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A CRITERION FOR HYPOELLIPTICITY OF SECOND ORDER DIFFERENTIAL OPERATORS

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Introduction and results. In this paper we give a sufficient condition for second order differential operators to be hypoelliptic. The condition is also necessary for a special class of differential operators.

Let Ω be an open set in R^n and let $P=p(x, D_x)$ be a second order differential operator with coefficients in $C^\infty(\Omega)$, that is,

$$p(x, D_x) = \sum_{j,k=1}^{n} a_{jk} D_{x_j} D_{x_k} + \sum_{j=1}^{n} i b_j D_{x_j} + c, \quad D_{x_j} = -i\partial_{x_j}, \tag{1}$$

where coefficients $a_{jk}(x)$, $b_j(x)$ and $c(x)$ belong to $C^\infty(\Omega)$. We assume that $a_{jk}(x)$, $b_j(x)$ are real valued and $a_{jk}(x)$ satisfy for any x in Ω

$$\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \geqq 0 \qquad \textit{for all } \xi\in R^n. \tag{2}$$

Let $\log\langle D_x\rangle$ denote a pseudodifferential operator with symbol $\log\langle\xi\rangle$, where $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$.

Theorem 1. *Assume that for any $\varepsilon>0$ and any compact set K of Ω there exists a constant $C_{\varepsilon,K}$ such that*

$$||(\log\langle D_x\rangle)^2 u|| \leqq \varepsilon ||Pu|| + C_{\varepsilon,K}||u||, \quad u\in C_0^\infty(K). \tag{3}$$

Then P is hypoelliptic in Ω. Furthermore we have

$$\mathrm{WF}\, Pv = \mathrm{WF}\, v \qquad \textit{for } v\in\mathcal{D}'(\Omega). \tag{4}$$

Corollary 2. *Assume that for any $\varepsilon>0$ and any compact set K of Ω there exists a constant $C_{\varepsilon,K}$ such that*

$$||(\log\langle D_x\rangle)u||^2 \leqq \varepsilon\, \mathrm{Re}\,(Pu, u) + C_{\varepsilon,K}||u||^2, \quad u\in C_0^\infty(K). \tag{5}$$

Then we have (4).

Proof of Corollary. For $u\in C_0^\infty(K)$ take $\phi, \psi\in C_0^\infty(\Omega)$ such that $\phi=1$ on K and $\psi=1$ on supp ϕ. Note

$$(\log\langle D_x\rangle)u = \psi(\log\langle D_x\rangle)u + (1-\psi)(\log\langle D_x\rangle)\phi\, u$$

and $(1-\psi)(\log\langle D_x\rangle)\phi\in S^{-\infty}$ (see Chapter 2 of [5]). Since $\psi(\log\langle D_x\rangle)u$ belongs to C_0^∞, in view of the above formula we may replace u in (5) by $(\log\langle D_x\rangle)u$. Since the principal symbol of $[P, \log\langle D_x\rangle]$ is purely imaginary we have

$$\mathrm{Re}([P, \log\langle D_x\rangle]u, (\log\langle D_x\rangle)u)\leqq C(\|(\log\langle D_x\rangle)^2u\|^2+\|u\|^2)\,.$$

In view of this it is clear that (5) implies (3). Q.E.D.

The estimate (3) is not always necessary for the hypoellipticity. We have a counter example $\mathcal{A}_0(x, D_x)=D_{x_1}^2+\exp(-1/|x_1|^\delta)D_{x_2}^2$ for $\delta\geqq 1$ given by [Fediĭ [2] (cf. [8]). In fact, $\mathcal{A}_0$ is hypoelliptic for any $\delta>0$, but when $\delta\geqq 1$ the estimate (3) does not hold for some small $\varepsilon>0$ (see Remark 3.1 in Section 3). However, for a class of differential operators, the estimate (3) is necessary to be hypoelliptic.

The result concerning the necessity can be discussed for some class of operators of higher order. Let m be an even positive integer and let P_0 be a differential operator of the form

$$P_0=D_t^m+\mathcal{A}(x, D_x) \qquad in\ R_t\times R_x^n\,,$$

where $\mathcal{A}(x, D_x)$ is a differential operator of order m with C^∞-coefficients. We assume that $\mathcal{A}(x, D_x)$ is formally self-adjoint in an open set Ω of R_x^n and bounded from below, that is, there exists a real c_0 such that

$$(6) \qquad (\mathcal{A}(x, D_x)u, u)\geqq c_0\|u\|^2 \qquad for\ u\in L^2(\Omega)\ satisfying\ \mathcal{A}u\in L^2(\Omega)\,.$$

Theorem 3. *Let P_0 be the above operator. Assume that P_0 is hypoelliptic in $R_t\times\Omega$. Then for any $x_0\in\Omega$ there exists a neighborhood ω of x_0 such that for any $\varepsilon>0$ the estimate*

$$(7) \qquad \|(\log\langle D_t, D_x\rangle)^{m/2}u\|^2\leqq\varepsilon\,\mathrm{Re}(P_0u,u)+C_\varepsilon\|u\|^2,\ u\in C_0^\infty(R_t\times\omega)$$

holds with a constant C_ε.

REMARK. When $m=2$ the estimate (5) follows from (7). In fact, for any compact set K of $R_t\times\Omega$, let K' be the projection of K to Ω, and take the partition of unity $\sum\phi_j^2(x)=1$ over K'. Since $\mathrm{Re}([P_0, \phi_j]u, \phi_ju)$ is majorated by a constant times of $\|u\|^2$, we have (7) for $u\in C_0^\infty(R_t\times K')$, which implies (5). In view of the proof of Corollary 2, the estimate (3) also follows from (7).

Our two theorems are applicable to the hypoellipticity of operators considered in [10] and [11]. Especially, an application shows that $D_t^2+D_{x_1}^2+\exp(-1/|x_1|^\delta)D_{x_2}^2$ $(\delta>0)$ is hypoelliptic in R^3 if and only if δ satisfies $\delta<1$ (cf. Theorem 8.41 of [6]). As another application we give:

Proposition 4. *Set $P_1=D_t^2+x_2^2D_{x_1}^2+D_{x_2}^2+D_{x_3}(\sigma(x_1)^2\zeta(x)^2)D_{x_3}$, where $\sigma, \zeta\in C^\infty$, $\sigma(s)>0$ $(s\neq 0)$, $\zeta>0$, $\sigma(0)=0$ and $s\sigma'(s)\geqq 0$. Then P_1 is hypoelliptic in R^4 if* *and only if* $\sigma(s)$ *satisfies*

(8) $$\lim_{s\to 0} |s^{1/2}\log\sigma(s)| = 0\,.$$

REMARK. If $\sigma(s)=\exp(-1/|s|^{\delta})$ for $\delta>0$ then (8) means $\delta<1/2$.

The plan of this paper is as follows: In Section 1 we prove Theorem 1. The idea of the proof is the same as in Section 5 of [11], though we employ the microlocalization method by Hörmander [4]. In Section 2 we prove Theorem 3 by using the interpolation method similar to the one in Métivier [7], where nonanalytic hypoellipticity for operators of the same form as (6) was studied (cf. Baouendi-Goulaouic [1]). The proof given in Section 2 is nothing but C^∞-version of [7]. Section 3 is devoted to the proof of Proposition 4.

We finally remark that the criterion of Theorem 1 applies to second order differential operators with finite degeneracy studied by Hörmander [3] and Oleinik-Radkevich [13], because for such operators we have the sub-elliptic estimate $\|u\|_{\delta}\leqq C(\|Pu\|+\|u\|)$, $\delta>0$.

1. Proof of Theorem 1

Before proving the theorem we introduce some notations. When $\phi, \psi\in C_0^\infty(R^n)$ satisfy $\psi=1$ in a neighborhood of supp ϕ, we write $\phi\Subset\psi$. For a pseudodifferential operator $Q=q(x, D_x)$ we denote by $\sigma(Q)$ the symbol $q(x, \xi)$. We denote by $Q_{(\beta)}^{(\alpha)}$ a pseudodifferential operator with symbol $q_{(\beta)}^{(\alpha)}(x,\xi)=D_x^\beta\partial_\xi^\alpha q(x,\xi)$ for multi-indices α and β.

Here and throughout the present paper $P=p(x, D_x)$ denotes the second order differential operator in Introduction satisfying the condition (2). For the brevity we assume $\Omega=R^n$. Without loss of generality we may assume that coefficients of P are defined in R^n and belong to $\mathscr{B}^\infty(R^n)$. As proved by [13], it follows from (2) that

(1.1) $$\sum_{|\alpha|=1}\|P^{(\alpha)}u\|^2\leqq C(\mathrm{Re}(Pu, u)+\|u\|^2),\quad u\in C_0^\infty(K)\,,$$

and

(1.2) $$\sum_{|\alpha|=1}\|\langle D_x\rangle^{-1}P_{(\alpha)}u\|^2\leqq C(\mathrm{Re}\sum_{j=1}^{n}(D_j\langle D_x\rangle^{-1}Pu, D_j\langle D_x\rangle^{-1}u)+\|u\|^2),\ u\in C_0^\infty(K)\,,$$

hold with some constant $C=C_K$, where $D_j=D_{x_j}$. In fact, (1.1) follows from (2.6.6) and (2.6.9) of [13], and (1.2) follows from (2.6.14) of [13].

Write $p(x, \xi)=\sum_{k=0}^{2}p_k(x, \xi)$, where p_k is positively homogeneous in ξ of degree k. Let $h(x)\in C_0^\infty(R_x^n)$ be 1 for $|x|\leqq 1/5$ and vanish for $|x|\geqq 7/24$. For $\gamma\equiv(x_0, \xi_0)\in R^n\times S^{n-1}$ we consider a microlocalized pseudodifferential operator

(1.3) $$P_\gamma = p_\gamma(\lambda y, \lambda D_y) = \sum_{k=0}^{2}p_k(x_0+\lambda y, \xi_0+\lambda D_y)h(\lambda D_y/3)\lambda^{-2k}$$

with a small parameter $\lambda>0$ (see [4] and Section 2 of [9]).

It is clear that for any muti-indices α and β we have

$$|\partial_\eta^\alpha D_y^\beta p_\gamma(\lambda y, \lambda\eta)|\leqq C_{\alpha,\beta}\lambda^{-4+|\alpha|+|\beta|},\ 0<\lambda\leqq 1\,, \tag{1.4}$$

with a constant $C_{\alpha,\beta}$ independent of λ.

