]^{(-1)^p} = [\mathfrak{o}_0, \mathfrak{h}_0]^{(-1)^0} \cdots [\mathfrak{o}_{n/2}, \mathfrak{h}_{n/2}]^{(-1)^{n/2}} \cdots [\mathfrak{o}_n, \mathfrak{h}_n]^{(-1)^n},$$

of the determinants $[\mathfrak{o}_p, \mathfrak{h}_p]$ of the change-base-matrices from $\mathfrak{h}_p$ to $\mathfrak{o}_p$. If we combine the terms symmetric with the middle term $[\mathfrak{o}_{n/2}, \mathfrak{h}_{n/2}]^{(-1)^{n/2}}$, torsion becomes

$$\left(\prod_{p=0}^{(n/2)-1} [\mathfrak{o}_p, \mathfrak{h}_p]^{(-1)^p} [\mathfrak{o}_{n-p}, \mathfrak{h}_{n-p}]^{(-1)^{n-p}}\right) [\mathfrak{o}_{n/2}, \mathfrak{h}_{n/2}]^{(-1)^{n/2}}.$$

Moreover, note that $[\mathfrak{o}_p, \mathfrak{h}_p]^{(-1)^p}[\mathfrak{o}_{n-p}, \mathfrak{h}_{n-p}]^{(-1)^{n-p}} = \{[\mathfrak{o}_p, \mathfrak{h}_p][\mathfrak{o}_{n-p}, \mathfrak{h}_{n-p}]\}^{(-1)^p}$ for n being even. Let $T_{\mathfrak{h}_p}^{\mathfrak{o}_p}, T_{\mathfrak{h}_{n-p}}^{\mathfrak{o}_{n-p}}$ denote the change-base-matrices from $\mathfrak{h}_p$ to $\mathfrak{o}_p$ of C_p, and from $\mathfrak{h}_{n-p}$ to $\mathfrak{o}_{n-p}$ of C_{n-p} respectively, i.e. $h_p^i = \sum_\alpha \left(T_{\mathfrak{h}_p}^{\mathfrak{o}_p}\right)_{\alpha i} o_p^\alpha$ and $h_{n-p}^j = \sum_\beta \left(T_{\mathfrak{h}_{n-p}}^{\mathfrak{o}_{n-p}}\right)_{\beta j} o_{n-p}^\beta$, where h_p^i is the i^{th}-element of the basis $\mathfrak{h}_p$.

If A and B are the matrices of $\omega_{p,n-p}$ in the bases $\mathfrak{h}_p, \mathfrak{h}_{n-p}$, and in the bases $\mathfrak{o}_p, \mathfrak{o}_{n-p}$, respectively, then $A = \left(T_{\mathfrak{h}_p}^{\mathfrak{o}_p}\right)^{\text{transpose}} B T_{\mathfrak{h}_{n-p}}^{\mathfrak{o}_{n-p}}$. By the ω-compatibility of the bases $\mathfrak{o}_p, \mathfrak{o}_{n-p}$, the matrix B is equal to $\mathrm{Id}_{k\times k}$, $\begin{bmatrix} 0_{m\times m} & \mathrm{Id}_{m\times m} \\ -\mathrm{Id}_{m\times m} & 0_{m\times m} \end{bmatrix}$ for $p \neq n/2$, $p = n/2$, respectively, where k is $\dim C_p = \dim C_{n-p}$ and $2m = \dim C_{n/2}$. Clearly, determinant of B is $1^k = (-1)^m(-1)^m$ or 1.

Hence, $\det A$ equals $\det T_{\mathfrak{h}_p}^{\mathfrak{o}_p} \det T_{\mathfrak{h}_{n-p}}^{\mathfrak{o}_{n-p}}$ or $[\mathfrak{o}_p, \mathfrak{h}_p][\mathfrak{o}_{n-p}, \mathfrak{h}_{n-p}]$ for all p. In particular, for $p = n/2$, it is $[\mathfrak{o}_{n/2}, \mathfrak{h}_{n/2}]^2$. Since $2m$ is even, and $\omega_{n/2,n/2}$ is non-degenerate skew-symmetric, the determinant $\det A_{n/2}$ is positive actually equals to $\operatorname{Pfaf}(\omega_{n/2,n/2})^2$, and thus $[\mathfrak{o}_{n/2}, \mathfrak{h}_{n/2}] = \pm\sqrt{\det A_{n/2}}$. Because $\mathfrak{o}_{n/2}, \mathfrak{h}_{n/2}$ are in the same orientation class, then $[\mathfrak{o}_{n/2}, \mathfrak{h}_{n/2}] = \sqrt{\det A_{n/2}}$.

The proof is finished by the fact $\omega_{p,n-p}(h_p^i, h_{n-p}^j) = [\omega_{p,n-p}](h_p^i, h_{n-p}^j)$. □

Before explaining the corresponding result for a general symplectic complex, we would like to recall the Pfaffian of a skew-symmetric matrix.

Let V be an even dimensional vector space over reals. Let $\omega\colon V \times V \to \mathbb{R}$ be a bilinear and anti-symmetric. If we fix a basis for V, then ω can be represented by a $2m \times 2m$ skew-symmetric matrix.

If A is any $2m \times 2m$ skew-symmetric matrix with real entries then, by the spectral theorem of normal matrices, one can easily find an orthogonal $2m \times 2m$-real matrix Q so that $QAQ^{-1} = \operatorname{diag}\left(\begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & a_m \\ -a_m & 0 \end{pmatrix}\right)$, where $a_1, \ldots, a_m$ are positive real. Thus, in particular, determinant of A is non-negative.

DEFINITION 2.1.7. For $2m \times 2m$ real skew-symmetric matrix A, *Pfaffian* of A will be $\sqrt{\det A}$.

Actually, if $A = [a_{ij}]$ is any $2m \times 2m$ skew-symmetric matrix and if we let $\omega_A = \sum_{i<j} a_{ij}\vec{e}_i \wedge \vec{e}_j$, then we can also define $\operatorname{Pfaf}(A)$ as the coefficient of $\vec{e}_1 \wedge \cdots \wedge \vec{e}_{2m}$ in the product $\overbrace{\omega_A \wedge \cdots \wedge \omega_A}^{m\text{-times}}/m!$.

For example, if A is the matrix $\operatorname{diag}\left(\begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & a_m \\ -a_m & 0 \end{pmatrix}\right)$, then ω_A is $\sum_{i=1}^m a_i \cdot \vec{e}_{2i-1} \wedge \vec{e}_{2i}$. An easy computation shows that $\underbrace{\omega_A \wedge \omega_A \wedge \cdots \wedge \omega_A}_{m\text{-times}}$ equals to $m!\,\underbrace{(a_1 \cdots a_m)}_{\text{Pfaffian of } A}\,\vec{e}_1 \wedge \cdots \wedge \vec{e}_{2m}$.

For a general $2m \times 2m$ skew-symmetric A, we can find an orthogonal matrix Q such that $QAQ^{-1} = \operatorname{diag}\left(\begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & a_m \\ -a_m & 0 \end{pmatrix}\right)$. As a result,

$$\underbrace{\omega_{QAQ^{-1}} \wedge \omega_{QAQ^{-1}} \wedge \cdots \wedge \omega_{QAQ^{-1}}}_{m\text{-times}}$$

equals to $m!\,\underbrace{(a_1 \cdots a_m)}_{\text{Pfaffian of } QAQ^{-1}}\,\vec{e}_1 \wedge \cdots \wedge \vec{e}_{2m}$ i.e. $\operatorname{Pfaf}(QAQ^{-1}) = \sqrt{\det(QAQ^{-1})}$ or $\sqrt{\det(A)}$.

On the other hand, one can easily prove that for any $2m \times 2m$ skew-symmetric matrix X and any $2m \times 2m$ matrix Y, $\operatorname{Pfaf}(YXY^t)$ is equal to $\operatorname{Pfaf}(A)\det(B)$. Consequently, since Q is orthogonal matrix, we can conclude that $\operatorname{Pfaf}(A)^2 = \det(A)$ for any skew-symmetric $2m \times 2m$ real matrix A. In other words, both definitions coincide.

Using Pfaffian, we can rephrase Theorem 2.1.6 as follows.

If C_* is a symplectic chain complex with ∂-zero, $\mathfrak{h}_p$ is a basis for $H_p(C)$, $\mathfrak{o}_p, \mathfrak{o}_{n-p}$ ω-compatible bases for C_p, C_{n-p} so that $\mathfrak{h}_{n/2}$ and $[\mathfrak{o}_{n/2}]$ are in the same orientation class, then

$$\operatorname{Tor}(C_*, \{\mathfrak{o}_p\}_{p=0}^n, \{\mathfrak{h}_p\}_{p=0}^n) = \left(\prod_{p=0}^{(n/2)-1} (\det[\omega_{p,n-p}])^{(-1)^p}\right) \cdot (\operatorname{Pfaf}[\omega_{n/2,n/2}])^{(-1)^{n/2}},$$

where $\mathrm{Pfaf}[\omega_{n/2,n/2}]$ is the Pfaffian of the matrix of the non-degenerate pairing $[\omega_{n/2,n/2}]\colon H_{n/2}(C) \times H_{n/2}(C) \to \mathbb{R}$ in bases $\mathfrak{h}_{n/2}, \mathfrak{h}_{n/2}$.

Theorem 2.1.8. *Let C_* be an exact symplectic chain complex. If $\mathfrak{c}_p, \mathfrak{c}_{n-p}$ are bases for C_p, C_{n-p}, respectively, then* $\mathrm{Tor}(C_*, \{\mathfrak{c}_p\}_{p=0}^n, \{0\}_{p=0}^n) = 1$.

Proof. From the exactness of C_*, we have $H_p(C) = 0$ or $\ker \partial_p = \mathrm{Im}\, \partial_{p+1}$. Using the short-exact sequence

$$0 \to \ker \partial_p \hookrightarrow C_p \twoheadrightarrow \mathrm{Im}\, \partial_p \to 0,$$

we also have $C_p = \ker \partial_p \oplus l_p(\mathrm{Im}\, \partial_p)$, where we consider the section $l_p(\partial_p x) = x$ for $\partial_p x \neq 0$.

