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J-GROUPS OF THE SUSPENSIONS OF THE STUNTED LENS SPACES MOD p

Dedicated to Professor Masahiro Sugawara on his 60th birthday

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1. Introduction

The purpose of this paper is to determine the J-groups of the suspensions of the stunted lens spaces. In order to state our theorem, we recall some notation in [3] and [7].

Let p be a prime and S^{2t+1} be the unit $(2t+1)$-sphere in the complex $(t+1)$-space. Then the $(2t+1)$-dimensional standard lens space mod p is the orbit space

$$L^t(p) = S^{2t+1}/Z_p, \quad Z_p = \{\exp(2\pi u\sqrt{-1}/p) \mid u = 0, 1, \cdots, p-1\}$$

where the action is given by $z(z_0, \cdots, z_t)=(zz_0, \cdots, zz_t)$. Let $[z_0, \cdots, z_t] \in L^t(p)$ denote the class of $(z_0, \cdots, z_t) \in S^{2t+1}$. The space $L^k(p)$ $(k \leqq t)$ is naturally imbedded in $L^t(p)$ by identifying $[z_0, \cdots, z_k]$ with $[z_0, \cdots, z_k, 0, \cdots, 0]$. Denote the subspace

$$L_0^k(p) = \{[z_0, \cdots, z_k] \in L^k(p) \mid z_k\text{: real, } z_k \geqq 0\} .$$

Then $L^k(p)-L_0^k(p)$ and $L_0^k(p)-L^{k-1}(p)$ $(k \leqq t)$ are $(2k+1)$- and $2k$-cells respectively, which make $L^t(p)$ a finite CW-complex.

Let $\nu_q(s)$ denote the exponent of the prime q in the prime power decomposition of s and $\mathfrak{m}(s)$ the function defined on positive integers as follows:

$$\nu_q(\mathfrak{m}(s)) = \begin{cases} 0 & \text{if } q \neq 2,\ s \not\equiv 0 \ (\mathrm{mod}(q-1)) \\ 1+\nu_q(s) & \text{if } q \neq 2,\ s \equiv 0 \ (\mathrm{mod}(q-1)) \\ 1 & \text{if } q=2,\ s \not\equiv 0 \ (\mathrm{mod}\ 2) \\ 2+\nu_2(s) & \text{if } q=2,\ s \equiv 0 \ (\mathrm{mod}\ 2)\,. \end{cases}$$

For non-negative integers r, t and n with $t>n$, we set

$$h(r, t, n) = \begin{cases} \min\{\nu_p(r)+1,\ [(t+r)/(p-1)]-[(n+r)/(p-1)]\} & (r>0) \\ [t/(p-1)]-[n/(p-1)] & (r=0)\,. \end{cases}$$

Main result is the following theorem.

Theorem. *For an odd prime p, we have*

(1) $\tilde{J}(S^{2r+1}(L_0^t(p)/L_0^n(p)))\cong 0$.

(2) $\tilde{J}(S^{2r}(L_0^t(p)/L_0^n(p)))\cong Z_{p^{h(r,t,n)}}$.

(3) $\tilde{J}(S^{2r+1}(L_0^t(p)/L^n(p)))\cong \tilde{J}(S^{2n+2r+3})$.

(4) i) *If* $n+r+1\not\equiv 0\ (\mathrm{mod}(p-1))$, *then*

$$\tilde{J}(S^{2r}(L_0^t(p)/L^n(p)))\cong Z_{p^{h(r,t,n+1)}}\oplus \tilde{J}(S^{2n+2r+2}).$$

ii) *If* $n+r+1\equiv 0\ (\mathrm{mod}(p-1))$, *then*

$$\tilde{J}(S^{2r}(L_0^t(p)/L^n(p)))\cong Z_{p^i}\oplus Z_{\mathrm{m}(n+r+1)p^{h(r,t,n+1)-i}}$$

where $i=\min\{h(r,t,n+1),\ \nu_p(n+1)\}$.