Let $(P^{(\alpha)})_\gamma$ and $(P_{(\alpha)})_\gamma$ be microlocalized operators defined from symbols of $P^{(\alpha)}$ and $P_{(\alpha)}$ by the similar formula as (1.3). From estimates (3), (1.1) and (1.2) we have the following:

Lemma 1.1. *For any real $s>0$ and any $\gamma\equiv(x_0, \xi_0)\in R^n\times S^{n-1}$ there exists a constant $C(s, \gamma)$ such that for $0<\lambda\leqq 1$*

$$(\log\lambda^{-s})^2\|Hv\|\leqq\|H_0P_\gamma v\|+C(s, \gamma)\|v\|,\ v\in\mathcal{S}_y\,, \tag{1.5}$$

where $H=h(\lambda D_y)h(\lambda y)$ and $H_0=h(\lambda D_y/2)h(\lambda y/2)$. Furthermore, for any $\gamma\in R^n\times S^{n-1}$ there exists a constant C_γ such that for $0<\lambda\leqq 1$

$$\sum_{|\alpha|=1}\|H(P^{(\alpha)})_\gamma v\|\leqq C_\gamma(\|HP_\gamma v\|+\|v\|),\ v\in\mathcal{S}_y\,, \tag{1.6}$$

and

$$\sum_{|\alpha|=1}\|\lambda^2H(P_{(\alpha)})_\gamma v\|\leqq C_\gamma(\|HP_\gamma v\|+\|v_\gamma\|),\ v\in\mathcal{S}_y\,. \tag{1.7}$$

Proof. Set $v(y)=(\exp(-i\lambda^{-2}x\cdot\xi_0)w(x))|_{x=\lambda y+x_0}$ for $w\in\mathcal{S}_x$. Then we have

$$\begin{aligned}&\exp(-i\lambda^{-2}x\cdot\xi_0)p(x, D_x)h((\lambda^2D_x-\xi_0)/3)w(x)\\&\quad=(P_\gamma v)(\lambda^{-1}(x-x_0))\end{aligned} \tag{1.8}$$

and for real s' we have

$$\begin{aligned}&\exp(-i\lambda^{-2}x\cdot\xi_0)|D_x|^{s'}h(\lambda^2D_x-\xi_0)w(x)\\&\quad=\lambda^{-2s'}(|\xi_0+\lambda D_y|^{s'}h(\lambda D_y)v)(\lambda^{-1}(x-x_0))\,.\end{aligned} \tag{1.9}$$

Indeed, both formulas are easily seen if we note the change of variables

$$x-x_0=\lambda y,\ \xi-\lambda^{-2}\xi_0=\lambda^{-1}\eta\,.$$

Furthermore we have

$$\begin{aligned}&\exp(-i\lambda^{-2}x\cdot\xi_0)(\log\langle D_x\rangle)^2h(\lambda^2D_x-\xi_0)w(x)\\&\quad=(4((\log\lambda^{-1}+r(D_y;\lambda))^2h(\lambda D_y)v)(\lambda^{-1}(x-x_0))\,,\end{aligned} \tag{1.10}$$

where $r(\eta;\lambda)=(\log(\lambda^4+|\lambda\eta+\xi_0|^2))/4$. It is clear that $\{r(\eta;\lambda)h(\lambda\eta);0<\lambda\leqq 1\}$ and $\{r(\eta;\lambda)^2h(\lambda\eta);0<\lambda\leqq 1\}$ are bounded sets in $S^0_{0,0}$, as pseudodifferential operators in R^n_y. Note that $\{h(\lambda^2\xi-\xi_0);0<\lambda\leqq 1\}$ is a bounded set in $S^2_{1\,0}$, as a pseudodifferential operator in R^n_x, because $\lambda^2\leqq(31/24)|\xi|^{-1}$ on supp $h(\lambda^2\xi-\xi_0)$. We shall prove (1.6). Substitute $u=h(x-x_0)h(\lambda^2D_x-\xi_0)w=h(x-x_0)h(\lambda^2D_x-\xi_0)$

$h((\lambda^2 D_x-\xi_0)/3)w$ for $w\in\mathcal{S}_x$ into (1.1). Then we have

$$\sum_{|\alpha|=1}\|h(\lambda^2 D_x-\xi_0)h(x-x_0)P^{(\alpha)}h((\lambda^2 D_x-\xi_0)/3)w\|^2 \tag{1.11}$$
$$\leqq C(\mathrm{Re}(h(\lambda^2 D_x-\xi_0)h(x-x_0)Ph((\lambda^2 D_x-\xi_0)/3)w, h(\lambda^2 D_x-\xi_0)h(x-x_0)w) + \|w\|^2),\ w\in\mathcal{S}_x,$$

In fact, we can majorate the terms concerning commutators among $h(x-x_0)$, $h(\lambda^2 D_x-\xi_0)$ and P appearing in the right hand side by a constant times of $\|w\|^2$, because, their symbols are purely imaginary. In view of (1.8) and the same formula with P replaced by $P^{(\alpha)}$ we have

$$\sum_{|\alpha|=1}\|H(P^{(\alpha)})_\gamma v\|^2\leqq C_\gamma(\mathrm{Re}(HP_\gamma v, Hv)+\|v\|^2),\ v\in\mathcal{S}_y, \tag{1.12}$$

which gives (1.6) together with Schwartz inequality. Similarly it follows from (1.2) that

$$\begin{aligned}&\sum_{|\alpha|=1}\||D_x|^{-1}h(\lambda^2 D_x-\xi_0)h(x-x_0)P_{(\alpha)}h((\lambda^2 D_x-\xi_0)/3)w\|^2\\ &\quad\leqq C(\mathrm{Re}\sum_{j=1}^n (D_j\langle D_x\rangle^{-1}h(\lambda^2 D_x-\xi_0)h(x-x_0)Ph((\lambda^2 D_x-\xi_0)/3)w\,,\\ &\qquad D_j\langle D_x\rangle^{-1}h(\lambda^2 D_x-\xi_0)h(x-x_0)w)+\|w\|^2)\,.\end{aligned} \tag{1.13}$$

From this we obtain (1.7) if we note (1.9) and

$$\begin{aligned}&\exp(-i\lambda^{-2}x\cdot\xi_0)D_j\langle D_x\rangle^{-1}h(\lambda^2 D_x-\xi_0)w(x)\\ &\quad=(r_j(D_y;\lambda)v)(\lambda^{-1}(x-x_0))\,,\end{aligned} \tag{1.9$'$}$$

where $r_j(\eta;\lambda)\equiv(\lambda\eta_j+\xi_{0j})(\lambda^4+|\lambda\eta+\xi_0|^2)^{-1/2}h(\lambda\eta)$ belongs to $S^0_{0,0}$ uniformly with respect to $0<\lambda\leqq 1$. We shall prove (1.5). Substituting $u=h(x-x_0)$ $h(\lambda^2 D_x-\xi_0)h(\lambda^2 D_x-\xi_0)/3)w$ into (3) we have for any $\varepsilon>0$ and some constant C_ε

$$\begin{aligned}&\|(\log\langle D_x\rangle)^2 h(\lambda^2 D_x-\xi_0)h(x-x_0)w\|\\ &\quad\leqq\varepsilon(\|h(\lambda^2 D_x-\xi_0)h(x-x_0)Ph((\lambda^2 D_x-\xi_0)/3)w\|\\ &\qquad+\|[h(\lambda^2 D_x-\xi_0),\ h(x-x_0)]Ph((\lambda^2 D_x-\xi_0)/3)w\|\\ &\qquad+\|[P,\ h(x-x_0)h(\lambda^2 D_x-\xi_0)]h((\lambda^2 D_x-\xi_0)/3)w\|)\\ &\qquad+C_\varepsilon\|w\|\equiv\varepsilon(I_1+I_2+I_3)+C_\varepsilon\|w\|\,.\end{aligned}$$

Note $h(\lambda^2 D_x-\xi_0)h(x-x_0)=h(\lambda^2 D_x-\xi_0)h(x-x_0)h((x-x_0)/2)$ and

$$\begin{aligned}&\sigma(h(\lambda^2 D_x-\xi_0)h(x-x_0))-h(x-x_0)h(\lambda^2\xi-\xi_0)\\ &\quad-\sum_{|\alpha|=1}D_x^\alpha h(x-x_0)\partial_\xi^\alpha h(\lambda^2\xi-\xi_0)\in S^{-2}_{1,0}\,,\end{aligned}$$

uniformly with respect to $0<\lambda\leqq 1$. Then we see that I_1 is estimated above by a constant times of

$$J\equiv\|h((\lambda^2 D_x-\xi_0)/2)h((x-x_0)/2)Ph((\lambda^2 D_x-\xi_0)/3)w\|+\|w\|\,,$$

because $h((\lambda^2\xi-\xi_0)/2)=1$ on supp $\partial_\xi^\alpha h(\lambda^2\xi-\xi_0)$. Since $D_x^\alpha h(x-x_0)\partial_\xi^\alpha h(\lambda^2 D_x-\xi_0)$ $\equiv D_x^\alpha h(x-x_0)\partial_\xi^\alpha h(\lambda^2 D_x-\xi_0)h((x-x)_0/2)$ mod $S^{-\infty}$, I_2 is also estimated above by J with a constant factor. Noting

$$[P, h(x-x_0)h(\lambda^2 D_x-\xi_0)] \\ -\sum_{|\alpha+\beta|=1}(-1)^{|\beta|}D_x^\alpha h(x-x_0)\partial_\xi^\beta h(\lambda^2 D_x-\xi_0)P_{(\beta)}^{(\alpha)}\in S_{1,0}^0$$

we see that I_2 is estimated above by a constant times of

$$||h((\lambda^2 D_x-\xi_0)/2)h((x-x_0)/2)P^{(\alpha)}h((\lambda^2 D_x-\xi_0)/3)w|| \\ +|||D_x|^{-1}h((\lambda^2 D_x-\xi_0)/2)h((x-x_0)/2)P_{(\alpha)}h((\lambda^2 D_x-\xi_0)/3)w||+||w||\,.$$

By substituting $u=h((x-x_0)/2)h((\lambda^2 D_x-\xi_0)/2)h((\lambda^2 D_x-\xi_0)/3)w$ into (1.1) and (1.2), we have (1.11) and (1.13) with $h(\lambda^2 D_x-\xi_0)h(x-x_0)$ replaced by $h((\lambda^2 D_x-\xi_0)/2)h((x-x_0)/2)$. Using these estimates together with Schwartz's inequality, from the estimations for I_j $(j=1, 2, 3)$ we have with a constant $c>0$ independent of ε

$$||(\log\langle D_x\rangle)^2 h(\lambda^2 D_x-\xi_0)h(x-x_0)w|| \\ \leqq c\varepsilon||h((\lambda^2 D_x-\xi_0)/2)h((x-x_0)/2)Ph((\lambda^2 D_x-\xi_0)/3)w||+C'_\varepsilon||w||\,.$$

In view of (1.8) and (1.10), we obtain

$$(\log\lambda^{-1})^2||Hv||\leqq\varepsilon||H_0P_\gamma v||+C''_\varepsilon||v|| \qquad \textit{if } 0<\lambda\leqq\lambda_1\,,$$

where $\lambda_1>0$ is a sufficiently small number such that for $0<\lambda\leqq\lambda_1$

$$||r(D_y;\lambda)h(\lambda D_y)v||\leqq(\log\lambda^{-1})||h(\lambda D_y)v||\,.$$

Taking $s=\varepsilon^{-2}$ we obtain (1.5) when $0<\lambda\leqq\lambda_1$. The estimate (1.5) for $\lambda_1<\lambda\leqq1$ is obvious. Q.E.D.

Estimates (1.6) and (1.7) can be strengthened to the following form:

Corollary. *For any $\gamma\in R^n\times S^{n-1}$ there exists a constant C'_γ such that for any $s>0$ estimates*

$$(1.6)' \quad \sum_{|\alpha|=1}||H(P^{(\alpha)})_\gamma v||\leqq C'_\gamma((\log\lambda^{-s})^{-1}||HP_\gamma v||+(\log\lambda^{-s})||Hv||+||v||)\,,$$

and

$$(1.7)' \quad \sum_{|\alpha|=1}||\lambda^2 H(P_{(\alpha)})_\gamma v||\leqq C'_\gamma((\log\lambda^{-s})^{-1}||HP_\gamma v||+(\log\lambda^{-s})||Hv||+||v||)$$

holds if $0<\lambda<1$.

Proof. The estimate (1.6)′ is a direct consequence of (1.12) because

$$\mathrm{Re}(HP_\gamma v, Hv)\leqq(\log\lambda^{-s})^{-2}||HP_\gamma v||^2+(\log\lambda^{-s})^2||Hv||^2\,.$$

We also have (1.7)′ by the similar estimate as (1.12) that is derived from (1.13). Q.E.D.