Let $\mathfrak{o}_p, \mathfrak{o}_{n-p}$ be ω-compatible bases for C_p, C_{n-p}, respectively. We can break $\mathfrak{o}_p = \mathfrak{o}_p^1 \sqcup \mathfrak{o}_p^3$, where $\mathfrak{o}_p^1$ generates $\ker \partial_p = \mathrm{Im}\, \partial_{p+1}$, and $\partial_p \mathfrak{o}_p^3$ generates $\mathrm{Im}\, \partial_p$. Similarly, $\mathfrak{o}_{n-p} = \mathfrak{o}_{n-p}^1 \sqcup \mathfrak{o}_{n-p}^3$, where $\mathfrak{o}_{n-p}^1$ generates $\ker \partial_{n-p} = \mathrm{Im}\, \partial_{n-p+1}$, and $\partial_{n-p} \mathfrak{o}_{n-p}^3$ generates $\mathrm{Im}\, \partial_{n-p}$. Since $\omega_{p,n-p}\colon C_p \times C_{n-p} \to \mathbb{R}$ is non-degenerate, ∂-compatible, then $\omega_{p,n-p}(\mathfrak{o}_p^1, \mathfrak{o}_{n-p}^1) = 0$, and $\omega_{p,n-p}(\mathfrak{o}_p^1, \mathfrak{o}_{n-p}^3)$ does not vanish. Also by the ω-compatibility of $\mathfrak{o}_p, \mathfrak{o}_{n-p}$, for every i there is unique j_i such that $\omega_{p,n-p}((\mathfrak{o}_p^1)_i, (\mathfrak{o}_{n-p}^3)_\alpha) = \delta_{j_i,\alpha}$. Likewise, for every k there is unique q_k such that $\omega_{p,n-p}((\mathfrak{o}_p^3)_k, (\mathfrak{o}_{n-p}^1)_\beta) = \delta_{q_k,\beta}$.

Recall torsion is independent of bases $\mathfrak{b}_p$ for $\mathrm{Im}\, \partial_p$ and section $\mathrm{Im}\, \partial_p \to C_p$. Let A_p be the determinant of the matrix of $\omega_{p,n-p}$ in bases $\mathfrak{c}_p, \mathfrak{c}_{n-p}$, and let O_p be the determinant of the matrix of $\omega_{p,n-p}$ in bases $\mathfrak{o}_p^1 \sqcup \mathfrak{o}_p^3, \mathfrak{o}_{n-p}^1 \sqcup \mathfrak{o}_{n-p}^3$. Since the set $\partial_p \mathfrak{o}_p^3 = \{\partial_p((\mathfrak{o}_p^3)_1), \ldots, \partial_p((\mathfrak{o}_p^3)_\alpha)\}$ generates $\mathrm{Im}\, \partial_p$, so does the set $\{\partial_p(A_p O_p (\mathfrak{o}_p^3)_1), \partial_p((\mathfrak{o}_p^3)_2), \ldots, \partial_p((\mathfrak{o}_p^3)_\alpha)\}$. Hence, image of the latter set under l_p, namely, $\tilde{\mathfrak{o}}_p^3 = \{A_p \cdot O_p \cdot (\mathfrak{o}_p^3)_1, (\mathfrak{o}_p^3)_2, \ldots, (\mathfrak{o}_p^3)_\alpha\}$ will also be basis for $l_p(\mathrm{Im}\, \partial_p)$. Keeping $\tilde{\mathfrak{o}}_{n-p}^3$ as $\mathfrak{o}_{n-p}^3$, we have

$$\begin{bmatrix} \omega_{p,n-p} \text{ in} \\ \mathfrak{o}_p^1 \sqcup \tilde{\mathfrak{o}}_p^3,\ \mathfrak{o}_{n-p}^1 \sqcup \mathfrak{o}_{n-p}^3 \end{bmatrix} = \left(T^{\mathfrak{c}_p}_{\mathfrak{o}_p^1 \sqcup \tilde{\mathfrak{o}}_p^3}\right)^{\text{transpose}} \begin{bmatrix} \omega_{p,n-p} \text{ in} \\ \mathfrak{c}_p,\ \mathfrak{c}_{n-p} \end{bmatrix} T^{\mathfrak{c}_{n-p}}_{\mathfrak{o}_{n-p}^1 \sqcup \mathfrak{o}_{n-p}^3}.$$

Determinant of left-hand-side is $A_p \cdot O_p \cdot O_p$, or A_p because of the determinant of $\omega_{p,n-p}$ in the ω-compatible bases $\mathfrak{o}_p, \mathfrak{o}_{n-p}$. Thus, for $p \neq n/2$, we obtained that $[\mathfrak{c}_p, \mathfrak{o}_p^1 \sqcup \tilde{\mathfrak{o}}_p^3][\mathfrak{c}_{n-p}, \mathfrak{o}_{n-p}^1 \sqcup \mathfrak{o}_{n-p}^3] = 1$.

For $p = n/2$, we can prove the same property as follows. Since $n/2$ is odd, $\omega_{n/2,n/2}\colon C_{n/2} \times C_{n/2} \to \mathbb{R}$ is non-degenerate and alternating, then the matrix of $\omega_{n/2,n/2}$ in any basis of $C_{n/2}$ will be an invertible $2m \times 2m$ skew-symmetric matrix X with real entries, where $2m = \dim C_{n/2}$. Actually, we can find an orthogonal $2m \times 2m$ matrix Q with real entries so that

$$Q X Q^{-1} = \mathrm{diag}\left(\begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & a_m \\ -a_m & 0 \end{pmatrix} \right).$$

So, the determinant of $\omega_{n/2,n/2}$ in any basis will be positive, in particular, the determinants $A_{n/2}$, $O_{n/2}$ of $\omega_{n/2,n/2}$ in basis $\mathfrak{c}_{n/2}$, $\mathfrak{o}^1_{n/2} \sqcup \mathfrak{o}^3_{n/2}$ respectively will be positive. Having noticed that, let $\tilde{\mathfrak{o}}^3_{n/2} = \{\sqrt{A_{n/2}} \cdot \sqrt{O_{n/2}} \cdot (\mathfrak{o}^3_{n/2})_1, (\mathfrak{o}^3_{n/2})_2, \dots, (\mathfrak{o}^3_{n/2})_\alpha\}$.

As explained above, on one side, we have that $\det\begin{bmatrix} \omega_{n/2,n/2} \text{ in} \\ \mathfrak{o}^1_{n/2} \sqcup \tilde{\mathfrak{o}}^3_{n/2} \end{bmatrix}$ is equal to $\sqrt{A_{n/2}} \cdot \sqrt{A_{n/2}}\sqrt{O_{n/2}} \cdot \sqrt{O_{n/2}} \det\begin{bmatrix} \omega_{n/2,n/2} \text{ in} \\ \mathfrak{o}^1_{n/2} \sqcup \mathfrak{o}^3_{n/2} \end{bmatrix}$ or $A_{n/2}$. On the other side, it is the product $[\mathfrak{c}_{n/2}, \mathfrak{o}^1_{n/2} \sqcup \tilde{\mathfrak{o}}^3_{n/2}] \cdot A_{n/2} \cdot [\mathfrak{c}_{n/2}, \mathfrak{o}^1_{n/2} \sqcup \tilde{\mathfrak{o}}^3_{n/2}]$. Consequently, $[\mathfrak{c}_{n/2}, \mathfrak{o}^1_{n/2} \sqcup \tilde{\mathfrak{o}}^3_{n/2}]^2$ is equal to 1.

If $\mathfrak{o}^1_{n/2} \sqcup \tilde{\mathfrak{o}}^3_{n/2}$ and $\mathfrak{c}_{n/2}$ are already in the same orientation class, then $[\mathfrak{c}_{n/2}, \mathfrak{o}^1_{n/2} \sqcup \tilde{\mathfrak{o}}^3_{n/2}] = 1$. If not, considering $\tilde{\mathfrak{o}}^3_{n/2}$ as $\{-\sqrt{A_{n/2}} \cdot \sqrt{O_{n/2}} \cdot (\mathfrak{o}^3_{n/2})_1, (\mathfrak{o}^3_{n/2})_2, \dots, (\mathfrak{o}^3_{n/2})_\alpha\}$, we still have $[\mathfrak{c}_{n/2}, \mathfrak{o}^1_{n/2} \sqcup \tilde{\mathfrak{o}}^3_{n/2}] = 1$.

As a result, we proved that

$$\begin{aligned} &\operatorname{Tor}(C_*, \{\mathfrak{c}_p\}_{p=0}^n, \{0\}_{p=0}^n) \\ &= \prod_{p=0}^{n} [\mathfrak{c}_p, \mathfrak{o}^1_p \sqcup \tilde{\mathfrak{o}}^3_p]^{(-1)^p} \\ &= \prod_{p=0}^{(n/2)-1} ([\mathfrak{c}_p, \mathfrak{o}^1_p \sqcup \tilde{\mathfrak{o}}^3_p][\mathfrak{c}_{n-p}, \mathfrak{o}^1_{n-p} \sqcup \mathfrak{o}^3_{n-p}])^{(-1)^p} \cdot [\mathfrak{c}_{n/2}, \mathfrak{o}^1_{n/2} \sqcup \tilde{\mathfrak{o}}^3_{n/2}]^{(-1)^{n/2}} = 1. \qquad \square \end{aligned}$$

Theorem 2.1.9. *For a general symplectic complex C_*, if $\mathfrak{c}_p$, $\mathfrak{h}_p$ are bases for C_p, $H_p(C)$, respectively, then*

$$\operatorname{Tor}(C_*, \{\mathfrak{c}_p\}_{p=0}^n, \{\mathfrak{h}_p\}_{p=0}^n) = \left(\prod_{p=0}^{(n/2)-1} (\det[\omega_{p,n-p}])^{(-1)^p} \right) \cdot \left(\sqrt{\det[\omega_{n/2,n/2}]}\right)^{(-1)^{n/2}},$$

where $\det[\omega_{p,n-p}]$ *is the determinant of the matrix of the non-degenerate pairing* $[\omega_{p,n-p}]\colon H_p(C) \times H_{n-p}(C) \to \mathbb{R}$ *in bases* $\mathfrak{h}_p$, $\mathfrak{h}_{n-p}$.