(5) $\tilde{J}(S^{2r+1}(L^t(p)/L_0^n(p)))\cong \tilde{J}(S^{2t+2r+2})$.

(6) $\tilde{J}(S^{2r}(L^t(p)/L_0^n(p)))\cong Z_{p^{h(r,t,n)}}\oplus \tilde{J}(S^{2t+2r+1})$.

(7) $\tilde{J}(S^{2r+1}(L^t(p)/(L^n(p)))\cong \tilde{J}(S^{2t+2r+2})\oplus \tilde{J}(S^{2n+2r+3})$.

(8) $\tilde{J}(S^{2r}(L^t(p)/L^n(p)))\cong \tilde{J}(S^{2r}(L_0^t(p)/L^n(p)))\oplus \tilde{J}(S^{2t+2r+1})$.

REMARK 1. The J-groups of the spheres are well known (cf. [3, Examples (3.5) and (3.6)] and [12]):

$$\tilde{J}(S^n)\cong\begin{cases} Z_{\mathrm{m}(n/2)} & (n\equiv 0\ (\mathrm{mod}\ 4)) \\ Z_2 & (n\equiv 1,\ 2\ (\mathrm{mod}\ 8)) \\ 0 & (\text{otherwise}). \end{cases}$$

REMARK 2. The partial results for the case $r=n=0$ in the parts (2) and (6) or the case $n=0$ in the part (2) have been obtained in [7, Theorem 2] and [10, Theorem 3.8]. The corresponding result for the case $p=2$ we have been shown in [11].

The paper is organized as follows. In section 2 we give preliminaries. In section 3 we give proofs of parts (2) and (4) i). In section 4 we prove the part (4) ii). The proofs of the other parts are given in the final section.

2. Preliminaries

In this section we prepare some lemmas which are needed to prove the theorem. From now on, p denotes an odd prime.

Lemma 2.1. *Let r be a positive integer and let k and j be integers with $k\equiv j$ (mod p), then*

$$k^r - j^r \equiv r(k-j)j^{r-1} \pmod{p^{\nu_p(r)+2}}.$$

Proof. Since $k \equiv j \pmod p$, we have

$$\begin{aligned} k^r - j^r &= (k-j)\textstyle\sum_{i=0}^{r-1} k^i j^{r-i-1} \\ &\equiv (k-j)\textstyle\sum_{i=0}^{r-1} j^{r-1} \pmod{p^2} \\ &= r(k-j)j^{r-1}. \end{aligned}$$

This proves the lemma for the case $\nu_p(r)=0$. Moreover

$$\begin{aligned} \textstyle\sum_{i=0}^{p-1}(k^r)^i(j^r)^{p-i-1} &\equiv \textstyle\sum_{i=0}^{p-1}(r(k-j)j^{r-1}+j^r)^i(j^r)^{p-i-1} && \pmod{p^2} \\ &\equiv \textstyle\sum_{i=0}^{p-1}(ir(k-j)j^{r-1}(j^r)^{i-1}+(j^r)^i)\,(j^r)^{p-i-1} && \pmod{p^2} \\ &= (p(p-1)/2)r(k-j)j^{r-1}(j^r)^{p-2}+p(j^r)^{p-1} \\ &\equiv p(j^r)^{p-1} \pmod{p^2}. \end{aligned}$$

Assume that

$$k^r - j^r \equiv r(k-j)j^{r-1} \pmod{p^{\nu_p(r)+2}}.$$

Then we have

$$\begin{aligned} k^{pr} - j^{pr} &= (k^r - j^r)\textstyle\sum_{i=0}^{p-1}(k^r)^i(j^r)^{p-i-1} \\ &\equiv r(k-j)j^{r-1}p(j^r)^{p-1} \pmod{p^{\nu_p(r)+3}} \\ &= pr(k-j)j^{pr-1}. \end{aligned}$$

Thus the lemma is proved by the induction on $\nu_p(r)$. q.e.d.