Note that $||v||\leqq||Hv||+||(1-H)v||$ and that for any $s>0$ and any $\gamma\in R^n\times S^{n-1}$ there exists a small positive number $\lambda_0\equiv\lambda_0(s,\gamma)<1$ such that

$$(1.14)\qquad (\log\lambda^{-s})^2\geqq 1+2(C(s,\gamma)+C'\log\lambda^{-s})\qquad \textit{if } 0<\lambda\leqq\lambda_0,$$

where $C(s,\gamma)$ and C'_γ are the same constants as in (1.5) and (1.6)′, respectively. Then it follows from (1.5) that

$$(1.15)\qquad (\log\lambda^{-s})^2||Hv||\leqq 2(||H_0P_\gamma v||+C(s,\gamma)||(1-H)v||),\ v\in\mathcal{S}_y,\quad \textit{if } 0<\lambda\leqq\lambda_0.$$

Note that for $|\alpha|=1$

$$(1.16)\qquad \begin{cases}\sigma(P_\gamma^{(\alpha)})\equiv\partial_\eta^\alpha p_\gamma(\lambda y,\lambda\eta)=\lambda^{-1}\sigma((P^{(\alpha)})_\gamma) & \textit{if } |\lambda\eta|\leqq 3/5,\\ \sigma(P_{\gamma(\alpha)})\equiv D_y^\alpha p_\gamma(\lambda y,\lambda\eta)=\lambda\sigma((P_{(\alpha)})_\gamma).\end{cases}$$

Since supp $h(\lambda\eta)$ is contained in $\{\eta;\ |\lambda\eta|\leqq 3/5\}$, we see that for $|\alpha|=1$, $H(\lambda P_\gamma^{(\alpha)}-(P^{(\alpha)})_\gamma)$ is L^2-bounded uniformly with respect to $0<\lambda\leqq\lambda_0$. Together with (1.14) and (1.15), the estimate (1.6)′ gives

$$(1.17)\qquad \begin{aligned}\sum_{|\alpha|=1}||H\lambda P_\gamma^{(\alpha)}v||\leqq\ & C'_\gamma(\log\lambda^{-s})^{-1}(||HP_\gamma v||+4||H_0P_\gamma v||)\\ &+(2C(s,\gamma)+C'_\gamma)||(1-H)v||,\quad v\in\mathcal{S}_y,\qquad \textit{if } 0<\lambda\leqq\lambda_0.\end{aligned}$$

From (1.7)′ we also have

$$(1.18)\qquad \sum_{|\alpha|=1}||H\lambda P_{\gamma(\alpha)}v||\leqq G,\quad v\in\mathcal{S}_y\qquad \textit{if } 0<\lambda\leqq\lambda_0,$$

where G denotes the right hand side of (1.17). For a while we assume $0<\lambda\leqq\lambda_0(s,\gamma)$ for fixed $s>0$ and $\gamma\in R^n\times S^{n-1}$.

For a real $\kappa>0$ and an integer $k>0$ we denote by $\Lambda_{\kappa,k}$ a pseudodifferential operator with a symbol $(1+\kappa\langle\xi\rangle)^{-k}$. It is easy to check that for any α the estimate

$$(1.19)\qquad |\partial_\xi^\alpha((1+\kappa\langle\xi\rangle)^{-k})|\leqq C_\alpha\langle\xi\rangle^{-|\alpha|}(1+\kappa\langle\xi\rangle)^{-k}$$

holds with a constant C_α independent of κ. Set

$$(1.20)\qquad k_\kappa(\eta;\lambda)=(1+\kappa\langle\lambda^{-2}\xi_0+\lambda^{-1}\eta\rangle)^{-k}h(\lambda\eta).$$

Then it follows from (1.19) that for any α the estimate

$$(1.21)\qquad |\partial_\eta^\alpha k_\kappa(\lambda;\eta)|\leqq C'_\alpha\lambda^{|\alpha|}k_\kappa(\eta;\lambda),\quad \lambda|\eta|\leqq 1/5,$$

holds with another constant C'_α independent of κ and λ.

Set $h_\delta(x)=h(x/\delta)$ for a small $0<\delta\leqq 1/10$. Fix an integer $N\geqq 3$. Take a sequence $\{h_\delta^j\}_{j=1}^{N-1}\subset C_0^\infty(R_x^n)$ such that

$$h_\delta = h_\delta^1 \Subset h_\delta^2 \Subset \cdots \Subset h_\delta^{N-2} \Subset h_\delta^{N-1} = h_{4\delta/3} \Subset h_{2\delta} \tag{1.22}$$

and for any α the estimate

$$|D_x^\alpha h_\delta^j| \leqq C_\alpha'' N^{|\alpha|} \tag{1.23}$$

holds with a constant C_α'' independent of N and j ($C_0''=1$).

Lemma 1.2. *Write*

$$\begin{aligned} &h_\delta^j(\lambda D_y)k_\kappa(D_y;\lambda)h_\delta^j(\lambda y)h_\delta^{j+1}(\lambda D_y)h_\delta^{j+1}(\lambda y) \\ &\quad = h_\delta^j(\lambda D_y)k_\kappa(D_y;\lambda)h_\delta^j(\lambda y)+r(y, D_y;\lambda)\,. \end{aligned} \tag{1.24}$$

Then for any integer l there exists a constant C_l independent of λ and N such that

$$\|r(y, D_y;\lambda)v\| \leqq C_l\lambda^{2l}N^{2l+2n+2}\|v\|\,, \quad v\in\mathcal{S}_y\,. \tag{1.25}$$

Proof. Consider the expansion formula of the simplified symbol of $h_\delta^j(\lambda D_y)k_\kappa(D_y;\lambda)h_\delta^j(\lambda y)\,(1-h_\delta^{j+1}(\lambda D_y)h_\delta^{j+1}(\lambda y))$ (See Chapter 2 of [5]). Noting (1.21) and (1.23) we obtain (1.25) by means of the Calderón Vaillancourt theorem (See Chapter 7 of [5]). Q.E.D.

To make clear the discussion below we prove the following simple lemma.

Lemma 1.3. *Let N be a fixed positive integer and let λ satisfy $0<\lambda\leqq 1$. For any finite sequence of positive numbers $\{C_j\}_{j=1}^l$ there exists a constant C_l' such that*

$$\sum_{j=1}^l C_j(N\lambda)^{2j} \leqq 1+C_l'(N\lambda)^{2l}\,. \tag{1.26}$$

Proof. Set $R=\max\limits_{1\leqq j\leqq l}\{C_j, 1\}$. When $N\lambda\leqq 1/2R$ we have

$$\sum_{j=1}^l C_j(N\lambda)^{2j} \leqq R\sum_{j=1}^l (1/2R)^{2j} \leqq \sum_{j=1}^\infty (1/2)^{2j}<1\,.$$

If $N\lambda\geqq 1/2R$ then we have

$$\sum_{j=1}^l C_j(N\lambda)^{2j} \leqq R\sum_{j=1}^l (N\lambda)^{-2l+2j}(N\lambda)^{2l} \leqq \Big(R\sum_{j=1}^l (2R)^{2l-2j}\Big)(N\lambda)^{2l}\,.$$

It suffices to set $C_l'=R\sum_{j=1}^l(2R)^{2l-2j}$. Q.E.D.

Set $\tilde{H}_\delta^j=h_\delta^j(\lambda D_y)k_\kappa(D_y;\lambda)h_\delta^j(\lambda y)$ and substitute $\tilde{H}_\delta^j v$ into (1.15). Then for any $s>0$ there exists a constant C_s independent of κ, λ and N such that

$$(\log\lambda^{-s})^2\|\tilde{H}_\delta^j v\| \leqq 2\|P_\gamma \tilde{H}_\delta^j v\|+C_s\lambda^s N^{s+2n+4}\|v\|\,, \quad v\in\mathcal{S}_y\,. \tag{1.27}$$

Indeed, considering the expansion formulas of the simplified symbols of $(1-H)$ $\tilde{H}^j_\delta$ and $(1-H_0)P_\gamma\tilde{H}^j_\delta$ and using (1.23), (1.21) and (1.4), by Calderon-Vaillantcourt theorem we see that for any $s>0$ there exists a constant C_s such that

$$||(1-H)\tilde{H}^j_\delta v||+||(1-H_0)P_\gamma\tilde{H}^j_\delta v||\leqq C_s\lambda^s N^{s-2n+4}||v||\,,\quad v\in\mathcal{S}_y\,.$$

Similarly, it follows from (1.17) and (1.18) that for any $s>0$ estimates

$$(1.28)\qquad \sum_{|\alpha|=1}||\lambda P_{\gamma(\alpha)}\tilde{H}^j_\delta v||\leqq 5C'_\gamma(\log\lambda^{-s})^{-1}||P_\gamma\tilde{H}^j_\delta v||+\tilde{C}_s\lambda^s N^{s+2n+4}||v||$$

and

$$(1.29)\qquad \sum_{|\alpha|=1}||\lambda P_{\gamma(\alpha)}\tilde{H}^j_\delta v||\leqq 5C'_\gamma(\log\lambda^{-s})^{-1}||P_\gamma\tilde{H}^j_\delta v||+\tilde{C}_s\lambda^s N^{s+2n+4}||v||\,,\quad v\in\mathcal{S}_y$$

hold with a constant $\tilde{C}_s$.

Lemma 1.4. *There exists a constant M independent of λ, κ and N such that for any $s>0$*

$$(1.30)\qquad ||P_\gamma\tilde{H}^j_\delta v||\leqq M||\tilde{H}_{2\delta}P_\gamma v||+MN(\log\lambda^{-s})^{-1}||P_\gamma\tilde{H}^{j+1}_\delta v||+C_s\lambda^s N^{s+2n+6}||v||\,,$$
$$v\in\mathcal{S}_y\,,\qquad \text{if}\quad \log\lambda^{-s}\geqq MN\,,$$

where C_s is a constant independent of λ, κ and N. Hence $\tilde{H}_\delta=h_\delta(\lambda D_y)k_\kappa(D_y;\lambda)$ $h_\delta(\lambda y)$.

Proof. It is clear that

$$(1.31)\qquad ||P_\delta\tilde{H}^j_\delta v||\leqq||\tilde{H}^j_\delta P_\gamma v||+||[P_\gamma,\tilde{H}^j_\delta]v||\,.$$

Noting $h^j_\delta(x)=h^j_\delta(x)h_{2\delta}(x)$ and considering the expansion formula of the simplified symbol of $\tilde{H}^j_\delta$, we have

$$||\tilde{H}^j_\delta P_\gamma v||=||\tilde{H}^j_\delta h_{2\delta}(\lambda y)P_\gamma v||\leqq(1+\sum_{q=1}^{[s/2]+1}C'_q(N\lambda)^{2q})||h_{2\delta}(\lambda D_y)k_\kappa h_{2\delta}(\lambda y)P_\gamma v||$$
$$+C_s\lambda^s N^{s+2n+6}||v||$$

for some constant C'_q and C_s. Here we used the estimate

$$||(h^j_\delta(\lambda D_y)k_\kappa)^{(\alpha)}v||\leqq C_\alpha(N\lambda^2)^{|\alpha|}||h_{2\delta}(\lambda D_y)k_\kappa v||\,,$$

which follows from (1.21), (1.23) and the fact that $h_{2\delta}(x)=1$ on supp $D^\alpha_x h^j_\delta(x)$. Using Lemma 1.3 we have

$$||\tilde{H}^j_\delta P_\gamma v||\leqq 2||\tilde{H}_{2\delta}P_\gamma v||+C_s\lambda^s N^{s+2n+6}||v||\,.$$

Here and throughout the proof of the lemma we denote by the same notation C_s different constants independent of λ, κ and N(, depending on s). We shall estimate the second term of the right hand side of (1.31). In view of Lemma 1.2 it suffices to estimate $||[P_\gamma,\tilde{H}^j_\delta]H^{j+1}_\delta v||$, where $H^{j+1}_\delta=h^{j+1}_\delta(\lambda D_y)h^{j+1}_\delta$ (λy). Write

$$[P_\gamma, \tilde{H}_\delta^j] = [P_\gamma, h_\delta^j(\lambda D_y)k_\kappa]h_\delta^j(\lambda y) + k_\kappa h_\delta^j(\lambda D_y)\,[P_\gamma, h_\delta^j(\lambda y)]\,.$$