Proof. Since C_p is disjoint union $\operatorname{Im}\partial_{p+1} \sqcup s_p(H_p(C)) \sqcup l_p(\operatorname{Im}\partial_p)$, any basis $\mathfrak{a}_p$ of C_p has three parts $\mathfrak{a}^1_p$, $\mathfrak{a}^2_p$, $\mathfrak{a}^3_p$, where $\mathfrak{a}^1_p$ is basis for $\operatorname{Im}\partial_{p+1}$, $\mathfrak{a}^2_p$ generates $s_p(H_p)$ the rest of $\ker\partial_p$ i.e. $[\mathfrak{a}^2_p]$ generates $H_p(C)$, and $\partial_p\mathfrak{a}^3_p$ is basis for $\operatorname{Im}\partial_p$, where $l_p\colon \operatorname{Im}\partial_p \to C_p$ is the section defined by $l_p(\partial_p x) = x$ for $\partial_p x \neq 0$, and $s_p\colon H_p \to \ker\partial_p$ by $s_p([x]) = x$.

If $\mathfrak{o}_p$, $\mathfrak{o}_{n-p}$ are ω-compatible bases for C_p and C_{n-p}, then we can also write $\mathfrak{o}_p = \mathfrak{o}^1_p \sqcup \mathfrak{o}^2_p \sqcup \mathfrak{o}^3_p$ and $\mathfrak{o}_{n-p} = \mathfrak{o}^1_{n-p} \sqcup \mathfrak{o}^2_{n-p} \sqcup \mathfrak{o}^3_{n-p}$. We may assume $[\mathfrak{o}_{n/2}]$ and $\mathfrak{h}_{n/2}$ are in the same orientation class. Otherwise, switch, say the first element $(\mathfrak{o}_{n/2})^1$ and the corresponding ω-compatible element $(\mathfrak{o}_{n/2})^{m+1}$ i.e. $\omega_{n/2,n/2}((\mathfrak{o}_{n/2})^1, (\mathfrak{o}_{n/2})^{m+1}) = 1$, where $2m = \dim H_{n/2}(C)$. In this way, we still have ω-compatibility and moreover we can guarantee that $[\mathfrak{o}_{n/2}]$, $\mathfrak{h}_{n/2}$ are in the same orientation class.

Using these ω-compatible bases $\mathfrak{o}_p$, as in Theorem 2.1.5, we have the ω-orthogonal splitting $C_* = C'_* \oplus C''_*$, where C'_p and C''_p are $\operatorname{Im}(\partial_{p+1}) \oplus l_p(\operatorname{Im}\partial_p)$, $s_p(H_p(C))$, and

$l_p\colon \operatorname{Im}\partial_p \to C_p$ is the section defined by $l_p(\partial_p x) = x$ for $\partial_p x \neq 0$, and $s_p\colon H_p \to \ker\partial_p$ by $s_p([x]) = x$.

C_p is the disjoint union $\operatorname{Im}\partial_{p+1} \sqcup s_p(H_p) \sqcup l_p(\operatorname{Im}\partial_p)$, so the basis $\mathfrak{c}_p$ of C_p has also three blocks $\mathfrak{c}_p^1, \mathfrak{c}_p^2, \mathfrak{c}_p^3$, where $\mathfrak{c}_p^1$ is a basis for $\operatorname{Im}\partial_{p+1}$, $\mathfrak{c}_p^2$ generates $s_p(H_p)$ the rest of $\ker\partial_p$, i.e. $[\mathfrak{c}_p^2]$ generates $H_p(C)$, and $\partial_p\mathfrak{c}_p^3$ is a basis for $\operatorname{Im}\partial_p$.

Consider the ∂-zero symplectic C''_* with the ω-compatible bases $\mathfrak{o}_p^2, \mathfrak{o}_{n-p}^2$. Note that by the ∂-zero property of C''_*, $H_p(C'')$ is $C''_p/0$ or $s_p(H_p(C))$. Hence $s_p(\mathfrak{h}_p)$ will be a basis $H_p(C'')$. Recall also that $[\mathfrak{o}_{n/2}^2]$ and $\mathfrak{h}_{n/2}^2$ are in the same orientation class. Therefore, by Theorem 2.1.6, we can conclude that

$$\operatorname{Tor}(C''_*, \{\mathfrak{o}_p^2\}_{p=0}^n, \{s_p(\mathfrak{h}_p)\}_{p=0}^n) = \left(\prod_{p=0}^{(n/2)-1} (\det[\omega_{p,n-p}])^{(-1)^p}\right)\cdot\left(\sqrt{\det[\omega_{n/2,n/2}]}\right)^{(-1)^{n/2}},$$

where $\det[\omega_{p,n-p}]$ is the determinant of the matrix of the non-degenerate pairing $[\omega_{p,n-p}]\colon H_p(C)\times H_{n-p}(C)\to\mathbb{R}$ in bases $\mathfrak{h}_p, \mathfrak{h}_{n-p}$.

On the other hand, if $\mathfrak{c}'_p$ is any basis for C'_p, then by Theorem 2.1.8 the torsion $\operatorname{Tor}(C'_*, \{c'_p\}_{p=0}^n, \{0\}_{p=0}^n)$ of the exact symplectic complex C'_* is equal to 1.

Let A_p be the determinant of the change-base-matrix from $\mathfrak{o}_p^2$ to $\mathfrak{c}_p^2$. If we consider the basis $\mathfrak{c}_p^1 \sqcup ((1/A_p)\mathfrak{c}_p^3)$ for the C'_p, then for the short-exact sequence

$$0 \to C''_* \hookrightarrow C_* = C'_* \oplus C''_* \twoheadrightarrow C'_* \to 0$$

the bases $\mathfrak{o}_p^2, \mathfrak{c}_p, \mathfrak{c}_p^1 \sqcup ((1/A_p)\mathfrak{c}_p^3)$ of C''_p, C_p, C'_p respectively will be compatible i.e. the determinant of the change-base-matrix from basis $\mathfrak{c}_p^1 \sqcup \mathfrak{o}_p^2 \sqcup ((1/A_p)\mathfrak{c}_p^3)$ to $\mathfrak{c}_p = \mathfrak{c}_p^1 \sqcup \mathfrak{c}_p^2 \sqcup \mathfrak{c}_p^3$ is 1.

Thus, by Milnor's result Theorem 1.1.3, $\operatorname{Tor}(C_*, \{\mathfrak{c}_p\}_{p=0}^n, \{\mathfrak{h}_p\}_{p=0}^n)$ is equal to the product of $\operatorname{Tor}(C''_*, \{\mathfrak{o}_p^2\}_{p=0}^n, \{s_p(\mathfrak{h}_p)\}_{p=0}^n)$, $\operatorname{Tor}(C'_*, \{\mathfrak{c}_p^1 \sqcup ((1/A_p)\mathfrak{c}_p^3)\}_{p=0}^n, \{0\}_{p=0}^n)$, and $\operatorname{Tor}(\mathcal{H}_*, \{s_p(\mathfrak{h}_p), \mathfrak{h}_p, 0\}_{p=0}^n, \{0\}_{p=0}^{3n+2})$, where $\mathcal{H}_*$ is the long-exact sequence $0 \to H_n(C'') \to H_n(C) \to H_n(C') \to H_{n-1}(C'') \to \cdots \to H_0(C'') \to H_0(C) \to H_0(C') \to 0$ obtained from the short-exact sequence of complexes. Since C'_* is exact, $H_p(C')$ are all zero. So, using the isomorphisms $H_p(C) \to H_p(C'') = C''_p/0$, namely s_p as section, we can conclude that $\operatorname{Tor}(\mathcal{H}_*, \{s_p(\mathfrak{h}_p), \mathfrak{h}_p, 0\}_{p=0}^n, \{0\}_{p=0}^{3n+2}) = 1$. From Theorem 2.1.8, we also obtain $\operatorname{Tor}(C'_*, \{\mathfrak{c}_p^1 \sqcup ((1/A_p)\mathfrak{c}_p^3)\}_{p=0}^n, \{0\}_{p=0}^n) = 1$.

Therefore, we verified that

$$\operatorname{Tor}(C_*, \{\mathfrak{c}_p\}_{p=0}^n, \{\mathfrak{h}_p\}_{p=0}^n) = \operatorname{Tor}(C''_*, \{\mathfrak{o}_p^2\}_{p=0}^n, \{s_p(\mathfrak{h}_p)\}_{p=0}^n).$$

This finishes the proof of Theorem 2.1.9. $\square$

3. Application

We will present an explanation of the relation between Reidemeister torsion and Pfaffian of Weil-Petersson form and hence Pfaffian of Thurston symplectic form [26] in this section.

3.1. Thurston and Weil-Petersson-Goldman symplectics forms.

In this section, we will explain the Teichmüller space of a hyperbolic surface, Weil-Petersson, Goldman and Thurston symplectic forms of the Teichmüller space. For more information about the subject, we refer the reader to [2] [13] [15] [16], and [27].

3.1.1. Teichmüller Space.

Let S be a fixed compact surface with negative Euler characteristic.

The Teichmüller space $\mathfrak{Teich}(S)$ of S is by definition the space of isotopy classes of *complex structures* on S. Recall that a complex structure on S is a homotopy equivalence of a homeomorphism $S \xrightarrow{f} M$, where M is a Riemann surface and where two such homeomorphisms $\begin{pmatrix} S \\ \downarrow f \\ M \end{pmatrix} \sim \begin{pmatrix} S \\ \downarrow f' \\ M' \end{pmatrix}$ are *equivalent*, if there is a conformal diffeomorphism $M \xrightarrow{g} M'$ such that $(f')^{-1} \circ g \circ f$ is isotopic to Id.