Lemma 2.2. *Let r be a positive integer with $r \equiv 0$ (mod $(p-1)$). Then, for each k prime to p, we have*

$$k^r - 1 \equiv r(1-k^{p-1}) \pmod{p^{\nu_p(r)+2}}.$$

Proof. Since $k^{p-1} \equiv 1 \pmod p$ for each k prime to p, we have

$$\begin{aligned} k^r - 1 &= (k^{p-1})^{r/(p-1)} - 1^{r/(p-1)} \\ &\equiv (r/(p-1))\,(k^{p-1}-1) && \pmod{p^{\nu_p(r)+2}} \\ &\equiv (1-p)\,(r/(p-1))\,(k^{p-1}-1) && \pmod{p^{\nu_p(r)+2}} \\ &= r(1-k^{p-1}) \end{aligned}$$

by Lemma 2.1 and the equality $\nu_p(r/(p-1)) = \nu_p(r)$. q.e.d.

The following equalities in the polynomial ring $Z[x]$ are obtained by making use of the binomial theorem.

(2.3)

(1) $\sum_{i=1}^{j}\binom{j}{i}\sum_{k=1}^{i}\binom{i}{k}(-1)^{i-k}x^k = x^j$.

(2) $\sum_{i=1}^{j}\binom{j}{i}(-1)^{j-i}\sum_{k=1}^{i}\binom{i}{k}x^k = x^j$.

In the rest of this paper we fix a positive integer t. Set $C^{2k+1}=L^k(p)-L_0^k(p)$ and $C^{2k}=L_0^k(p)-L^{k-1}(p)$ for $0\leqq k\leqq t$. Then the lens space $L^t(p)$ has the cell decomposition

$$L^t(p)=C^0\cup C^1\cup C^2\cup\cdots\cup C^{2t+1},$$
$$\partial(C^{2k+1})=0,\ \partial(C^{2k})=pC^{2k-1}.$$

Denote by c^k the dual cochain of C^k. Then we have the following lemma.

Lemma 2.4. *For each integer n with $0\leqq n<t$, we have*

(1) $H^*(L_0^t(p), L_0^n(p))\cong\sum_{i=n+1}^t Z_p\{c^{2i}\}$

where $Z_p\{c^{2i}\}$ means the cyclic group of order p generated by c^{2i}.

(2) $H^*(L_0^t(p), L_0^n(p); Z_2)\cong 0$.

The following lemma can be obtained by making use of Lemma 2.4 and the Atiyah-Hirzebruch spectral sequence for K-theory and KO-theory (cf. [8]).

Lemma 2.5. *The orders of $\tilde{K}^{-r}(L_0^t(p)/L_0^n(p))$ and $\widetilde{KO}^{-r}(L_0^t(p)/L_0^n(p))$ are divisors of p^{t-n}. Precisely,*

(1) $\operatorname{ord}\tilde{K}^{-r}(L_0^t(p)/L_0^n(p))=\begin{cases}1 & (r\colon odd)\\ p^{t-n} & (r\colon even),\end{cases}$

(2) $\operatorname{ord}\widetilde{KO}^{-r}(L_0^t(p)/L_0^n(p))=\begin{cases}1 & (r\colon odd)\\ p^{[(2t+r)/4]-[(2n+r)/4]} & (r\colon even)\end{cases}$

where ord G *means the order of a finite group G.*

Considering the Z_p-action on $S^{2t+1}\times \boldsymbol{C}$ given by

$$\exp(2\pi\sqrt{-1}/p)\,(z,u)=(z\cdot\exp(2\pi\sqrt{-1}/p),\ u\cdot\exp(2\pi\sqrt{1-}/p))$$

for $(z,u)\in S^{2t+1}\times\boldsymbol{C}$, we have a complex line bundle

$$\eta\colon(S^{2t+1}\times\boldsymbol{C})/Z_p\to L^t(p).$$

Set

$$\sigma=\eta-1\in\tilde{K}(L^t(p)).$$

We also denote by σ the restriction of σ to $L_0^t(p)$. Then the following proposition is well known.