Note that the expansion formula

$$[P_\gamma, h_\delta^j(\lambda D_y)k_\kappa] = \sum_{0<|\alpha|\leqq[s/2]+1}(-1)^{|\alpha|}(h_\delta^j(\lambda D_y)k_\kappa)^{(\alpha)}P_{\gamma(\alpha)}/\alpha! + R(y, D_y; \lambda)\,,$$

where R is a negligible operator, in the sense of

$$||Rv|| \leqq C_s\lambda^s N^{s+2n+6}||v||\,.$$

In view of (1.4), (1.21) and (1.23), we see that there exist constants M_1 and M_2 independent of s, κ, λ and N such that

$$\begin{aligned} ||[P_\gamma, h_\delta^j(\lambda D_y)k_\kappa]h_\delta^j(\lambda y)H_\delta^{j+1}v|| &\leqq M_1 N\sum_{|\alpha|=1}||k_\kappa\lambda P_{\gamma(\alpha)}h_\delta^j(\lambda y)H_\delta^{j+1}v|| \\ &+ M_2N^2\Big(1+\sum_{q=1}^{[s/2]+1}C_q''N^q\lambda^{2q}\Big)||k_\kappa h_\delta^j(\lambda y)H_\delta^{j+1}v|| + C_s\lambda^sN^{s+2n+6}||v|| \end{aligned} \tag{1.32}$$

holds with some constants C_q''. Consider the expansion formula of the simplified symbol of $k_\kappa h_\delta^j(\lambda y)$ and use Lemma 1.3. Then the second term of the right hand side of (1.32) is estimated above by

$$2M_2N^2||\tilde{H}_\delta^{j+1}v|| + C_s\lambda^sN^{s+2n+6}||v||\,. \tag{1.33}$$

For $|\alpha|=1$ the estimate

$$\begin{aligned} &||k_\kappa\lambda P_{\gamma(\alpha)}h_\delta^j(\lambda y)H_\delta^{j+1}v|| \\ &\quad\leqq ||\lambda P_{\gamma(\alpha)}k_\kappa H_\delta^{j+1}v|| + M_3N||\tilde{H}_\delta^{j+1}v|| + C_s\lambda^sN^{s+2n+6}||v|| \end{aligned}$$

holds with a constant M_3 independent of s, κ, λ and N. Here we used Lemma 1.3 to estimate terms corresponding to $[k_\kappa\lambda P_{\gamma(\alpha)}, h_\delta^j(\lambda y)]$ and $[\lambda P_{\gamma(\alpha)}, k_\kappa]$. From (1.29) we obtain

$$\begin{aligned} ||k_\kappa\lambda P_{\gamma(\alpha)}h_\delta^j(\lambda y)H_\delta^{j+1}v|| &\leqq 2C_\gamma'(\log\lambda^{-s})^{-1}||P_\gamma\tilde{H}_\delta^{j+1}v|| + M_3N||\tilde{H}_\delta^{j+1}v|| \\ &+ C_s\lambda^sN^{s+2n+6}||v||\,, \quad |\alpha| = 1 \end{aligned} \tag{1.34}$$

From (1.32)–(1.34) we see that the estimate

$$\begin{aligned} &||[P_\gamma, h_\delta^j(\lambda D_y)k_\kappa]h_\delta^j(\lambda y)H_\delta^{j+1}v|| \\ &\quad\leqq (M_4N)^2||\tilde{H}_\delta^{j+1}v|| + M_4N(\log\lambda^{-s})^{-1}||P_\delta\tilde{H}_\delta^{j+1}v|| + C_s\lambda^sN^{s+2n+6}||v|| \end{aligned}$$

holds with a suitable constant M_4 larger than C_γ' and M_j ($j=1, 2, 3$). If we use (1.27) with j replaced by $j+1$ to estimate the first term of the right hand side, we can estimate $||[P_\gamma, h_\delta^j(\lambda D_y)k_\kappa]h_\delta^j(\lambda D_y)H_\delta^{j+1}v||$ by the right hand side of (1.30) with another suitable M larger than M_4, because $(MN)^2/(\log\lambda^{-s})^2 \leqq MN/\log\lambda^{-s}$ if $\log\lambda^{-s} \geqq MN$. Noting $||k_\kappa h_\delta^j(\lambda D_y)\tilde{v}|| \leqq ||k_\kappa h_{2\delta}(\lambda D_y)\tilde{v}||$ for $\tilde{v}\in\mathcal{S}_y$, we can also estimate the term $||k_\kappa h_\delta^j(\lambda D_y)\,[P_\gamma, h_\delta^j(\lambda y)]H_\delta^{j+1}v||$ by using (1.28) instead of (1.29). We have estimated the second term of the right hand side of (1.31). So we obtain the desired estimate. Q.E.D.

From (1.27) and (1.30) we have

Lemma 1.5. *For any integer $N\geqq 3$ there exists a constant M independent of N, λ and κ such that for any $s>0$*

$$(1.35)\quad (\log\lambda^{-s})^N\|\tilde{H}_\delta v\|\leqq(\log\lambda^{-s})^N\|\tilde{H}_{2\delta}P_\gamma v\|+(MN)^N\|v\|+C_s N! N^{s+2n+6}\|v\|, \quad v\in\mathcal{S}_y, \quad \textit{if } 0<\lambda\leqq\lambda_0(s,\gamma),$$

where C_s is a constant independent of κ, λ and N.

Proof. In view of $\tilde{H}_\delta=\tilde{H}_\delta^1$ it follows from (1.27) that

$$(\log\lambda^{-s})^N\|\tilde{H}_\delta v\|/2\leqq(\log\lambda^{-s})^{N-2}(\|P_\delta\tilde{H}_\delta^1 v\|+C_s\lambda^s N^{s+2n+4}\|v\|).$$

Applying (1.30) to the first term of the right hand side. Then we have

$$(\log\lambda^{-s})^N\|\tilde{H}_\delta v\|/2\leqq M(\log\lambda^{-s})^{N-2}\|\tilde{H}_{2\delta}P_\gamma v\|+MN(\log\lambda^{-s})^{N-3}\|P_\gamma\tilde{H}_\delta^2 v\| +2C_s(\log\lambda^{-s})^{N-2}\lambda^s N^{s+2n+6}\|v\| \qquad \textit{if } \log\lambda^{-s}\geqq MN.$$

Use (1.30) for the second term of the right hand side and use repeatedly (N–3) times. Then we obtain

$$(1.36)\quad \begin{aligned}(\log\lambda^{-s})^N\|\tilde{H}_\delta v\|/2 &\leqq M\sum_{j=0}^{N-3}(\log{}^{-s})^{N-j-2}(MN)^j\|\tilde{H}_{2\delta}P_\gamma v\| \\ &+(MN)^{N-2}\|P_\gamma\tilde{H}_\delta^{N-1}v\|+(\log\lambda^{-s})^{N-2}\Big(1+\sum_{j=0}^{N-2}(\log\lambda^{-s})^{-j}(MN)^j\Big) \\ &\times C_s\lambda^s N^{s+2n+6}\|v\|, \qquad \textit{if } \log\lambda^{-s}\geqq MN.\end{aligned}$$

Note that $\tilde{H}_\delta^{N-1}=\tilde{H}_{4\delta/3}$ and $h_{4\delta/3}\Subset h_{2\delta}$. By means of similar formulas as (1.6) and (1.7) (together with (1.16) we have

$$\|P_\gamma\tilde{H}_\delta^{N-1}v\|\leqq M(\|\tilde{H}_{2\delta}P_\gamma v\|+\|v\|),$$

taking another larger M if necessary. If $\log\lambda^{-s}\geqq MN$, it follows from (1.36) that

$$(\log\lambda^{-s})^N\|\tilde{H}_\delta v\|\leqq(\log\lambda^{-s})^N\|\tilde{H}_{2\delta}P_\gamma v\|+(MN)^N\|v\| +C_s(\log\lambda^{-s})^N\lambda^s N^{s+2n+6}\|v\|.$$

When $\log\lambda^{-s}\leqq MN$ this estimate still holds because of the second term of the right hand side. Noting $(\log\lambda^{-s})^N\lambda^s=(\log\lambda^{-s})^N\exp(-\log\lambda^{-s})\leqq N!$, we obtain (1.35). Q.E.D.

The estimate (1.35) with $N\leqq 2$ also follows from (1.27) with $j=1$ because for a suitable constant M' we have

$$\|P_\gamma\tilde{H}_\delta^1 v\|\leqq M'(\|\tilde{H}_{2\delta}P_\gamma v\|+\|v\|)$$

similarly as the estimate after (1.36). Thus, (1.35) holds for any $N=0,1,2,\cdots$.

Let τ be a small parameter chosen later on. Multiply both sides of (1.35) by $\tau^N/N!$ and sum up with respect to $N=0, 1, 2, \cdots$. Then we obtain

$$\lambda^{-s\tau}\|\tilde{H}_\delta v\|\leqq\lambda^{-s\tau}\|\tilde{H}_{2\delta}P_\gamma v\|+\Big(\sum_{N=0}^{\infty}(MN\tau)^N/N!+C_s\sum_{N=0}^{\infty}\tau^N N^{s+2n+6}\Big)\|v\|$$

because $\sum(\tau\log\lambda^{-s})^N/N!=\lambda^{-s\tau}$. Choose τ such that $Me\,\tau<1$ and $0<\tau<1$. Then, by using Stirling formula $N^N/N!\leqq e^N$ we have

$$\lambda^{-s\tau}\|\tilde{H}_\delta v\|\leqq\lambda^{-s\tau}\|\tilde{H}_{2\delta}P_\gamma v\|+C'_s\|v\|\,,\quad v\in\mathcal{S}_y\,, \tag{1.37}$$

for another constant C'_s. Note that τ is independent of s because M is so. Hence we can replace $s\tau$ in (1.37) by $2s'+2s''$ for any real $s', s''>0$. Multiply $\lambda^{2s''}$ by (1.37) with $s\tau$ replaced by $2s'+2s''$. Then we see that there exists a constant $C_0=C_0(s', s'', \gamma)$ independent of κ and λ such that

$$\begin{aligned}\lambda^{-2s'}\|\tilde{H}_\delta v\|&\leqq\lambda^{-2s'}\|\tilde{H}_{2\delta}P_\gamma v\|+C_0\lambda^{2s''}\|v\|\,,\\ &v\in\mathcal{S}_y \qquad \textit{if } \lambda>0 \textit{ is sufficiently small}\,.\end{aligned} \tag{1.38}$$

Taking another large C_0 if necessary, we may assume that (1.38) holds for $0<\lambda\leqq1$. Note that for any $\xi_0\in S^{n-1}$, any $0<\delta'\leqq1$ and any real $\tilde{s}$ the estimate

$$C^{-1}\|h_{\delta'}(\lambda D_y)v\|\leqq\||\xi_0+\lambda D_y|^{\tilde{s}}\|h_{\delta'}(\lambda D_y)v\|\leqq C\|h_{\delta'}(\lambda D_y)v\|\,,\quad v\in\mathcal{S}_y$$

holds for some constant $C=C_{\tilde{s},\delta'}$ because

$$C^{-1}\leqq|\xi_0+\xi|^{\tilde{s}}\leqq C \textit{ on } \operatorname{supp} h_{\delta'}(\xi)\,.$$

Substitute $v(y)=h(\lambda D_y)\tilde{v}(y)$ into (1.38) for $\tilde{v}(y)=\exp(-i\lambda^{-2}x\cdot\xi_0)w(x)|_{x=\lambda y+x_0}$, $w\in\mathcal{S}_x$. Then in view of (1.8), (1.20), (1.9) and the above estimate, we see that there exists a constant C'_0 such that

$$\begin{aligned}&\|h_\delta(\lambda^2D_x-\xi_0)|D_x|^{s'}\Lambda_{\kappa,k}h_\delta(x-x_0)w\|^2\\ &\leqq C'_0(\|h_{2\delta}(\lambda^2D_x-\xi_0)|D_x|^{s'}\Lambda_{\kappa,k}h_{2\delta}(x-x_0)P(x, D_x)w\|^2\\ &\quad+\|h(\lambda^2D_x-\xi_0)|D_x|^{-s''}w\|^2+\lambda\|w\|^2_{-s''}),\ w\in\mathcal{S}_x\,,\quad \textit{if } 0<\lambda\leqq1\,.\end{aligned} \tag{1.39}$$

Here we used the fact that

$$\{\lambda^{-1/2}h_\delta(\lambda^2D_x-\xi_0)h_\delta(x-x_0)\,(1-h(\lambda^2D_x-\xi_0));\ 0<\lambda\leqq1\}$$
$$\{\lambda^{-1/2}h_{2\delta}(\lambda^2D_x-\xi_0)h_{2\delta}(x-x_0)P(x, D_x)\,(1-h((\lambda^2D_x-\xi_0)/3));\ 0<\lambda\leqq1\}$$

are contained in a bounded set of $S^{-s''}_{1,0}$.