Fix a complex a structure on S, and conformally identify S with $\mathbb{H}^2/\Gamma$, where Γ is a discrete group of conformal transformations of the upper half-plane $\mathbb{H}^2 \subset \mathbb{C}$. The deformation of the complex structure will produce Beltrami-differential.

Namely, if $\left\{S \xrightarrow{f_t} S_t\right\}$ is a path in $\mathfrak{Teich}(S)$ differentiable with respect to t, and if we consider the composition maps $S_0 \xrightarrow{f_0^{-1}} S \xrightarrow{f_t} S_t$, then these can be extended to quasi-conformal maps $\mathbb{H}^2 \xrightarrow{g_t} \mathbb{H}^2$ such that $(\partial g_t/\partial \overline{z})/(\partial g_t/\partial z)$ is a tensor of type $(\partial/\partial z) \otimes d\overline{z}$ with measurable coefficient and finite L^∞-norm. In other words, we have a differentiable path in the complex Banach space $B(\Gamma)$ of Γ-invariant Beltrami differentials, where $\Gamma \cong \pi_1(S)$. Then, $(d/dt)((\partial g_t/\partial \overline{z})/(\partial g_t/\partial z))|_{t=0}$ is also in $B(\Gamma)$. Recall that a Beltrami differential is an element of the complex-Banach space of Γ-invariant tensors of type $\mu(z)(\partial/\partial z) \otimes d\overline{z}$ with measurable coefficients and finite L^∞-norm and satisfying that $\forall \gamma \in \Gamma,\ \mu \circ \gamma(\overline{d\gamma/dz}) = \mu(d\gamma/dz)$.

By the uniformization theorem, Teichmüller space $\mathfrak{Teich}(S)$ of S can also be interpreted as the space of isotopy classes of hyperbolic metrics on S (i.e. Riemannian metrics with constant -1 curvature), or as the space of conjugacy classes of all discrete faithful homomorphisms from the fundamental group $\pi_1(S)$ to the group $\mathrm{Isom}^+(\mathbb{H}^2) \cong \mathrm{PSL}_2(\mathbb{R})$ of orientation-preserving isometries of upper-half lane $\mathbb{H}^2 \subset \mathbb{C}$ as follows.

A complex structure on S lifts to a complex structure on the universal covering $\tilde{S}$ of S. Since S has genus at least 2, then by the uniformization theorem, $\tilde{S}$ is biholomorphic to the upper-half-plane $\mathbb{H}^2 \subset \mathbb{C}$. Recall that every biholomorphic homeomorphism of $\mathbb{H}^2$ is of the form $f(z) = (az + b)/(cz + d)$, where a, b, c, d are real numbers with

$ad - bc = 1$. This defines a representation from the fundamental group $\pi_1(S)$ of S into $\mathrm{PSL}_2(\mathbb{R})$ which is discrete, faithful and well-defined up to conjugation by the orientation preserving isometries of $\mathbb{H}^2$. This enables us to identify $\mathfrak{Teich}(S)$ as the set of all conjugacy classes of discrete faithful representations of $\pi_1(S)$ into $\mathrm{PSL}_2(\mathbb{R})$.

If we set $\mathfrak{R} = \mathrm{Hom}_{\mathrm{df}}(\pi_1(S), \mathrm{PSL}_2(\mathbb{R}))/\mathrm{PSL}_2(\mathbb{R})$, where $\mathrm{Hom}_{\mathrm{df}}(\pi_1(S), \mathrm{PSL}_2(\mathbb{R}))$ is the set of Discrete Faithful representations of $\pi_1(S)$ into $\mathrm{PSL}_2(\mathbb{R})$, then it is a well known fact that the image of the embedding $\mathfrak{Teich}(S) \to \mathfrak{R}$ is open ([30] [23]).

3.1.2. The Goldman symplectic form. Consider the real-analytic identification of $\mathfrak{Teich}(S)$, i.e.

$$\mathfrak{R} = \mathrm{Hom}_{\mathrm{df}}(\pi_1(S), \mathrm{PSL}_2(\mathbb{R}))/\mathrm{PSL}_2(\mathbb{R}).$$

Fix a point $\varrho \in \mathfrak{Teich}(S) \subset \mathfrak{R}$. The standard deformation of representation will enable us to identify the tangent space $\mathrm{T}_\varrho\mathfrak{Teich}(S) = \mathrm{T}_\varrho\mathfrak{R}$ to the first cohomology space $H^1(S; \mathrm{Ad}_\varrho)$ of S with coefficients in the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ of $\mathrm{PSL}_2(\mathbb{R})$ twisted by the adjoint representation $\mathrm{Ad}_\varrho\colon \pi_1(S) \to \mathrm{Aut}(\mathfrak{sl}_2(\mathbb{R}))$.

For the sake of completeness, we will roughly describe this identification. We refer the reader to [31] [23] [14] for details.

Take a path $\{\varrho_t\} \subset \mathfrak{R}$ through ϱ and differentiable with respect to the real variable t. Thus, for each $\gamma \in \pi_1(S)$, we have a differentiable path $\{\varrho_t(\gamma)\}_t$ through $\varrho(\gamma) \in \mathrm{PSL}_2(\mathbb{R})$. By the fact that the inversion in a Lie group is also a differentiable map, we can get a differentiable path $\{\varrho(\gamma)^{-1}\varrho_t(\gamma)\}_t$ through $I \in \mathrm{PSL}_2(\mathbb{R})$. Then, $(d/dt)(\varrho(\gamma)^{-1}\varrho_t(\gamma))|_{t=0} \in H^1(S; \mathrm{Ad}_\varrho)$ is in the first cohomology space of S with coefficients twisted by adjoint representation.

The first twisted-cohomology space $H^1(S; \mathrm{Ad}_\varrho)$ can be defined as follows. The action of $\pi_1(S)$ on the universal cover S turns the group of the chain complex $C_*(\tilde{S}; \mathbb{Z})$ into $\mathbb{Z}[\pi_1(S)]$-module. Similarly, the adjoint action by Ad_ϱ makes $\mathfrak{sl}_2(\mathbb{R})$ a $\mathbb{Z}[\pi_1(S)]$-module, where $\mathbb{Z}[\pi_1(S)]$ is the integral-group-ring.

The twisted cohomology modules $H^*(S, \mathrm{Ad}_\varrho)$ are defined as the homology of the complex $C^*(S; \mathrm{Ad}_\varrho) = \mathrm{Hom}_{\mathbb{Z}[\pi_1(S)]}(C_*(\tilde{S}), \mathfrak{sl}_2(\mathbb{R})) = \mathfrak{sl}_2(\mathbb{R}) \otimes_{\mathbb{Z}[\pi_1(S)]} C_*(\tilde{S})$. Namely, $C^n(\tilde{S}; \mathrm{Ad}_\varrho)$ is the group homomorphisms $C_n(\tilde{S}, \mathbb{Z}) \to \mathfrak{sl}_2(\mathbb{R})$ that commute with the action of $\pi_1(S)$.

Since the Cartan-Killing bilinear form $B\colon \mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{sl}_2(\mathbb{R}) \to \mathbb{R}$, defined by $B(t_1, t_2) = 4\,\mathrm{Trace}(t_1 t_2)$, is invariant under adjoint action, then one can define a cup product $\smile_B\colon C^1(S; \mathrm{Ad}_\varrho) \times C^1(S; \mathrm{Ad}_\varrho) \to C^2(S; \mathbb{R})$ by assigning $\varphi, \psi \in C^1(S; \mathrm{Ad}_\varrho)$ to $\varphi \smile \psi \in C^2(S, \mathbb{R})$. More precisely, if $\Delta \in C_2(S; \mathbb{R})$ is a two-simplex in S, and $\tilde{\Delta}$ is a fix a lift Δ in the universal covering $\tilde{S}$, then $(\varphi \smile_B \psi)(\Delta) = B(\varphi(\tilde{\Delta}_{\mathrm{front}}), \psi(\tilde{\Delta}_{\mathrm{back}}))$, where $\tilde{\Delta}_{\mathrm{front}}$, $\tilde{\Delta}_{\mathrm{back}}$ denote the front and back faces of $\tilde{\Delta}$. The well-defineteness will follow from the invariance of B under conjugation. The product also induces an ant-symmetric bilinear form $\omega_{\mathrm{Goldman}}\colon H^1(S; \mathrm{Ad}_\varrho) \times H^1(S; \mathrm{Ad}_\varrho) \to H^2(S; \mathbb{R}) \cong \mathbb{R}$, where the isomorphism $H^2(S; \mathbb{R}) \cong \mathbb{R}$ is obtained from the integral of the fundamental class of the oriented surface S.

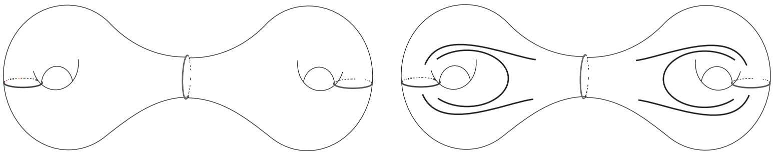

Fig. 1. Geodesic lamination with 3 leaves. Maximal geodesic lamination obtained from pant-decomposition.

In [14], W.M. Goldman proved that for the isomorphism $\mathrm{T}_{\varrho}\mathfrak{Teich}(S) \cong H^1(S;\mathrm{Ad}_{\varrho})$, the Weil-Petersson form coincides with the Weil-Petersson form ω_{WP} of $\mathrm{T}_{\varrho}\mathfrak{Teich}(S)$, up to a multiplicative constant. More precisely,

Theorem 3.1.1 (Goldman, [14]). *If $u, v \in H^1(S;\mathrm{Ad}_{\varrho})$ are two cohomology clases with coefficients in $\mathfrak{sl}_2(\mathbb{R})$, then $\omega_{\mathrm{WP}}[S] = -8\omega_{\mathrm{Goldman}}(u, v)$, where $[S] \in H_1(S;\mathbb{Z})$ is the fundamental class of the oriented surface S.*

3.1.3. The Thurston Symplectic Form. Endow the surface S with a hyperbolic metric m_0, namely with a Riemannian metric of constant curvature -1.