Proposition 2.6 (Kambe [6, Theorem 1 and Lemma 2.5]).

(1) $K(L_0^t(p))\cong Z[\sigma]/(\sigma^{t+1}, (\sigma+1)^p-1)$.
(2) *$\tilde{K}(L_0^t(p))$ is the direct sum of cyclic groups generated by $\sigma,\sigma^2,\cdots,\sigma^{p-1}$. The order of σ^i is $p^{[(t-i)/(p-1)]+1}$.*

From this we obtain the following result.

Corollary 2.7. *Let u be a positive integer with $u=s(p-1)+j$ for $1\leqq j\leqq p-1$. Then, in $\tilde{K}(L_0^t(p))$,*

$$\sigma^u\equiv(-p)^s\sigma^j$$

modulo the subgroup generated by

$$\{p^{s+1}\sigma^1, \cdots, p^{s+1}\sigma^j, p^s\sigma^{j+1}, \cdots, p^s\sigma^{p-1}\}\ .$$

Proof. By making use of the relation $(\sigma+1)^p=1$, we obtain inductively

$$\sigma^{j+p-1}=\textstyle\sum_{i=1}^{j-1}(\sum_{k=1}^{i}(-1)^{j-k-1}\binom{p+j-k-1}{j-k}\binom{p}{i-k+1})\sigma^i$$
$$+\textstyle\sum_{i=j}^{p-1}(-1)^j\binom{p-i+j-2}{j-1}\binom{p+j-1}{i}\sigma^i\ ,$$

for $1\leqq j\leqq p-1$. Set integers $B_{i,j}$ $(1\leqq i\leqq p-1,\ 1\leqq j\leqq p-1)$ by

$$B_{i,j}=\begin{cases}\sum_{k=1}^{i}(-1)^{j-k-1}\binom{p+j-k-1}{j-k}\binom{p}{i-k+1} & (1\leqq i<j)\\ (-1)^j\binom{p-i+j-2}{j-1}\binom{p+j-1}{i} & (j\leqq i\leqq p-1)\,.\end{cases}$$

Then we have

$$\sigma^{j+p-1}=\textstyle\sum_{i=1}^{p-1}B_{i,j}\sigma^i$$

and

$$B_{i,j}\equiv\begin{cases}0 & (\mathrm{mod}\ p^2)\quad(1\leqq i<j)\\ -p & (\mathrm{mod}\ p^2)\quad(i=j)\\ 0 & (\mathrm{mod}\ p)\quad(j<i\leqq p-1)\,.\end{cases}$$

This proves the case $s=1$. Now suppose the result true for some value of s, that is

$$\sigma^{j+s(p-1)}=\textstyle\sum_{i=1}^{p-1}A_i\sigma^i$$

with

$$A_i\equiv\begin{cases}0 & (\mathrm{mod}\ p^{s+1})\quad(1\leqq i<j)\\ (-p)^s & (\mathrm{mod}\ p^{s+1})\quad(i=j)\\ 0 & (\mathrm{mod}\ p^s)\quad(j<i\leqq p-1)\,.\end{cases}$$

Then we have

$$\begin{aligned}\sigma^{j+(s+1)(p-1)}&=\textstyle\sum_{i=1}^{p-1}A_i\sigma^{i+p-1}\\ &=\textstyle\sum_{i=1}^{p-1}A_i(\sum_{k=1}^{p-1}B_{k,i}\sigma^k)\\ &=\textstyle\sum_{k=1}^{p-1}(\sum_{i=1}^{p-1}A_iB_{k,i})\sigma^k\ .\end{aligned}$$

It follows from the inductive hypothesis that

$$A_iB_{k,i} \equiv \begin{cases} 0 & (\text{mod } p^{s+2}) \quad (k<i \text{ or } i<j) \\ (-p)^{s+1} & (\text{mod } p^{s+2}) \quad (i=k=j) \\ 0 & (\text{mod } p^{s+1}) \quad (\text{otherwise}). \end{cases}$$

Hence

$$\sum_{i=1}^{p-1} A_iB_{k,i} \equiv \begin{cases} 0 & (\text{mod } p^{s+2}) \quad (1\leqq k<j) \\ (-p)^{s+1} & (\text{mod } p^{s+2}) \quad (k=j) \\ 0 & (\text{mod } p^{s+1}) \quad (j<k\leqq p-1). \end{cases}$$

Thus the proof is completed by the induction with respect to s.