To complete the proof of Theorem 1 we prepare the followings:

Definition 1.6. For $\delta>0$ and $\xi_0\in S^{n-1}$ we say that a function $\psi(\xi)\in C^\infty(R^n)$ belongs to Ψ_{δ,ξ_0} if ψ satisfies

$$\begin{cases} 0 \leqq \psi \leqq 1,\ \psi(\xi) = 1 & \text{for } |\xi/|\xi| - \bar{\xi}_0| \leqq \delta/12 \text{ and } |\xi| \geqq 1/3\,, \\ \psi(\xi) = 0 & \text{for } |\xi/|\xi| - \bar{\xi}_0| \geqq \delta/10 \text{ or } |\xi| \leqq 1/2\,, \\ \psi(t\xi) = \psi(\xi) & \text{for } t \geqq 1 \text{ and } \xi \in S^{n-1}\,. \end{cases}$$

Proposition 1.7 (cf. Proposition 2.2 of [9]). *Let $\bar{\xi}_0 \in S^{n-1}$ and let $h(x) \in C_0^\infty$ be a function defined at the beginning of this section. Set $h_\delta(x) = h(x/\delta)$ for a $\delta > 0$. If ψ_δ and $\tilde{\psi}_\delta$ belong to $\Psi_{\delta,\bar{\xi}_0}$ and $\Psi_{7\delta,\bar{\xi}}$, respectively, then for any $s > 0$ there exists a constant $C_s > 0$ such that*

$$\begin{aligned} C_s^{-1}\|\psi_\delta(D_x)u\|^2 &\leqq \int_0^1 \|h_\delta(\lambda^2 D_x - \bar{\xi}_0)u\|^2/\lambda d\lambda + \|u\|_{-s}^2 \\ &\leqq C_s(\|\tilde{\psi}_\delta(D_x)u\|^2 + \|u\|_{-s}^2)\,, \qquad u \in \mathcal{S}_x \end{aligned} \tag{1.40}$$

Proof. Set $r = |\xi|$, $\theta = \xi/|\xi|$. Then

$$\int_0^1 \|h_\delta(\lambda^2 D_x - \bar{\xi}_0)u\|^2/\lambda d\lambda = \int_{S^{n-1}} d\theta \int_0^1 d\lambda \int_0^\infty h_\delta(\lambda^2 r\theta - \bar{\xi}_0)^2 |\hat{u}(r\theta)|^2/\lambda (r^{n-1}dr)\,.$$

It is easy to see that $h_\delta(\lambda^2 r\theta - \bar{\xi}_0) = 1$ on

$$\{(\theta, r, \lambda) \in S^{n-1} \times R_+ \times [0, 1];\ \theta \in \operatorname{supp} \psi_\delta \quad \textit{and} \quad |\lambda^2 r - 1| \leqq \delta/10\}\,.$$

Therefore the integral is estimated below by

$$\int_{S^{n-1}} \psi_\delta(\theta)^2 d\theta \int_{1-\delta/10}^\infty |\hat{u}(r\theta)|^2 r^{n-1} dr \int_{\sqrt{(1-\delta/10)/r}}^{\sqrt{(1+\delta/10)/r}} d\lambda/\lambda\,.$$

This give the first inequality of (1.40). Another inequality easily follows if we note that supp $h_\delta(\lambda^2 r\theta - \bar{\xi}_0)$ is contained in

$$\{(\theta, r, \lambda);\ \tilde{\psi}_\delta(\theta) = 1 \quad \textit{and} \quad |\lambda^2 r - 1| \leqq 7\delta/24\}\,. \qquad \text{Q.E.D.}$$

Apply Proposition 1.7 to (1.39). Then we see that for any $\gamma = (x_0, \bar{\xi}_0) \in R^n \times S^{n-1}$, any $s', s'' > 0$, any integer $k > 0$ and any $\kappa > 0$ there exists a constant $C'' = C''(\gamma, s', s'', k)$ independent of κ such that

$$\begin{aligned} &\|\psi_\delta(D_x)\Lambda_{\kappa,k} h_\delta(x - x_0)w\|_{s'}^2 \\ &\leqq C''(\|\tilde{\psi}_\delta(D_x)\Lambda_{\kappa,k} h_{2\delta}(x - x_0)P(x, D_x)w\|_{s'}^2 + \|w\|_{-s''}^2)\,, \quad w \in \mathcal{S}_x \end{aligned} \tag{1.41}$$

if $\psi_\delta(\xi) \in \Psi_{\delta,\bar{\xi}_0}$ and $\tilde{\psi}_\delta(\xi) \in \Psi_{14\delta,\bar{\xi}_0}$.

From now on we shall prove (4). Let $(x_0, \xi_0) \in T^*R^n \backslash 0$ and let $u \in \mathcal{D}'(R^n)$. Set $\bar{\xi}_0 = \xi_0/|\xi_0|$. Suppose that $(x_0, \xi_0) \notin \mathrm{WF}\ Pu$. Then there exists a $\delta > 0$ such that $\tilde{\psi}_\delta(D_x)h_{2\delta}(x - x_0)Pu \in H_{s'}$ for any real $s' > 0$ if $\tilde{\psi}_\delta(\xi) \in \Psi_{14\delta,\xi_0}$. Since $h_{4\delta}(x - x_0)u \in \mathcal{E}'$ we have $h_{4\delta}(x - x_0)u \in H_{-s''}$ for some $s'' > 0$. Choose $k > 0$ in (1.41) such that $k \geqq s' + s'' + 2$. Then, by taking a sequence $\{w_j\}_{j=1}^\infty \subset \mathcal{S}_x$ such that

$$w_j \to h_{4\delta}(x - x_0)u \text{ in } H_{-s''}\,,$$

from (1.42) we see that

$$
(1.42)\qquad \begin{aligned} &\|\Lambda_{\kappa,k}\psi_\delta(D_x)h_\delta(x-x_0)u\|_{s'}^2 \\ &\qquad \leqq C''(\|\tilde\psi_\delta(D_x)h_{2\delta}(x-x_0)Pu\|_{s'}^2+\|h_{4\delta}(x-x_0)u\|_{-s''}^2)\,, \end{aligned}
$$

if $\psi_\delta(\xi)\in\Psi_{\delta,\xi_0}$ and $\tilde\psi_\delta(\xi)\in\Psi_{14\delta,\xi_0}$. Here we used the fact that $\|\Lambda_{\kappa,k}w\|\leqq\|w\|$ for $w\in L^2$. Letting κ tend to 0 in (1.42), we have $\psi_\delta(D_x)h_\delta(x-x_0)u\in H_{s'}$. Since s' is arbitrary, we have $(x_0,\xi_0)\notin \mathrm{WF}\,u$. Now the proof of Theorem 1 has been completed.

2. Proof of Theorem 3

As stated in Introduction, the method used here is only a version of the one in [7]. Let P_0 be the differential operator in Introduction, that is,

$$
(2.1)\qquad P_0=D_t^m+\mathcal{A}(x,D_x)\quad in\ R_t\times R_x^n\,,
$$

where $\mathcal{A}(x,D_x)$ is formally self-adjoint in an open set Ω of R^n and bounded below. Following [7], for $s\geqq1$ we introduce $G^s(\bar\Omega;\mathcal{A})$ the space of $u\in L^2(\Omega)$ such that $\mathcal{A}^ku\in L^2(\Omega)$ for $k=1,2,\cdots$ and moreover there exists a constant M satisfying

$$
(2.2)\qquad \|\mathcal{A}^ku\|_{L_2(\Omega)}\leqq M^{k+1}(k!)^{sm},\ k=1,2,\cdots.
$$

We also introduce the space $G^s(\Omega;\mathcal{A})$ of $u\in L^2_{\mathrm{loc}}(\Omega)$ whose restriction in any $\Omega_1\Subset\Omega$ is in $G^s(\bar\Omega_1;\mathcal{A})$.

Proposition 2.1. *Assume that $G^1(\Omega;\mathcal{A})\not\subset C^\infty(\Omega)$. Then P_0 is not hypoelliptic in $R_t\times\Omega$* (cf. Corollaries 3.6–3.7 of [7] and see also [1]).

Proof. There exists a $u_0\in G^1(\Omega;\mathcal{A})$ such that $u_0\notin C^\infty(\Omega)$. The series

$$
u(t,x)=\sum_{k=0}^\infty(it)^{mk}(-\mathcal{A})^ku_0(x)/(mk)!
$$

is strongly convergent in $L^2(\tilde\Omega)$ for some $\tilde\Omega=I_\delta\times\Omega_1$, where $I_\delta=(-\delta,\delta)\subset R_t$ and $\Omega_1\Subset\Omega$. We have $P_0u=0$ and u is not C^∞ in $\tilde\Omega$ because $u_0=u(0,\cdot)$ is not C^∞ in Ω. Q.E.D.

Note that for any open set $\omega\Subset\Omega$

$$
\mathrm{Re}(P_0u,u)=\|D_t^{m/2}u\|^2+(\mathcal{A}u,u),\ u\in C_0^\infty(R_t\times\omega)\,.
$$

For the proof of Theorem 3 it suffices to show:

Proposition 2.2. *Assume that $G^1(\Omega;\mathcal{A})\subset C^\infty(\Omega)$. Then for any $x_0\in\Omega$ there exists a neighborhood of ω of x_0 such that for any $\varepsilon>0$ the estimate*

$$
(2.3)\qquad \|(\log\langle D_x\rangle)^{m/2}u\|^2\leqq\varepsilon(\mathcal{A}u,u)+C_\varepsilon\|u\|^2,\ u\in C_0^\infty(\omega)
$$

holds with a constant C_ε. (cf. Theorem 3.5 of [7]).

In the proof of Proposition 2.2 we may replace $\mathcal{A}$ by $\mathcal{A}+\mu$ for any real μ because $G^1(\Omega; \mathcal{A})=G^1(\Omega; \mathcal{A}+\mu)$. Taking a large $\mu>0$, in view of (6) we may assume that $(\mathcal{A}u, u)>0$ for $u\in L^2(\Omega)$ satisfying $\mathcal{A}u\in L^2(\Omega)$. Therefore, we have the Friedrichs extension $(A, D(A))$ in $L^2(\Omega)$ of $\mathcal{A}(x, D_x)$, as a positive self-adjoint realization.