A *geodesic lamination* is a closed subset of S which can be decomposed as a union of disjoint complete geodesics which have no self-intersection points. Such a notion is actually a topological object, independent of the metric, in the sense that there is a natural identification between m-geodesic laminations and m'-geodesic laminations for any two negatively curved metrics m and m'. A geodesic lamination is *maximal* if it is maximal for inclusion among all geodesic laminations, which is equivalent to the property that the complement $S - \lambda$ consists of finitely many infinite triangles. See Fig. 1.

A fundamental example of a maximal geodesic lamination is obtained as follows. Start with a family λ_1 of disjoint simple closed geodesics decomposing S into pairs of pants. Each pair of pants can be divided into two infinite triangles by two infinite geodesics spiralling around some boundary components. The union of λ_1 and of these spiralling geodesics forms a maximal geodesic lamination λ.

A *transverse cocycle* σ for λ on S is a real-valued function on the set of all arcs k transverse to (the leaves) of λ with the following properties:

- σ is finitely additive, i.e. $\sigma(k) = \sigma(k_1) + \sigma(k_2)$, whenever the arc k transverse to λ is decomposed into two subarcs k_1, k_2 with disjoint interiors, and
- σ is invariant under the homotopy of arcs transverse to λ, i.e. $\sigma(k) = \sigma(k')$ whenever the transverse arc k is deformed to arc k' by a family of arcs which are all transverse to the leaves of the geodesic lamination λ.

The transverse cocycles for the geodesic lamination λ form a fnite dimensional real-vector space $\mathcal{H}(\lambda)$, whose dimension can explicitly be computed from the topology of λ, see [5]. In particular, if the geodesic lamination is maximal, then $\mathcal{H}(\lambda)$ is isomorphic to $\mathbb{R}^{|\chi(S)|}$, where $|\chi(S)|$ denotes the Euler characteristic of S. This computation is done by using a (fattened) train-track $\Phi \subset S$ carrying the lamination λ.

Recall that a (*fattened*) *train track* Φ on the surface S is a family of finitely many 'long' rectangles $e_1, \ldots, e_n$ which are foliated by arcs parallel to the 'short' sides and which meet only along arcs (possibly reduced to a point) contained in their short sides. In addition, a train track Φ must satisfy the following:

- each point of the 'short' side of a rectangle also belongs to another rectangle, and each component of the union of the short sides of all rectangles is an arc, as opposed to a closed curve;
- note that the closure $\overline{S - \Phi}$ of the complement $S - \Phi$ has a certain number of 'spikes', corresponding to the points where at least 3 rectangles meet; we require that no component of $\overline{S - \Phi}$ is a disc with 0, 1 or 2 spikes or an annulus with no spike.

The rectangles are called the *edges* of Φ. The foliations of the edges of Φ induce a foliation of Φ, whose leaves are the *ties* of the train track. The finitely many ties where several edges meet are the *switches* of the train track Φ. A tie which is not a switch is *generic*. The geodesic lamination λ is *carried* by the train track Φ if it is contained in the interior of Φ and if its leaves are transverse to the ties of Φ. There are several constructions which provide a train track Φ carrying λ; see for instance [21] [6].

For a fixed train-track Φ, let $\mathcal{W}(\Phi)$ be the vector space of all *edge weight systems* for Φ. More precisely, maps a assigning a weight $a(e) \in \mathbb{R}$ to each edge e of Φ and satisfying, for each switch s of Φ, the following *switch relation*

$$\sum_{i=1}^{p} a(e_i) = \sum_{j=p+1}^{p+q} a(e_j),$$

where $e_1, \ldots, e_p$ are the edges adjacent to one side of the switch s and $e_{p+1}, \ldots, e_{p+q}$ are the edges adjacent to other side.

If the geodesic lamination λ is carried by the train-track Φ, a transverse cocycle $\sigma \in \mathcal{H}(\lambda)$ defines an edge weight system $a_\sigma \in \mathcal{W}(\Phi)$ by the property that $a_\sigma(e) = \sigma(k_e)$, where k_e is an arbitrary tie of the edge e. This gives an injective additive map [5]. Moreover, this map gives isomorphism $\mathcal{H}(\lambda) \cong \mathcal{W}(\Phi)$, if Φ snuggly carries the lamination λ, a technical condition that can be realized when λ is maximal.

It is also possible that we can arrange the train-track Φ so that it is *generic* in the sense that each switch is adjacent to exactly 3 edges. Thus, at each switch s of Φ, there are one incoming e_s^{in} touching the switch s on one side and two outgoing e_s^{left}, e_s^{right} touching s on the other side, where as seen from the incoming edge e_s^{in} and for the orientation of the surface S, e_s^{left} branches out to the left and e_s^{right} branches out to the right.

The *Thurston symplectic form* on $\mathcal{W}(\Phi)$ is the anti-symmetric bilinear form $\omega_{\text{Thurston}}\colon \mathcal{W}(\Phi) \times \mathcal{W}(\Phi) \to \mathbb{R}$ defined by

$$\omega_{\text{Thurston}}(a, b) = \frac{1}{2} \sum_{s} \det\begin{bmatrix} a(e_s^{\text{left}}) & a(e_s^{\text{right}}) \\ b(e_s^{\text{left}}) & b(e_s^{\text{right}}) \end{bmatrix},$$

where the sum is over all switches of the train-track Φ, where $a(e_s^{\text{left}})$, $a(e_s^{\text{right}})$ denote the multiplicities assigned to the edges diverging respectively to the left and to the right at the switch s, and where 'det' is the determinant of 2×2 matrices.

Using the isomorphism $\mathcal{H}(\lambda) \cong \mathcal{W}(\Phi)$, this induces the *Thurston symplectic form* on $\omega_{\text{Thurston}}\colon \mathcal{H}(\lambda) \times \mathcal{H}(\lambda) \to \mathbb{R}$ defined by

$$\omega_{\text{Thurston}}(\sigma_1, \sigma_2) = \frac{1}{2} \sum_{s} \det\begin{bmatrix} \sigma_1(e_s^{\text{left}}) & \sigma_1(e_s^{\text{right}}) \\ \sigma_2(e_s^{\text{left}}) & \sigma_2(e_s^{\text{right}}) \end{bmatrix},$$

where $\sigma_i(e) \in \mathbb{R}$ is the weight associated to the edge e by the transverse cocycle σ_i.

It can be proved that τ is actually independent of the train-track Φ. In fact, τ also has a homological interpretation as an algebraic intersection number. See [21] [3].

3.1.4. Shearing coordinates of Teichmüller space. Let λ be a maximal geodesic lamination on the surface S. The shearing coordinates for Teichmüller space $\mathfrak{Teich}(S)$ of S, as developed in [3], define a real-analytical embedding $\varphi_\lambda\colon \mathfrak{Teich}(S) \to \mathcal{H}(\lambda)$. For $\rho \in \mathfrak{Teich}(S)$, the transverse cocycle $\varphi_\lambda(\rho)$ associates to each transverse arc k a number $\varphi_\lambda(\rho)(k)$, which, intuitively, measures the 'shift to the left' between the two ideal triangles in $S = \mathbb{H}^2/\rho(\pi_1(S))$ corresponding to the components of $S - \lambda$ that contain the end points of k.

The precise definition of φ_λ can be somewhat technical, but we only need to understand its tangent map, which induces an isomorphism between the tangent space $T_\rho\mathfrak{Teich}(S) \cong H^1(S; \mathrm{Ad}_\rho)$ and the vector space of transverse cocycles $\mathcal{H}(\lambda)$.

For this, it is convenient to lift the situation to the universal $\tilde{S}$ of S. Fix an isometric identification between $\tilde{S}$ endowed with the hyperbolic metric corresponding to $\rho \in \mathfrak{Teich}(S)$ and the hyperbolic plane $\mathbb{H}^2$, and choose the geodesic lamination λ as geodesic lamination for this metric. Let $\tilde{\lambda}$ be the preimage of λ in $\tilde{S}$. If $\tilde{k}$ is an arc transverse to $\tilde{\lambda}$ and $\sigma \in \mathcal{H}(\lambda)$, we define $\sigma(k) = \sigma(\tilde{k})$, where k is the projection of $\tilde{k}$.

If we differentiate the explicit formula for φ_λ^{-1} given in [3] §5, we obtain the following formula

Lemma 3.1.2 ([27]). *If $\sigma \in \mathcal{H}(\lambda)$ is a transverse cocycle for the maximal geodesic lamination λ, then the element $T_\rho\varphi_\lambda^{-1}(\sigma) \in T_\rho\mathfrak{Teich}(S) \cong H^1(S; \mathrm{Ad}_\rho)$ is repre-*

sented by a cocycle $u_\sigma \in C^1(S; \mathrm{Ad}_\rho)$ *such that, for every oriented arc* $\tilde{k}$ *transverse to* $\tilde{\lambda}$

$$u_\sigma(\tilde{k}) = \sigma(\tilde{k}) t_{g^-_{d^+}} + \sum_{d \neq d^+, d^-} \sigma(\tilde{k}_d)(t_{g^-_d} - t_{g^+_d}),$$

where the sum is over all components d *of* $\tilde{k} - \tilde{\lambda}$ *that are distinct from the components* d^+ *and* d^- *respectively containing the positive and the negative end points of* $\tilde{k}$, *where* $\tilde{k}_d$ *is a subarc of* $\tilde{k}$ *joining the negative end of* $\tilde{k}$ *to an arbitrary point in the component* d, *where* g^+_d *and* g^-_d *are the leaves of* $\tilde{\lambda}$ *respectively passing through the positive and negative end points of* d *and are oriented to the left of* $\tilde{k}$, *and where* $t_g \in \mathfrak{sl}_2(\mathbb{R})$ *is the hyperbolic translation along the oriented geodesic* g *of* $\tilde{S} \cong \mathbb{H}^2$.