We define the function

$$\mu\colon Z \to Z \tag{2.8}$$

by setting $\mu(k)$ to be the remainder of k divided by p for every $k\in Z$. Set

(2.9)
(1) $x_i = I^r(\eta^i-1)\in\tilde{K}(S^{2r}L_0^t(p))$ for each integer i,
(2) $y_i = I^r(\eta-1)^i\in\tilde{K}(S^{2r}L_0^t(p))$ for each positive integer i

where I denotes the isomorphism defined by the Bott periodicity. Then, following properties are obtained by the proof of [10, Theorem 3.8] and the equalities of (2.3).

(2.10)
(1) $x_i = x_{\mu(i)}$.
(2) $y_i = \sum_{j=1}^{i}\binom{i}{j}(-1)^{i-j}x_j \quad (i>0)$.
(3) $x_i = \sum_{j=1}^{i}\binom{i}{j}y_j \quad (i>0)$.
(4) *For Adams operation* ψ^k, *we have*

$$\psi^k(x_i) = k^r x_{ki}.$$

For each i prime to p, $N(i)$ denote the integer chosen to satisfy the property

$$iN(i) \equiv 1 \pmod{p^t} \tag{2.11}$$

Let w be the remainder of r divided by $p-1$ and set $v=p-1-w$. Then $1\leqq v \leqq p-1$, and

$$\begin{aligned} \sum_{i=1}^{v}\binom{v}{i}(-1)^{v-i}N(i^{pr}) &\equiv \sum_{i=1}^{v}\binom{v}{i}(-1)^{v-i}i^{v} \pmod{p} \\ &= v! \\ &\not\equiv 0 \pmod{p} \end{aligned}$$

by [10, Lemma 3.7]. For $1\leqq j\leqq p-1$, we put

$$Y_j=y_j-\sum_{i=1}^{j}\binom{j}{i}(-1)^{j-i}N(i^{pr})N_v y_v \tag{2.12}$$

where $N_v=N(\sum_{i=1}^{v}\binom{v}{i}(-1)^{v-i}N(i^{pr}))$. Then we have the following.

Lemma 2.13. *Let j be an integer with $1\leqq j\leqq p-1$ and k an integer prime to p, then we have*

(1) $Y_j\equiv y_j \pmod{py_v} \quad (j>v)$.

(2) $Y_v=0$.

(3) $Y_1=-N_v\sum_{i=1}^{v}\binom{v}{i}(-1)^{v-i}N(i^{pr})(\psi^{i^p}-1)y_1$.

(4) $Y_j=\sum_{i=1}^{j}\binom{j}{i}(-1)^{j-i}N(i^{pr})((\psi^{i^p}-1)y_1+Y_1)$.

(5) $(\psi^k-1)(y_j)-k^r\sum_{i=1}^{j}\binom{j}{i}(-1)^{j-i}\sum_{u=1}^{\mu(ki)}\binom{\mu(ki)}{u}Y_u+Y_j$

$$\equiv\begin{cases}0 & \pmod{p^{\nu_p(r)+1}y_v} \quad (j<v)\\ (1-k^{p-1})ry_v & \pmod{p^{\nu_p(r)+2}y_v} \quad (j=v)\\ 0 & \pmod{p^{\nu_p(r)+2}y_v} \quad (j>v).\end{cases}$$

Proof. (1) Since

$$\sum_{i=1}^{j}\binom{j}{i}(-1)^{j-i}N(i^{pr})\equiv\sum_{i=1}^{j}\binom{j}{i}(-1)^{j-i}i^v=0 \pmod p$$

by [10, Lemma 3.7], (1) is obtained by the definition (2.12).