For the proof of (2.3) it suffices to show that for any $\varepsilon>0$ and any $r>0$ there exists a $C_{\varepsilon,r}$ such that

$$\|(\log\langle D_x\rangle)^{mr}u\|^2\leqq\varepsilon\|A^r u\|^2+C_{\varepsilon,r}|u|^2, \quad u\in C_0^\infty(\omega)\,. \tag{2.4}$$

In fact, the estimate (2.3) follows immediately from (2.4) with $r=1/2$. From now on we shall prove (2.4). We may assume that x_0 is the origin. We use the same notation as in [7] p. 840–849. Let $\psi\in C_0^\infty(\Omega)$ equal 1 in $\Pi=((-a, a))^n\Subset\Omega$. The hypothesis of the proposition implies that $u\in D_\delta^1(A)\Rightarrow\psi u\in\mathcal{S}$ for a fixed $\delta>0$ because $D^1(A)\equiv\bigcup_{\delta>0}D_\delta^1(A)\subset G^1(\Omega; \mathcal{A})$. The Banach closed graph theorem shows that for any integer $k>0$ there exists a constant M_k such that

$$\sup_\xi |\langle\xi\rangle^{2k}\widehat{\psi u}(\xi)|\leqq M_k(N_\delta^1(u))^{1/2}, \quad u\in D_\delta^1(A)\,. \tag{2.5}$$

In view of (3.4) of [7], it is clear that for any k there exists a constant $M_k'(\geqq1)$ such that

$$J_k^L(u)\leqq e^{2k}\|(L+1)^k u\|_{L^2(\Pi)}^2\leqq M_k'\|\langle\xi\rangle^{2k}\widehat{\psi u}\|^2\,, \tag{2.6}$$

where $J_k^L(u)$ denotes $J_k(u)$ defined from the spectrum resolution of L. Here $(L, D(L))$ is the realization of Legendre operator

$$\mathcal{L}=\sum_{j=1}^n\partial_{x_j}(x_j^2-a^2)\partial_{x_j}$$

defined in [5] p. 845. In what follows, to make clear the correspondance we often use the superscript A or L such as $J_k^A(u)$, $J_k^L(u)$. Set $K_k=\{\xi;\ \langle\xi\rangle\geqq M_k'M_{k+2}$. Then from (2.5) and (2.6) we have

$$\begin{aligned}J_k^L(u)&\leqq\|(M_k'M_{k+2}/\langle\xi\rangle)M_{k+2}^{-1}\langle\xi\rangle^{2k+2}\widehat{\psi u}\langle\xi\rangle^{-1}\|_{L^2(K_k)}^2+M_k'\|\langle\xi\rangle^{2k}\widehat{\psi u}\|_{L^2(R_\xi^n\setminus K_k)}^2\\&\leqq N_\delta^1(u)+C_k\|u\|_{L^2(\Omega)}^2, \quad u\in D_\delta^1(A)\,,\end{aligned} \tag{2.7}$$

with a constant C_k. Set $u(t)=F^A(t)u$. Then the estimate (2.7) and Lemma 3.1 of [7] show that for any $r>0$ and $k>0$

$$\begin{aligned}I_{r,k}(u(\cdot))&\equiv\int_1^\infty\{\exp(-\delta(et)^{1/m})J_k^L(u(t))+\|u(t)\|_{L^2(\Pi)}^2\}t^{2r}\frac{dt}{t}\\&\leqq2J_r^A(u)+C_k'\|u\|_{L^2(\Omega)}^2, \quad u\in D(A^r)\end{aligned} \tag{2.8}$$

holds with a constant C_k'. We need replace Lemma 3.2 of [7] by

Lemma 2.3. *Let $t \to u(t)$ be a measurable mapping from $[1, \infty)$ to $L^2(\Pi)$ and let $I_{r,k}(u(\cdot))$ denote the integral defined by the formula (2.8). Assume that for reals $\delta>0$, $r>0$ and an integer $k>0$ the integral $I_{r,k}(u(\cdot))$ is bounded. Then the integral $u=\int_1^\infty u(t)\frac{dt}{t}$ is convergent, $u \in D((\log(L+1))^{mr})$ and for a constant C independent of k we have*

$$k^{2mr}\|(\log(L+1))^{mr}u\|^2_{L^2(\Pi)} \leqq CI_{r,k}(u(\cdot)) . \tag{2.9}$$

The proof of the lemma is parallel to the one of Lemma 3.2 of [7] if we set $\sigma(t, \lambda)=\exp(2k \log \lambda - \delta e^{1/m} t^{1/m})$ and $t(\lambda)=e^{-1}((k/\delta)\log \lambda)^m$. We remark that the estimate

$$\|(\log(L+1))^r u\|^2_{L^2(\Pi)} \leqq \int_1^\infty (\log \lambda)^{2r} \|F^L(\lambda)u\|^2_{L^2(\Pi)} \frac{d\lambda}{\lambda}$$

holds similarly to (3.4) of [7]. The detail is omitted.

Set $\omega=((-a/2, a/2))^n$. Then for the proof of Proposition 2.2 it remains to show

$$\|(\log\langle D_x\rangle)^{mr}u\|^2 \leqq C(\|(\log(L+1))^{mr}u\|^2+\|u\|^2), \quad u \in C_0^\infty(\omega) . \tag{2.10}$$

In fact, from (2.8)–(2.10) we have (2.4) since we can take any large k.

From now on we shall prove (2.10). Let $\{\lambda_j;\ 0<\lambda_1<\lambda_2<\cdots\}$ be the set of eigenvalues of $(L, D(L))$ and let $P_j(x)$ be the normalized eigenfunction (Legendre polynomial) associated with λ_j. Then, for $u \in C_0^\infty(\omega)$, $(\log(L+1))^{mr}u$ is defined by

$$(\log(L+1))^{mr}u = \sum_{j=1}^\infty (\log(\lambda_j+1))^{mr}(P_j, u)_{L^2(\Pi)}P_j(x). \tag{2.11}$$

Here we remark that $v \in C^\infty(\Pi)$ belongs to $D^\infty(L) \equiv \bigcap_{j=1}^\infty D(L^j)$ and hence to $D((\log(L+1))^{mr})$ because $\sum (\log(\lambda_j+1))^{2mr}|(P_j, v)|^2 \leqq C \sum \lambda_j^2 |(P_j, v)|^2$. Note that $(\log(L+1))^{mr}=(L+1)(L+1)^{-1}(\log(L+1))^{mr}$ and let Γ be a contour of Figure 1:

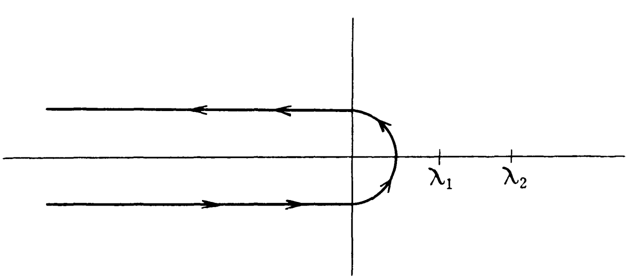


Fig. 1.

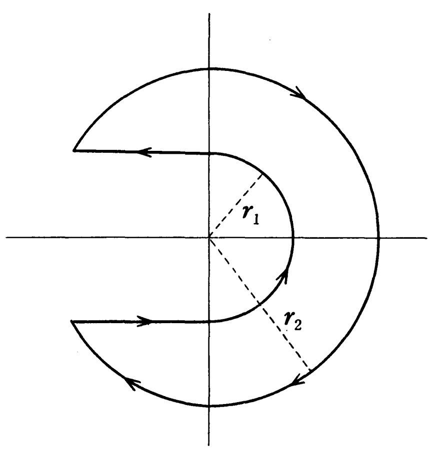
Since the similar formulas to (2.11) hold for $(L+1)^{-1}(\log(L+1))^{mr}$ and $(L-\zeta)^{-1}$, by the residue caculus we have

$$(2.12)\qquad \begin{aligned}&(L+1)^{-1}(\log(L+1))^{mr}\,u\\ &=\frac{1}{2\pi i}\int_{\Gamma}(\zeta+1)^{-1}(\log(\zeta+1))^{mr}(L-\zeta)^{-1}\,ud\zeta\,,\quad u\in C_0^\infty(\omega)\,.\end{aligned}$$

We shall approximate $(L-\zeta)^{-1}$ by a pseudodifferential operator by using the argument in Chapter 8 of [5]. Let $\tilde{\mathcal{L}}$ be a second order differential operator with real valued $\mathcal{B}^\infty$-coefficients such that $\tilde{\mathcal{L}}=\mathcal{L}$ in a neighborhood of $\bar{\omega}$. We may assume that the symbol $\tilde{l}(x,\xi)$ of $\tilde{\mathcal{L}}$ satisfies $C_0^{-1}\langle\xi\rangle^2<\tilde{l}(x,\xi)<C_0\langle\xi\rangle^2$ for large $|\xi|$. Then we have

$$(2.13)\qquad \begin{aligned}&|\tilde{l}_{(\beta)}^{(\alpha)}(x,\xi)\,(\tilde{l}(x,\xi)-\zeta)^{-1}|\leqq C_{\alpha\beta}\langle\xi\rangle^{-|\alpha|}\\ &\qquad\qquad \textit{for}\ \text{large}\ |\xi|\quad \textit{and}\quad \zeta\in\Omega_\xi^C\equiv\boldsymbol{C}\backslash\Omega_\xi\\ &\qquad\qquad\qquad \text{(cf. (1.4) of Chapter 8 of [5])}\,.\end{aligned}$$

Here Ω_ξ denotse the interior of clockwise-oriented Jordan curve Γ_ξ^0 that is defined as in Figure 2:



$r_1=c_0^{-1}\langle\xi\rangle^2$,
$r_2=c_0\langle\xi\rangle^2$.

Fig. 2.

By means of Lemma 2.2 of Chapter 8 of [5] we have a parametrix $Q(\zeta)=Q(x,D_x;\zeta)$ of $\tilde{\mathcal{L}}-\zeta$ such that

$$(\tilde{\mathcal{L}}-\zeta)\,Q(\zeta)=I+R(x,D_x;\zeta)\,,$$

where symbols $q(\zeta)=q(x,\xi;\zeta)$ and $r(\zeta)=r(x,\xi;\zeta)$ are analytic with respect to $\zeta\in\Omega_\xi^C$ and satisfy for large $|\xi|$ and $\zeta\in\Omega_\xi^C$

$$(2.14)\qquad q(\zeta)=(\tilde{l}(x,\xi)-\zeta)^{-1}(1+\tilde{q}(x,\xi;\zeta))\,,$$

$$(2.15)\qquad \begin{cases} |q^{(\alpha)}_{(\beta)}(\zeta)| \leqq C_{\alpha\beta}(\langle\xi\rangle^2+|\zeta|)^{-1}\langle\xi\rangle^{-|\alpha|}, \\ |\tilde{q}^{(\alpha)}_{(\beta)}(\zeta)| \leqq C'_{\alpha\beta}\langle\xi\rangle^{-|\alpha|}, \end{cases}$$

$$(2.16)\qquad |r^{(\alpha)}_{(\beta)}(\zeta)| \leqq C_{\alpha\beta,N}(\langle\xi\rangle^2+|\zeta|)^{-1}\langle\xi\rangle^{-N}$$

for any $N>0$. In Π we have

$$(2.17)\qquad \begin{aligned}(L-\zeta)\,((L-\zeta)^{-1}u-Q(\zeta)\,u) &= u-(\mathcal{L}-\zeta)\,Q(\zeta)\,u \\ &= (\tilde{\mathcal{L}}-\mathcal{L})\,Q(\zeta)\,u-R(\zeta)\,u \equiv \tilde{R}(\zeta)\,u\,,\quad u\in C_0^\infty(\omega)\,,\end{aligned}$$

where the symbol of $\tilde{R}(\zeta)$ satisfies the inequality similar to (2.16). In fact, for $\phi(x)\in C_0^\infty(\Pi)$ such that $\phi=1$ in a neighborhood of ϖ, we see that $(\tilde{\mathcal{L}}-\mathcal{L})\,Q(\zeta)\phi$ is a regularizer in the sense of (2.16). It follows from (2.17) that

$$(L-\zeta)^{-1}u = Q(\zeta)\,u+(L-\zeta)^{-1}\,\tilde{R}(\zeta)u \quad in\ \Pi\,.$$

From this and (2.12) we have

$$(2.18)\qquad \begin{aligned}(\log(L+1))^{mr}\,u &= (L+1)\,\frac{1}{2\pi i}\int_\Gamma(\zeta+1)^{-1}(\log(\zeta+1))^{mr}\,Q(\zeta)u\,d\zeta \\ &\quad +\frac{1}{2\pi i}\int_\Gamma(\log(\zeta+1))^{mr}(L-\zeta)^{-1}\,\tilde{R}(\zeta)u\,d\zeta \quad in\ \Pi\,.\end{aligned}$$