Using these coordinates, in [27], we also proved that up to a multiplicative constant, $\omega_{\mathrm{Thurston}}$ is the same as $\omega_{\mathrm{Goldman}}$ and hence is in the same equivalence class of ω_{WP}. More precisely,

Theorem 3.1.3 ([27]). *Let* S *be a closed oriented surface with negative Euler charactersistic (i.e. of genus at least two), and let* λ *be a (fixed) maximal geodesic lamination on the surface* S. *For the identification* $\mathrm{T}_\rho\mathfrak{Teich}(S) \cong \mathcal{H}(\lambda; \mathbb{R})$, *we have the following commutative diagram*

$$\begin{tikzcd} H^1(S;\mathrm{Ad}_\rho) \times H^1(S;\mathrm{Ad}_\rho) \arrow[r, "\smile_B"] \arrow[d] & H^2(S;\mathbb{R}) \arrow[d] \\ \mathcal{H}(\lambda;\mathbb{R}) \times \mathcal{H}(\lambda;\mathbb{R}) \arrow[r, "2\tau"] \arrow[ur, phantom, "\circlearrowleft"] & \mathbb{R}. \end{tikzcd}$$

3.2. Proof of Appication. In this section, we will apply the ideas explained so far to the complex $C_*(K; \mathrm{Ad}_\rho)$, where S is compact hyperbolic surface without boundary, $\rho\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$ is a discrete faithful representation, and K is a fine cell-decomposition of S so that the adjoint bundle $\tilde{S} \times_\rho \mathfrak{sl}_2(\mathbb{R})$ is trivial over each cell.

The twisted chain complex

$$0 \to C_2(K;\mathrm{Ad}_\rho) \to C_1(K;\mathrm{Ad}_\rho) \to C_0(K;\mathrm{Ad}_\rho) \to 0$$

gives us the twisted homologies $H_*(S;\mathrm{Ad}_\rho)$, which are independent of K. Moreover, $H_2(S;\mathrm{Ad}_\rho)$, $H_0(S;\mathrm{Ad}_\rho)$ both vanish for ρ being discrete, faithful and thus in particular irreducible.

Recall that $C_p(K;\mathrm{Ad}_\rho) = C_p(\tilde{K};\mathbb{Z}) \otimes_\rho \mathfrak{sl}_2(\mathbb{R})$ denotes the quotient $C_p(\tilde{K};\mathbb{Z}) \otimes \mathfrak{sl}_2(\mathbb{R})/\!\sim$, where the orbit $\{\gamma \bullet \sigma \otimes \gamma \bullet t;\ \gamma \in \pi_1(S)\}$ of $\sigma \otimes t$ is identified and where the action of the fundamental group in the first slot by deck transformations, and in the second slot by the conjugation with $\rho(\cdot)$. Let $\{e^p_1, \ldots, e^p_{m_p}\}$ be basis for the $C_p(K;\mathbb{Z})$, then $c_p := \{\tilde{e}^p_1, \ldots, \tilde{e}^p_{m_p}\}$ is a $\mathbb{Z}[\pi_1(S)]$-basis for $C_i(\tilde{K};\mathbb{Z})$, where $\tilde{e}^p_j$ is a lift of e^p_j. If we choose a $\mathbb{R}$-basis $\mathcal{A} = \{\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3\}$ of $\mathfrak{sl}_2(\mathbb{R})$, then $\mathfrak{c}_p := c_p \otimes_\rho \mathcal{A}$ will be an $\mathbb{R}$-basis for $C_p(K, \mathrm{Ad}_\rho)$, called a *geometric* for $C_p(K; \mathrm{Ad}_\rho)$. Let $\mathfrak{h}_p$ be a basis for $H_p(S; \mathrm{Ad}_\rho)$.

We defined the torsion $\mathrm{Tor}(C_*(K; \mathrm{Ad}_\rho), \{\mathfrak{c}_p\}_{p=0}^2, \{\mathfrak{h}_p\}_{p=0}^2)$ is the *Reidemeister torsion* of the triple K, Ad_ρ, and $\{\mathfrak{h}_p\}_{p=0}^2$. We proved in Lemma 2.0.5 that $\mathrm{Tor}(C_*)$ is independent of the cell-decomposition.

For the rest of the paper, we consider the $\mathbb{R}$-basis $\mathcal{A} = \{t_1, t_2, t_3\}$ of $\mathfrak{sl}_2(\mathbb{R})$ as $\left\{(1/\sqrt{8})\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, (1/\sqrt{8})\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, (1/\sqrt{8})\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$. Note that the matrix of the Cartan-Killing for B of $\mathfrak{sl}_2(\mathbb{R})$ in this basis is $\mathrm{Diag}(1, -1, 1)$ where $B(a, b) = 4\,\mathrm{Trace}(ab)$.

Let K' be the dual cell-decomposition of S corresponding to the cell decomposition K. Since torsion is invariant under subdivision, it is not loss of generality to assume that cells $\sigma \in K$, $\sigma' \in K'$ can meet at most once and moreover the diameter of each cell has diameter less than, say, half of the injectivity radius of S. If we denote $C_* = C_*(K; \mathrm{Ad}_\rho)$, $C'_* = C_*(K'; \mathrm{Ad}_\rho)$, then by the invariance of torsion under subdivision, $\mathrm{Tor}(C_*(K; \mathrm{Ad}_\rho), \{c_p \otimes_\rho \mathcal{A}\}_{p=0}^2, \{\mathfrak{h}_p\}_{p=0}^2) = \mathrm{Tor}(C_*(K'; \mathrm{Ad}_\rho), \{c'_p \otimes_\rho \mathcal{A}\}_{p=0}^2, \{\mathfrak{h}_p\}_{p=0}^2)$. Let D_* be the complex $C_* \oplus C'_*$, then by considering the inclusion $C_* \hookrightarrow D_*$ and the projection $D_* \twoheadrightarrow C'_*$, we clearly obtain the short-exact sequence

$$0 \to C_* \hookrightarrow D_* = C_* \oplus C'_* \twoheadrightarrow C'_* \to 0.$$

Considering the inclusion $s\colon C'_* \to D_*$ as a section, we can conclude that bases $\mathfrak{c}_p$ of C_p, $\mathfrak{c}_p \oplus \mathfrak{c}'_p$ of D_* and $\mathfrak{c}'_p$ of C'_* are compatible in the sense that determinant of the change-base-matrix from $\mathfrak{c}_p \oplus s(\mathfrak{c}'_p)$ to $\mathfrak{c}_p \oplus \mathfrak{c}'_p$ is (clearly) 1. Therefore, by Milnor's result Theorem 1.1.3, $\mathrm{Tor}(D_*, \{\mathfrak{c}_p \oplus \mathfrak{c}'_p\}_{p=0}^2, \{\mathfrak{h}_p \oplus \mathfrak{h}_p\}_{p=0}^2)$ equals to the product of $\mathrm{Tor}(C_*, \{\mathfrak{c}_p\}_{p=0}^2, \{\mathfrak{h}_p\}_{p=0}^2)$, $\mathrm{Tor}(C'_*, \{\mathfrak{c}'_p\}_{p=0}^2, \{\mathfrak{h}_p\}_{p=0}^2)$, and $\mathrm{Tor}(\mathcal{H}_*)$, where $\mathcal{H}_*$ is the long exact-sequence obtained the above short-exact sequence of complexes, more precisely

$$\begin{aligned}\mathcal{H}_*\colon 0 &\to H_2(C_*) \to H_2(D_*) = H_2(C_*) \oplus H_2(C'_*) \to H_2(C'_*)\\ &\to H_1(C_*) \to H_1(D_*) = H_1(C_*) \oplus H_1(C'_*) \to H_1(C'_*)\\ &\to H_0(C_*) \to H_0(D_*) = H_0(C_*) \oplus H_0(C'_*) \to H_0(C'_*) \to 0.\end{aligned}$$

As ρ discrete, faithful, it is irreducible, and hence $H_2(C_*)$, $H_2(C'_*)$, $H_0(C_*)$, $H_0(C'_*)$ are all zero. Thus, $\mathcal{H}_*$ is actually

$$0 \to H_1(C_*) \to H_1(D_*) = H_1(C_*) \oplus H_1(C'_*) \to H_1(C'_*) \to 0.$$

If we consider the inclusion as section $H_1(C'_*) \to H_1(D_*)$, then we can conclude that $\mathrm{Tor}(\mathcal{H}_*) = 1$ and thus we proved that:

Lemma 3.2.1. *Let $\mathfrak{c}_p$, $\mathfrak{c}'_p$ be the geometric bases of $C_* = C_p(K; \mathrm{Ad}_\rho)$, $C'_* = C_p(K'; \mathrm{Ad}_\rho)$ respectively, and let $\mathfrak{h}_1$ be a basis for $H_1(S; \mathrm{Ad}_\rho)$. Then,*

$$\mathrm{Tor}(D_*, \{\mathfrak{c}_p \oplus \mathfrak{c}'_p\}_{p=0}^2, \{0 \oplus 0, \mathfrak{h}_1 \oplus \mathfrak{h}_1, 0 \oplus 0\}) = [\mathrm{Tor}(C_*, \{\mathfrak{c}_p\}_{p=0}^2, \{0, \mathfrak{h}_1, 0\})]^2.$$

We will now explain how the complex $D_* = C_* \oplus C'_*$ can be considered as a symplectic complex. Following the notations of §1.3, let $(\,\cdot\,,\,\cdot\,)_{p,2-p}\colon C_p \times C'_{2-p} \to \mathbb{R}$ be the intersection form defined by

$$(\sigma_1 \otimes t_1, \sigma_2 \otimes t_2)_{p,2-p} = \sum_{\gamma\in\pi_1(S)} \sigma_1 \,\#\, (\gamma \bullet \sigma_2) B(t_1, \gamma \bullet t_2),$$

where the action of γ on t_2 by $\mathrm{Ad}_{\rho(\gamma)}$, and on σ_2 as deck transformation, "#" denotes the intersection number form and B is the Cartan-Killing form of $\mathfrak{sl}_2(\mathbb{R})$.