Since it follows from (2.16) for $\sigma(\tilde{R}(\zeta))$ that

$$\|(L-\zeta)^{-1}\,\tilde{R}(\zeta)u\|_{L^2(\Pi)} \leqq C|\zeta|^{-2}\|u\|\,,$$

the $L^2(\Pi)$ norm of the second term of the right hand side of (2.18) is estimated above by constant times of $\|u\|$. In view of (2.14) and (2.15), the residue calculus shows that

$$\frac{1}{2\pi i}\int_\Gamma(\zeta+1)^{-1}(\log(\zeta+1))^{mr}\,Q(\zeta)\,d\zeta$$

is a pseudodifferential operator with principal symbol $(\tilde{l}_0+1)^{-1}(\log(\tilde{l}_0+1))^{mr}$, where $\tilde{l}_0=\tilde{l}_0(x,\xi)$ is the principal symhol of $\tilde{\mathcal{L}}$ and $\tilde{l}_0(x,\xi)=\sum_{j=1}^n(a^2-x^2)\xi_j^2$ in a neighborhood of ϖ. Therefore, noting the product formula of pseudodifferential operators, from (2.18) we obtain

$$\|\psi(\log\langle D_x\rangle)^{mr}u\| \leqq C(\|\psi(\log(L+1))^{mr}u\|+\|u\|)\,,\quad u\in C_0^\infty(\omega)\,,$$

where $\psi\in C_0^\infty(\Pi)$ satisfies $0\leqq\psi\leqq1$ and $\psi=1$ in a neighborhood ω_1 of ϖ. Since $(1-\psi)\,(\log\langle D_x\rangle)^{mr}\phi\in S^{-\infty}$ for $\phi\in C_0^\infty(\omega_1)$ satisfying $\phi=1$ on ϖ, we have (2.10).

3. Proof of Proposition 4

First we shall prove the sufficiency of (8). For the proof it suffices to show the estimate (5) by Corollary 2 in Introduction. Note that

$$\text{(3.1)} \qquad \mathrm{Re}(P_1 u, u) = \|x_2 D_{x_1} u\|^2 + \|D_{x_2} u\|^2 + \|\sigma(x_1)\,\zeta(x)\,D_{x_3} u\|^2 + \|D_{x_4} u\|^2\,, \quad u \in \mathcal{S}\,,$$

Here and in what follows we denote the variable t of P_1 by x_4. Let $|||u|||^2$ denote the right hand side of (3.1) and let $f(\xi)\in C^\infty(R^4)$ be a symbol in $S^0_{1,0}$ such that

$$\begin{cases} 0\leqq f\leqq 1\,,\ f=1 \text{ on } \{|\xi|\leqq 2|\xi_3|\}\cap\{|\xi|\geqq 1\}\,, \\ \operatorname{supp} f\subset\{|\xi|\leqq 3|\xi_3|\}\cap\{|\xi|\geqq 1/2\}\,. \end{cases}$$

For any compact set K of R^4 the estimate

$$\|\langle D_x\rangle^{1/2}(1-f(D_x))\,u\|^2\leqq C_K(|||u|||^2+\|u\|^2)\,,\quad u\in C_0^\infty(K)\,,$$

holds with a constant C_K because for some constants C'_K and C''_K we have

$$\text{(3.2)} \qquad |||fu|||^2\leqq |||u|||^2+C'_K\|u\|^2\,,\quad u\in C_0^\infty(K)\,,$$

$$\text{(3.3)} \qquad \|\langle D_{x_1}\rangle^{1/2} u\|^2\leqq C''_K(\|x_2D_{x_1}u\|^2+\|D_{x_2}u\|^2+\|u\|^2)\,,\quad u\in C_0^\infty(K)\,,$$

(see [3], [14]). Hence, to derive (1) it suffices to show that for any $\varepsilon>0$ and compact set any K there exists a constant $C_{\varepsilon,K}$ such that

$$\text{(3.4)} \qquad \|(\log\langle D_{x_3}\rangle)\,fu\|^2\leqq\varepsilon|||u|||^2+C_{\kappa,K}\|u\|^2\,,\quad u\in C_0^\infty(K)\,.$$

In deriving (3.4) for a fixed K we may assume that σ and ζ belong to $\mathcal{B}^\infty$, and $\sigma(x_1)\geqq\sigma_0$ ($|x_1|\geqq 1$), $\zeta(x)\geqq\zeta_0$ for constants $\sigma_0, \zeta_0>0$. Let $\phi_0(t)$, $\phi_1(t)$, $\phi_2(t)$ and $\phi_3(t)$ be C^∞-functions in $[0, \infty)$ such that $0\leqq\phi_j\leqq 1$,

$$\begin{cases} \operatorname{supp}\phi_0\subset[0, 1)\quad, & \phi_0=1 \textit{ on } [0, 1/2]\,, \\ \operatorname{supp}\phi_1\subset[0, 2)\quad, & \phi_1=1 \textit{ on } [0, 3/2]\,, \\ \operatorname{supp}\phi_2\subset(3/2, \infty), & \phi_2=1 \textit{ in } [2, \infty)\,, \\ \operatorname{supp}\phi_3\subset(1, \infty)\quad, & \phi_3=1 \textit{ in } [3/2, \infty) \end{cases}$$

and

$$\text{(3.5)} \qquad \phi_1+\phi_2=1 \textit{ in } [0, \infty)\,.$$

Let κ be a small positive constant such that $\kappa\leqq 1/4$ and set $\chi_j(x_1, \xi)=\phi_j(\sigma(x_1)\langle\xi\rangle^{2\kappa})$ $(j=0, \cdots, 3)$.

Lemma 3.1. *It follows that $\chi_j(x_1, D_x)$ belongs to $S^0_{1\,\kappa}$. Furthermore we have*

$$\text{(3.6)} \qquad \chi_1+\chi_2=I\,.$$

Proof. (3.6) is the direct consequence of (3.5). Since σ is non-negative we have

$$(3.7) \qquad |d\sigma(x_1)/dx_1| \leqq C_0\, \sigma(x_1)^{1/2}$$

and hence for any j

$$(3.8) \qquad |d^j\, \sigma(x_1)/dx_1^j| \leqq C_j\, \sigma(x_1)^{(1-j/2)_+}$$

where a_+ denotes max $(a, 0)$. From the Leibniz formula we have $|\alpha+\beta| \neq 0$

$$\begin{aligned} \chi_{j(\beta)}^{(\alpha)}(x_1, \xi) &= \sum_{0<k\leqq|\alpha+\beta|} C_k \phi_j^{(k)}(\sigma(x_1)\, h) \\ &\times \sum_{\substack{\alpha^1+\cdots+\alpha^k=\alpha \\ \beta^1+\cdots+\beta^k=\beta}} C_{\alpha^1,\dots,\alpha^k,\beta^1,\dots,\beta^k}\, \sigma_{(\beta^1)}\cdots\sigma_{(\beta^k)}\, h^{(\alpha^1)}\cdots h^{(\alpha^k)}\,, \end{aligned}$$

where $h=\langle\xi\rangle^{2\kappa}$. Using (3.8) and $|h^{(\tilde{\alpha})}| \leqq C_{\tilde{\alpha}}\, h\langle\xi\rangle^{-|\tilde{\alpha}|}$, we obtain

$$|\chi_{j(\beta)}^{(\alpha)}| \leqq C \sum_{0<k\leqq|\alpha+\beta|} \phi^{(k)}(\sigma(x_1)\, h)\, \sigma(x_1)^{(k-|\beta|/2)_+}\, h^k\langle\xi\rangle^{-|\alpha|}\,.$$

Since for $k\neq 0$ we have

$$1/2 \leqq \sigma(x_1)\, h \leqq 2 \quad on \quad \text{supp}\, \phi_j^{(k)}(\sigma(x_1)\, h)\,,$$

we see $\chi_j(x_1, \xi) \in S_{1,\kappa}^0$. Q.E.D.

Lemma 3.2. *For any real s and any $N>0$ there exists a constant $C=C(s, N)$ such that for $j=2, 3$*

$$(3.9) \qquad \|\chi_j\, u\|_s \leqq C(\|\sigma(x_1)\, \zeta(x)\, u\|_{s+2\kappa} + \|u\|_{-N}\,, \quad u \in \mathcal{S}\,.$$

Proof. Let $a(x, \xi)$ denote the simplified symbol of a pseudo-differential operator

$$\langle D_x\rangle^{2\kappa}\, \sigma(x_1)\, \zeta(x) + \chi_0(x_1, D_x)\,.$$

Then $a(x, \xi)$ belongs to $S_{1,\kappa}^2$ and satisfies the (H)-condition in the following sence:

i) There exists a constant $C_0>0$ such that

$$(3.10) \qquad a(x, \xi) \geqq C_0 \quad \textit{for large } |\xi|\,.$$

ii) For any α and β there exists a constant $C_{\alpha\beta}$ such that

$$(3.11) \qquad |a_{(\beta)}^{(\alpha)}(x, \xi)/a(x, \xi)| \leqq C_{\alpha\beta}\langle\xi\rangle^{\kappa|\beta|-|\alpha|} \quad \textit{for large } |\xi|\,.$$

Indeed, if $a_0(x, \xi)=\sigma(x_1)\, \zeta(x)\langle\xi\rangle^{2\kappa}+\chi_0(x_1, \xi)$ then $a(x, \xi)-a_0(x, \xi) \in S_{1,\kappa}^{2\kappa-1}$ and hence it suffices to show that $a_0(x, \xi)$ satisfies the (H)-condition. In view of (3.7) and (3.8) it is not difficult to check (3.11) for a_0, by the same way as in the proof of Proposition 5.3 of [8]. Since $a(x, \xi)$ satisfies the (H)-condition there exists a parametrix $b(x, D_x) \in S^0_{1,\kappa}$ such that

$$b(x, D_x)\, a(x, D_x) \equiv I \mod S^{-\infty}$$

(cf. Theorem 5.4 in Chapter 2 of [5]). Note that for $j=2, 3$

$$\chi_j \equiv \chi_j ba \equiv \chi_j b \langle D_x \rangle^{2\kappa} \sigma(x_1)\, \zeta(x) \mod S^{-\infty} \tag{3.12}$$

because $\operatorname{supp} \phi_0 \cap \operatorname{supp} \phi_j = \emptyset$. Since $\chi_j b \langle D_x \rangle^{2\kappa} \in S^{2\kappa}_{1,\kappa}$ the estimate (3.9) is the direct consequence of (3.12). Q.E.D.

Substituting $D_{x_3}u$ instead of u into (3.9) with $s=-2\kappa$ we have

$$\begin{aligned} &\|\langle D_x \rangle^{-2\kappa} D_{x_3} \chi_j u\|^2 \\ &\leqq C(\|\sigma(x_1)\, \zeta(x)\, D_{x_3} u\|^2 + \|u\|^2)\,, \ j=2, 3, u \in S\,, \end{aligned} \tag{3.13}$$

for some constant C. In order to show for a fixed compact set K

$$\|(\log\langle D_{x_3}\rangle)\, \chi_1 f u\|^2 \leqq \varepsilon |||u|||^2 + C_\varepsilon \|u\|\,, \quad u \in C_0^\infty(K) \tag{3.14}$$

we prepare the following lemma.

Lemma 3.3. *Set $I_\mu = \{t \in R^1;\ |t| < \mu\}$ for $\mu > 0$. Then for any $s > 0$ there exists a constant C_s independent of μ such that*

$$\|v\| \leqq C_s\, \mu^s \|v\|_s \quad \textit{for} \quad v \in C_0^\infty(I_\mu)\,. \tag{3.15}$$

The lemma seems to be fairly well-known, but we give the proof for the convenience of the reader.

Proof. First we shall prove that for any $\varepsilon > 0$ there exists a $\mu_0 > 0$ such that

$$\|v\| \leqq \varepsilon \|v\|_s \quad \text{for} \quad v \in C_0^\infty(I_{\mu_0})\,. \tag{3.16}$$

Suppose that there exist an $\varepsilon_0 > 0$ and $\{v_j\}_{j=1}^\infty \subset C_0^\infty$ such that

$$\begin{aligned} &\operatorname{supp} v_j \subset I_{\mu_j}\,, \quad \mu_j \to 0\ (j \to \infty)\,, \\ &\|v_j\|_s = 1\,, \quad \|v_j\| > \varepsilon_0\,. \end{aligned}$$

In view of the weak compactness of the Hilbert space we have a subsequence $\{v_{j_k}\}_{k=1}^\infty$ such that v_{j_k} weakly converges some v_0 in H_s. Note that $\{v_{j_k}\}$ is a compact set in L^2 by means of the Rellich theorem. Taking a subsequence of $\{v_{j_k}\}$ if necessary, we may assume that v_{j_k} converges some v_0' in L^2. We have $v_0 = v_0'$ because both convergences are the one in S'. It follows from $\operatorname{supp} v_0 = \{0\}$ that v_0 is a linear sum of derivatives of Dirac δ. In view of $v_0 \in H_s$ we have

$v_0=0$, which is contradictory to $\|v_0\|\geqq\varepsilon_0$. From (3.16) we have for some $\mu_0>0$

$$\|\hat{v}(\tau)\|\leqq\||\tau|^s\hat{v}(\tau)\| \quad \text{for} \quad v\in C_0^\infty(I_{\mu_0}), \tag{3.17}$$

where $\hat{v}$ is the Fourier transform of v. The estimate (3.15) easily follows from (3.17) if we take the change of variable from t to $\mu_0 t/\mu$. Q.E.D.

Let $\phi_4(t)$ be a C^∞-function in $[0, \infty)$ such that $\phi_4=1$ in [0,2] and supp $\phi_4\subset$ [0,3). Set

$$\chi_4(x_1, \xi_3) = \phi_4(\sigma(x_1)\langle\xi_3\rangle^{2\kappa}).$$

It follows from the condition (8) that for any $\varepsilon>0$ there exists a $M_\varepsilon>0$ such that

$$|x_1|\leqq\varepsilon(\log\langle\xi_3\rangle)^{-2} \quad \text{on supp } \chi_4(x_1, \xi_3) \qquad \text{if} \quad |\xi_3|\geqq M_\varepsilon, \tag{3.18}$$

because $(x_1, \xi_3)\in\text{supp } \chi_4$ implies $\sigma(x_1)\langle\xi_3\rangle^{2\kappa}\leqq 3$. Let $\check{u}$ denote the Fourier transform of $u\in\mathcal{S}_x$ with respect to x_3 variable. Setting $v(x_1)=\chi_4(x_1, \cdot)\,\check{u}(x_1, \cdot)$ in (3.15) with $s=1/2$, in view of (3.18) we have

$$\|\chi_4(x_1, \xi_3)\,\check{u}\|^2_{L^2(R_{x_1})}\leqq C_1\,\varepsilon(\log\langle\xi_3\rangle)^{-2}\|\langle D_{x_1}\rangle^{1/2}\chi_4\,\check{u}\|^2_{L^2(R_{x_1})}, \quad \textit{if} \quad |\xi_3|\geqq M_\varepsilon,$$

for some constant C_1 independent of ε. Multiplying both sides by $(\log\langle\xi_3\rangle)^2$ and integrating with respect to x_2, x_4 and ξ_3 we have

$$\|(\log\langle D_{x_3}\rangle)\,\chi_4(x_1, D_{x_3})\,u\|^2\leqq C_1\,\varepsilon\|\langle D_{x_1}\rangle^{1/2}\chi_4(x_1, D_{x_3})\,u\|^2 + C_\varepsilon\|u\|^2.$$

Noting $\chi_4(x_1, D_{x_3})\,\chi_1(x_1, D_x) = \chi_1(x_1, D_x)$, we obtain

$$\begin{aligned}&\|(\log\langle D_{x_3}\rangle)\,\chi_1 fu\|^2\\ &\quad\leqq\varepsilon C_1\|\langle D_{x_1}\rangle^{1/2}\chi_1 fu\|^2+C_\varepsilon\|u\|^2, \quad u\in\mathcal{S}.\end{aligned} \tag{3.19}$$

In view of (3.3) we have for a fixed compact set K

$$\begin{aligned}\|\langle D_{x_1}\rangle^{1/2}\chi_1 fu\|^2&\leqq C_K(\|x_2\,D_{x_1}fu\|^2+\|D_{x_2}fu\|^2\\ &\quad+\|(D_{x_1}\chi_1)fu\|^2+\|u\|^2), \quad u\in C_0^\infty(K).\end{aligned} \tag{3.20}$$

Since $(D_{x_1}\chi_1)(x, \xi)\in S^\kappa_{1,\kappa}$ and $\chi_3=1$ on supp $D_{x_1}\chi_1$, we obtain

$$\begin{aligned}\|(D_{x_1}\chi_1)fu\|^2&\leqq C(\|\langle D_x\rangle^\kappa\chi_3 fu\|^2+\|u\|^2)\\ &\leqq C'(\|\langle D_x\rangle^{-2\kappa}D_{x_3}\chi_3 fu\|^2+\|u\|^2), \quad u\in\mathcal{S}.\end{aligned} \tag{3.21}$$

Using (3.13) with $j=3$ and (3.2), from (3.20) and (3.21) we have

$$\|\langle D_{x_1}\rangle^{1/2}\chi_1 fu\|^2\leqq C_K(|||u|||^2+\|u\|^2), \quad u\in C_0^\infty(K).$$

Combinig this and (3.19) we obtain (3.14). The estimate (3.4) follows from (3.14) and (3.13) with $j=2$. We have proved the estimate (5) for P_1.

From now on we shall prove the necessity of (8). Suppose that (8) is not satisfied but P_1 is hypoelliptic. Then there exists a $\delta>0$ and a sequence $\{s_k\}_{k=1}^{\infty}$ such that

$$\begin{cases} s_k \to 0 \quad (k\to\infty), \\ \sigma(s_k)\exp(2\delta^{-1}|s_k|^{-1/2})\leqq 1 . \end{cases} \tag{3.22}$$

Without loss of generality, we may assume $s_k>0$. Set $\lambda_k=\exp(-\delta^{-1}s_k^{-1/2})$. Then it follows from (3.22) that

$$\lambda_k^{-2}\,\sigma(s/(\log\lambda_k^{-\delta})^2)\leqq 1 \qquad \textit{for} \quad 0\leqq s\leqq 1 , \tag{3.23}$$

because σ is non-decreasing in R_+. By means of Theorem 3 and its remark we see that for any $\varepsilon>0$ and any compact set K of R^4 there exists a constant $C_{\varepsilon,K}$ such that

$$\|(\log\langle D_x\rangle)\,u\|^2\leqq\varepsilon\,\mathrm{Re}(P_1 u, u)+C_{\varepsilon,K}\|u\|^2 , \quad u\in C_0^\infty(K) . \tag{3.24}$$

(Recall that the variable t of P_4 is denoted by x_4 in this section.) Note

$$\|(\log\langle D_{x_3}\rangle)\,\phi_0(\lambda_k^2 D_{x_3}-1)\,u\|\leqq\|(\log\langle D_x\rangle)\,u\| , \quad u\in C_0^\infty(K) .$$

Since ζ is bounded on K, in view of (3.1) we have

$$\begin{aligned} &\|(\log\langle D_{x_3}\rangle)\,\phi_0(\lambda_k^2 D_{x_3}-1)\,u\| \\ &\leqq\varepsilon(\|x_2 D_{x_1}u\|+\|D_{x_2}u\|+\|\sigma(x_1)\,D_{x_3}u\|+\|D_{x_4}u\|) \\ &\quad +C_{\varepsilon,K}\|u\| , \quad u\in C_0^\infty(K) . \end{aligned} \tag{3.25}$$

Set $v(y)=\prod_{j=1}^{4}\phi_0(2|y_j-1/2|)$ and consider the change of variables

$$\begin{aligned} &y_1=(\log\lambda_k^{-\delta})^2\,x_1 , \quad y_2=(\log\lambda_k^{-\delta})\,x_2 , \\ &y_3=\lambda_k^{-1}\,x_3 , \quad y_4=x_4 . \end{aligned}$$

Let $u_0(x)$ denote the function v after the above change of variables. Then the support of $u_0(x)$ is contained in $\{|x|\leqq 4\}$ if λ_k is small enough. Substitute $\exp(i\lambda_k^{-2}x_3)\,u_0(x)$ into (3.25) and take the change of variables from x to y. Then, by using the similar formula as (1.10) we have

$$\begin{aligned} &2\log\lambda_k^{-1}\|\phi_0(\lambda_k D_{y_3})\,v\| \\ &\quad\leqq\varepsilon(\delta\log\lambda_k^{-1}(\|y_2 D_{y_1}v\|+\|D_{y_2}v\|) \\ &\qquad +\lambda_k^{-1}\|\sigma(y_1/(\log\lambda_k^{-\delta})^2)\,D_{y_3}v\| \\ &\qquad +\lambda_k^{-2}\|\sigma(y_1/(\log\lambda_k^{-\delta})^2)\,v\|+\|D_{y_4}v\|)+C_\varepsilon\|v\| , \end{aligned} \tag{3.26}$$

because a pseudodifferential operator in R_{y_3} with a symbol $(\log(\lambda_k^4+(\lambda_k\eta_3+1)^2)^{1/2})$ $\phi_0(\lambda_k\eta_3)$ is L^2-bounded unifromly with respect to λ_k. Note $\phi_0(\lambda_k D_{y_3})v$ converges v in L^2 when λ_k tends to 0. Then, there exists a $c_0>0$ such that

$$||\phi_0(\lambda_k D_{y_3})v||\geqq c_0$$
$$\textit{if}\quad \lambda_k\leqq\lambda_0\quad \textit{for a sufficiently small}\quad \lambda_0\,.$$

Since it follows from (3.23) that $\lambda_k^{-2}\sigma(y_1/(\log\lambda_k^{-\delta})^2)\leqq 1$ on supp v and also on supp $D_{y_3}v$, there exist constants c_1, c_2 independent of ε and C'_ε such that

$$2c_0\log\lambda_k^{-1}\leqq c_1\,\varepsilon\log\lambda_k^{-1}+c_2\,\varepsilon+C'_\varepsilon\qquad \textit{if}\quad \lambda_k\leqq\lambda_0\,.$$

Setting $\varepsilon=c_0/c_1$ we have a contradiction when λ_k tends to 0.

REMARK 3.1. By the similar way as in the proof of the necessity of (8), we can show that the estimate (3) dose not hold with some small $\varepsilon_0>0$ for $\mathcal{A}_0(x, D_x)=D_{y_1}^2+\exp(-1/|x_1|^\delta)D_{x_2}^2$ when $\delta\geqq 1$. This fact also can be seen by considering the eigenvalue problem for a differential operator $-d^2/dx^2+\exp(-1/|x|^\delta)\eta^2$ with Dirichlet boundary condition. It was proved in [10] that the smallest eigenvalue is estimated above by $(\log\eta)^2$ with a constant factor.

REMARK 3.2. Let P_3 be a differential operator

$$D_t^2+x_2^2D_{x_1}^2+D_{x_2}^2+\sigma(x_1)^2(x_4^2D_{x_3}^2+D_{x_4}^2)\quad\text{in}\quad R^5\,,$$

where $\sigma\in C^\infty$, $\sigma(0)=0$, $\sigma(s)>0$ $(s\neq 0)$ and $s\sigma'(s)\geqq 0$. By the same way as in this section we can prove that P_3 is hypoelliptic in R^5 if and only if σ satisfies (8).

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Adding to the proof: After the present paper had been submitted, the author received a preprint [15] from Mr. Hoshiro. In [15], one can see almost the same result as Proposition 2.2.

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