Recall that $\#\colon C_0 \times C'_2 \to \mathbb{R}$ is the map

$$\alpha \,\#\, \beta = \begin{cases} 1, & \text{if} \quad \alpha \in \beta; \\ 0, & \text{otherwise} \end{cases}$$

$\#\colon C_2 \times C'_0 \to \mathbb{R}$ is defined as

$$\beta \,\#\, \alpha = \begin{cases} 1, & \text{if} \quad \alpha \in \beta; \\ 0, & \text{otherwise} \end{cases}$$

and $\#\colon C_1 \times C'_1 \to \mathbb{R}$ is the map $\alpha \,\#\, \beta = 0, 1, -1,$ where α, β are in the corresponding generating sets. So, $\#\colon C_p \times C'_{2-p} \to \mathbb{R}$ satisfies $\alpha \,\#\, \beta = (-1)^p \beta \,\#\, \alpha$. Note also that intersection number form "#" is compatible with boundary operator in the sense that for $p = 0, 1, 2,$ $(\partial\alpha) \,\#\, \beta = (-1)^{p+1} \alpha \,\#\, (\partial\beta)$.

Since the action of $\pi_1(S)$ on $\tilde{S}$ properly, discontinuously, and freely, and σ_1, σ_2 are compact, the set $\{\gamma \in \pi_1(S);\, \sigma_1 \cap (\gamma \bullet \sigma_2)\}$ is finite. Note that because intersection number form "#" is anti-symmetric and B is invariant under adjoint action, $(\,\cdot\,,\,\cdot\,)_{p,2-p}$ is anti-symmetric. Moreover, as # is boundary compatible, so are $(\,\cdot\,,\,\cdot\,)_{p,2-p}$. Define $(\,\cdot\,,\,\cdot\,)_{p,2-p}$ on $C_p \times C_{2-p}$ and $C'_p \times C'_{2-p}$ as 0. If $\omega_{p,2-p} : D_p \times D_{2-p} \to \mathbb{R}$ are map defined using $(\,\cdot\,,\,\cdot\,)_{p,2-p}$, then D_* becomes a symplectic complex.

The existence of ω-compatible bases can be obtained from the natural bases. Recall the cells of K and K' can meet at most once. So, if $\{e_1^p, \ldots, e_{k_p}^p\}$ is a bases for p-dimensional cells in K, then the corresponding dual $\{(e_1^p)', \ldots, (e_{k_p}^p)'\}$ will generate $(2-p)$-dimensional cells in K'. e_i^p meets with $(e_i^p)'$ exactly once and never with the other $(e_j^p)'$. Fix the lifts $\{\tilde{e}_1^p, \ldots, \tilde{e}_{k_p}^p\}$ of $\{e_1^p, \ldots, e_{k_p}^p\}$ so that the corresponding dual $\{\widetilde{(e_1^p)'}, \ldots, \widetilde{(e_{k_p}^p)'}\}$ is already fixed. Recall that $\mathcal{A} = \{t_1, t_2, t_3\}$ denotes the basis $\left\{(1/\sqrt{8})\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, (1/\sqrt{8})\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, (1/\sqrt{8})\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$ for $\mathfrak{sl}_2(\mathbb{R})$. Note that the matrix of the Cartan-Killing for B of $\mathfrak{sl}_2(\mathbb{R})$ is in this basis is $\mathrm{Diag}(1, -1, 1)$, where $B(a, b) = 4\,\mathrm{Trace}(ab)$.

By the property that the size of the cells are less than half of the injectivity radius, the intersection $\big(\widetilde{(e_i^p)} \otimes x, \widetilde{(e_j^p)'} \otimes y\big)_{p,2-p}$ becomes $B(x, y) \cdot \underbrace{\widetilde{(e_i^p)} \,\#\, \widetilde{(e_j^p)'}}_{=\delta_{ij}}$. The

ω-compatible bases are obtained by using the following. For $p = 0, 1, 2$, let $\{\tilde{e}_1^p \otimes t_1, \ldots, \tilde{e}_{k_p}^p \otimes t_1; \tilde{e}_1^p \otimes t_2, \ldots, \tilde{e}_{k_p}^p \otimes t_2; \tilde{e}_1^p \otimes t_3, \ldots, \tilde{e}_{k_p}^p \otimes t_3\}$ be basis for C_p and $\left\{\widetilde{(e_1^p)'} \otimes t_1, \ldots, \widetilde{(e_{k_p}^p)'} \otimes t_1; \widetilde{(e_1^p)'} \otimes (-t_2), \ldots, \widetilde{(e_{k_p}^p)'} \otimes (-t_2); \widetilde{(e_1^p)'} \otimes t_3, \ldots, \widetilde{(e_{k_p}^p)'} \otimes t_3\right\}$ be basis for C'_{2-p}. Recall that torsion will be the same (i.e. the well-definiteness) if we change the basis $\mathcal{A}$ of $\mathfrak{sl}_2(\mathbb{R})$ as long as the change-base-matrix has determinant ± 1.

Therefore, we can apply Lemma 2.1.9.

Theorem 3.2.2. *If $\mathfrak{c}_p$, $\mathfrak{c}'_p$ are the geometric bases of $C_* = C_p(K; \mathrm{Ad}_\rho)$, $C'_* = C_p(K'; \mathrm{Ad}_\rho)$ respectively, and if $\mathfrak{h}_1$ is a basis for $H_1(S; \mathrm{Ad}_\rho)$, then*

$$\mathrm{Tor}(D_*, \{\mathfrak{c}_p \oplus \mathfrak{c}'_p\}_{p=0}^2, \{0 \oplus 0, \mathfrak{h}_1 \oplus \mathfrak{h}_1, 0 \oplus 0\}) = (\mathrm{Pfaf}([\omega]_{1,1}))^{-1}$$

where $[\omega]_{1,1} \colon H_1(D_*) \times H_1(D_*) \to \mathbb{R}$ is the map $\begin{bmatrix} 0 & (\,\cdot\,,\,\cdot\,)_{1,1} \\ -(\,\cdot\,,\,\cdot\,)_{1,1} & 0 \end{bmatrix}$, where $(\,\cdot\,,\,\cdot\,)_{1,1} \colon H_1(C_*) \times H_1(C'_*) \to \mathbb{R}$ is the extension of the intersection form

$$(\,\cdot\,,\,\cdot\,)_{1,1} \colon C_1(K; \mathrm{Ad}_\rho) \times C_1(K'; \mathrm{Ad}_\rho) \to \mathbb{R},$$

and where $\mathrm{Pfaf}([\omega]_{1,1}) = \sqrt{\det\begin{bmatrix} [\omega]_{1,1} \\ \textit{in basis } \mathfrak{h}_1 \oplus \mathfrak{h}_1 \end{bmatrix}}$.

Recall $H_1(D_*) = H_1(C_*) \oplus H_1(C'_*)$ and each component is canonically isomorphic to $H_1(S; \mathrm{Ad}_\rho)$. So, we can consider

$$(\,\cdot\,,\,\cdot\,)_{1,1} \colon H_1(C_*) \times H_1(C'_*) \to \mathbb{R}$$

as $(\,\cdot\,,\,\cdot\,)_{1,1} \colon H_1(S; \mathrm{Ad}_\rho) \times H_1(S; \mathrm{Ad}_\rho) \to \mathbb{R}$, and thus $[\omega]_{1,1} \colon H_1(D_*) \times H_1(D_*) \to \mathbb{R}$ can be considered as $[\omega]_{1,1} \colon H_1(S; \mathrm{Ad}_\rho) \oplus H_1(S; \mathrm{Ad}_\rho) \times H_1(S; \mathrm{Ad}_\rho) \oplus H_1(S; \mathrm{Ad}_\rho) \to \mathbb{R}$. Note that because $(\,\cdot\,,\,\cdot\,)_{1,1} \colon H_1(S; \mathrm{Ad}_\rho) \times H_1(S; \mathrm{Ad}_\rho) \to \mathbb{R}$ is non-degenerate skew-symmetric, $\det(\,\cdot\,,\,\cdot\,)_{1,1}$ in basis $\mathfrak{h}_1$, which actually is $\mathrm{Pfaf}((\,\cdot\,,\,\cdot\,)_{1,1})^2$, is positive. Thus, $\mathrm{Pfaf}([\omega]_{1,1})$ equals to $\sqrt{\left(\det\begin{bmatrix} (\,\cdot\,,\,\cdot\,)_{1,1} \\ \text{in basis } \mathfrak{h}_1 \end{bmatrix}\right)^2}$, or $\det\begin{bmatrix} (\,\cdot\,,\,\cdot\,)_{1,1} \\ \text{in basis } \mathfrak{h}_1 \end{bmatrix}$.

Therefore, Theorem 3.2.2 says if $\mathfrak{c}_p$, $\mathfrak{c}'_p$ are the geometric bases of $C_* = C_p(K; \mathrm{Ad}_\rho)$, $C'_* = C_p(K'; \mathrm{Ad}_\rho)$ respectively, and if $\mathfrak{h}_1$ is a basis for $H_1(S; \mathrm{Ad}_\rho)$, then

$$\mathrm{Tor}(D_*, \{\mathfrak{c}_p \oplus \mathfrak{c}'_p\}_{p=0}^2, \{0 \oplus 0, \mathfrak{h}_1 \oplus \mathfrak{h}_1, 0 \oplus 0\}) = \left(\det\begin{bmatrix} (\,\cdot\,,\,\cdot\,)_{1,1} \\ \text{in basis } \mathfrak{h}_1 \end{bmatrix}\right)^{-1}.$$

On the other hand, by Lemma 3.2.1, we also have

$$\mathrm{Tor}(D_*, \{\mathfrak{c}_p \oplus \mathfrak{c}'_p\}_{p=0}^2, \{0 \oplus 0, \mathfrak{h}_1 \oplus \mathfrak{h}_1, 0 \oplus 0\}) = [\mathrm{Tor}(C_*, \{\mathfrak{c}_p\}_{p=0}^2, \{0, \mathfrak{h}_1, 0\})]^2,$$

and thus $\mathrm{Tor}(C_*, \{\mathfrak{c}_p\}_{p=0}^2, \{0, \mathfrak{h}_1, 0\}) = \pm\sqrt{\det\left[\begin{smallmatrix}(\,\cdot\,,\ \cdot\,)_{1,1}\\ \text{in basis } \mathfrak{h}_1\end{smallmatrix}\right]}$. Let $H = [h_{ij}]$ be the non-degenerate skew-symmetric matrix of $(\,\cdot\,,\ \cdot\,)_{1,1}$ in basis $\mathfrak{h}_1$, i.e. $h_{ij} = ((\mathfrak{h}_1)_i, (\mathfrak{h}_1)_j)_{1,1}$, where $(\mathfrak{h}_1)_i$ denotes the i^{th} element of the basis $\mathfrak{h}_1$.

Recall the commutative diagram of §1.3

$$\begin{tikzcd}
H^1(S;\mathrm{Ad}_\rho) \times H^1(S;\mathrm{Ad}_\rho) \arrow[r, "\smile_B"] \arrow[dr, phantom, "\circlearrowleft"] & H^2(S;\mathbb{R}) \\
H_1(S;\mathrm{Ad}_\rho) \times H_1(S;\mathrm{Ad}_\rho) \arrow[u, "\mathrm{PD}"] \arrow[r, "(\cdot,\cdot)_{1,1}"] & \mathbb{R}, \arrow[u]
\end{tikzcd}$$

where $\mathbb{R} \to H^2(S;\mathbb{R})$ is the mapping sending 1 to the fundamental class of $H^2(S;\mathbb{R})$ and the inverse of this the map $\mathbb{R} \to H^2(S;\mathbb{R})$ is integration over the surface, where B is the Cartan-Killing form of $\mathfrak{sl}_2(\mathbb{R})$.

If $\mathfrak{h}^1$ is the basis of $H^1(S;\mathrm{Ad}_\rho)$ corresponding to the basis $\mathfrak{h}_1$ of $H_1(S;\mathrm{Ad}_\rho)$, then from the commutative diagram, $h_{ij} = ((\mathfrak{h}_1)_i, (\mathfrak{h}_1)_j)_{1,1}$ equals to $\int_S (\mathfrak{h}^1)_i \smile_B (\mathfrak{h}^1)_j$. The last term is actually $\omega_{\text{Goldman}}((\mathfrak{h}^1)_i, (\mathfrak{h}^1)_j)$, where ω_{Goldman} is the Goldman symplectic form on Teichmüller space $\mathrm{Teich}(S)$ of S, namely

$$H^1(S;\mathrm{Ad}_\rho) \times H^1(S;\mathrm{Ad}_\rho) \xrightarrow{\smile_B} H^2(S;\mathbb{R}) \xrightarrow{\int_S} \mathbb{R}.$$

So, the non-degenerate skew-symmetric matrix $H = [h_{ij}]$ is also the matrix of the anti-symmetric ω_{Goldman} in basis $\mathfrak{h}^1$ of $H^1(S;\mathrm{Ad}_\rho)$. Let $A = [a_{ij}]$ be the skew-symmetric matrix $(H^{\text{transpose}})^{-1}$. Consider the 2-form ω_A associated to A defined by $\sum_{i<j} a_{ij}(\mathfrak{h}^1)_i \wedge (\mathfrak{h}^1)_j$. Recall that, using the de Rham theory, elements of $H^1(S;\mathrm{Ad}_\rho)$ can be considered (locally) as $\alpha \otimes t$, where $\alpha \in H^1(S;\mathbb{R})$, and $t \in \mathfrak{sl}_2(\mathbb{R})$. If $\alpha_1 \otimes t_1$, $\alpha_2 \otimes t_2$ are in $H^1(S;\mathrm{Ad}_\rho)$, then $\alpha_1 \otimes t_1 \wedge \alpha_2 \otimes t_2$ is nothing but $\alpha_1 \wedge \alpha_2 B(t_1, t_2) \in H^2(S;\mathbb{R})$, i.e. $\alpha_1 \otimes t_1 \smile_B \alpha_2 \otimes t_2$.

Note that $\mathrm{Pfaf}(\omega_A)$, which is $\omega_A \wedge \cdots \wedge \omega_A/(3g-3)!$, is $\det(A)$. Combining all these, we can conclude that $\mathrm{Tor}(C_*, \{\mathfrak{c}_p\}_{p=0}^2, \{0, \mathfrak{h}_1, 0\}) = \pm\sqrt{\det(H)^{-1}} = \pm\sqrt{\det(A)} = \pm\,\mathrm{Pfaf}(\omega_A)$. Actually, by Theorem 2.1.9 and the existence of ω-compatible bases obtained from the natural bases, we have

$$\mathrm{Tor}(C_*, \{\mathfrak{c}_p\}_{p=0}^2, \{0, \mathfrak{h}_1, 0\}) = \mathrm{Pfaf}(\omega_A).$$

Consider $\omega_H \in H^2(S;\mathbb{R})$ associated to the matrix H by $\sum_{i<j} h_{ij}(\mathfrak{h}^1)_i \wedge (\mathfrak{h}^1)_j$, then $\omega_A = \alpha\omega_H$ for $H^2(S;\mathbb{R})$ being 1-dimensional. Integrating both sides over S and recalling that $\int_S (\mathfrak{h}^1)_i \smile_B (\mathfrak{h}^1)_j = ((\mathfrak{h}_1)_i, (\mathfrak{h}_1)_j)_{1,1}$, i.e. h_{ij}, we obtain $\sum_{i<j} a_{ij}h_{ij} = \alpha\sum_{i<j} h_{ij}h_{ij}$, or $\sum_{i<j} a_{ij}H_{ji}^{\text{transpose}} = \alpha\sum_{i<j} h_{ij}H_{ji}^{\text{transpose}}$, or $\sum_{i=1}^{6g-6}(A \cdot H^{\text{transpose}})_{ii} = \alpha\sum_{i=1}^{6g-6}(H \cdot H^{\text{transpose}})_{ii}$, thus $\alpha = (6g-6)/\|H\|^2$, where $\|H\|^2$ is the inner product $\langle H, H\rangle = \mathrm{Tr}(HH^{\text{transpose}})$.

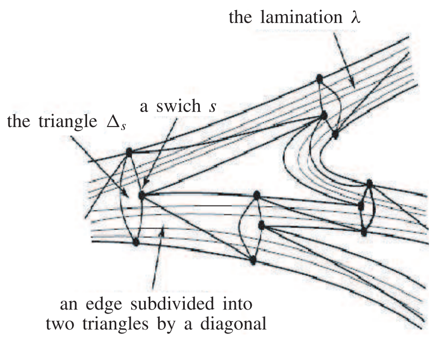

Fig. 2.

Thus, $\mathrm{Pfaf}(\omega_A)$ equals to $((6g-6)/\|H\|^2)^{3g-3}\cdot\mathrm{Pfaf}(\omega_H)$ i.e. $((6g-6)/\|H\|^2)^{3g-3}\cdot\sqrt{\det(H)}$, where $h_{ij}=((\mathfrak{h}_1)_i,(\mathfrak{h}_1)_j)_{1,1}=\omega_{\mathrm{Goldman}}((\mathfrak{h}^1)_i,(\mathfrak{h}^1)_j)$.

Therefore, we have proved that

Theorem 3.2.3. *If $\mathfrak{h}^1$ is a basis for $H^1(S;\mathrm{Ad}_\rho)$, and for $p=0,1,2$, $\mathfrak{c}_p$ are the geometric bases of $C_p(K;\mathrm{Ad}_\rho)$, then*

$$\mathrm{Tor}(C_*,\{\mathfrak{c}_p\}_{p=0}^2,\{0,\mathfrak{h}_1,0\})=\left(\frac{6g-6}{\|H\|^2}\right)^{3g-3}\mathrm{Pfaf}(\omega_{\mathrm{Goldman}}),$$

where $\mathrm{Pfaf}(\omega_{\mathrm{Goldman}})$ denotes $\sqrt{\det(H)}$, and H is the matrix $[\omega_{\mathrm{Goldman}}((\mathfrak{h}^1)_i,(\mathfrak{h}^1)_j)]$.

Let λ be a maximal geodesic lamination on the surface S. Let $K_\lambda=K_\Phi$ triangulation of the surface by using the maximal geodesic lamination (see [27] for details.) Namely, let Φ be a fattened train-track carrying the maximal geodesic lamination. For each switch s of Φ, choose in the incoming edge e_s^{in} an arc s' transverse to λ with the same end points as s but interior disjoint s. Then, $s\cup s'$ will bound in e_s^{in} a triangle Δ_s whose edges are s', $s\cap e_s^{\mathrm{left}}$, and $s\cap e_s^{\mathrm{right}}$ see Fig. 2. The complement in Φ of all these triangles Δ_s is a disjoint union of rectangles. Split each rectangle into two triangles by a diagonal transverse to λ so that we have a triangulation of Φ whose edges are all transverse to the leaves of λ. Extend this triangulation arbitrarily to a triangulation of the surface S.

Considering the above triangulation of S and by Theorem 3.1.3, we conclude the proof of Theorem 0.0.4.

Theorem 3.2.4. *Let S be a compact hyperbolic surface, λ be a fixed maximal geodesic lamination on S, and let K_λ be the corresponding triangulation of the sur-*

face obtained from λ. *For* $p = 0, 1, 2$, *let* $\mathfrak{c}_p$ *be the corresponding geometric bases for* $C_p(K_\lambda; \mathcal{A}d_\rho)$, *and let* $\mathfrak{h}$ *be a basis for* $\mathcal{H}(\lambda; \mathbb{R})$.

$$\mathrm{Tor}(C_*, \{\mathfrak{c}_p\}_{p=0}^2, \{0, \mathfrak{h}, 0\}) = \frac{(6g-6)\cdot\sqrt{2^{6g-6}}}{4\cdot\|T\|^2}\mathrm{Pfaff}(\tau),$$

where $\mathrm{Pfaff}(\tau)$ *is the* Pfaffian *of the matrix* $T = [\tau(\mathfrak{h}_i, \mathfrak{h}_j)]$, $\|T\|^2 = \mathrm{Trace}(T\,T^{\mathrm{transpose}})$, *and* $\tau\colon \mathcal{H}(\lambda; \mathbb{R}) \times \mathcal{H}(\lambda; \mathbb{R}) \to \mathbb{R}$ *is the Thurston symplectic form.*

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