(2) From the definition of N_v, we have

$$Y_v=y_v-(\sum_{i=1}^{v}\binom{v}{i}(-1)^{v-i}N(i^{pr}))N_v y_v=0.$$

(3) By making use of the properties (2.10) and the definition of $N(i)$, we have

$$\begin{aligned}Y_1&=y_1-N_v y_v\\ &=-N_v(y_v-\sum_{i=1}^{v}\binom{v}{i}(-1)^{v-i}N(i^{pr})y_1)\\ &=-N_v\sum_{i=1}^{v}\binom{v}{i}(-1)^{v-i}(x_i-N(i^{pr})y_1)\\ &=-N_v\sum_{i=1}^{v}\binom{v}{i}(-1)^{v-i}N(i^{pr})(i^{pr}x_i-x_1)\\ &=-N_v\sum_{i=1}^{v}\binom{v}{i}(-1)^{v-i}N(i^{pr})(\psi^{i^p}-1)x_1.\end{aligned}$$

(4) Similarly, we have

$$Y_j=y_j-\sum_{i=1}^{j}\binom{j}{i}(-1)^{j-i}N(i^{pr})(y_1-Y_1)$$

$$= \sum_{i=1}^{j}\binom{j}{i}(-1)^{j-i}(x_i - N(i^{pr})(x_1 - Y_1))$$

$$= \sum_{i=1}^{j}\binom{j}{i}(-1)^{j-i}N(i^{pr})((\psi^{i^p}-1)x_1 + Y_1)$$

by the properties (2.10).

(5) It follows from the definition (2.12), by making use of the properties (2.10) and the equalities (2.3), that

$$\sum_{u=1}^{j}\binom{j}{u}Y_u = x_j - \sum_{u=1}^{j}\binom{j}{u}\sum_{i=1}^{u}\binom{u}{i}(-1)^{u-i}N(i^{pr})N_v y_v$$
$$= x_j - N(j^{pr})N_v y_v\,.$$

Hence we have

$$(\psi^k - 1)y_j = \left(\sum_{i=1}^{j}\binom{j}{i}(-1)^{j-i}\psi^k x_i\right) - y_j$$
$$= \left(\sum_{i=1}^{j}\binom{j}{i}(-1)^{j-i}k^r x_{ki}\right) - y_j$$
$$= \left(\sum_{i=1}^{j}\binom{j}{i}(-1)^{j-i}k^r\left(\sum_{u=1}^{\mu(ki)}\binom{\mu(ki)}{u}Y_u + N(\mu(ki)^{pr})N_v y_v\right)\right) - y_j$$
$$= k^r\sum_{i=1}^{j}\binom{j}{i}(-1)^{j-i}\sum_{u=1}^{\mu(ki)}\binom{\mu(ki)}{u}Y_u$$
$$+ k^r\sum_{i=1}^{j}\binom{j}{i}(-1)^{j-i}N(\mu(ki)^{pr})N_v y_v$$
$$- Y_j - \sum_{i=1}^{j}\binom{j}{i}(-1)^{j-i}N(i^{pr})N_v y_v$$
$$= k^r\sum_{i=1}^{j}\binom{j}{i}(-1)^{j-i}\sum_{u=1}^{\mu(ki)}\binom{\mu(ki)}{u}Y_u - Y_j$$
$$+ \sum_{i=1}^{j}\binom{j}{i}(-1)^{j-i}(k^r N(\mu(ki)^{pr}) - N(i^{pr}))N_v y_v$$

by the properties (2.10). Lemma 2.1 shows

$$(\psi^k - 1)y_j - k^r\sum_{i=1}^{j}\binom{j}{i}(-1)^{j-i}\sum_{u=1}^{\mu(ki)}\binom{\mu(ki)}{u}Y_u + Y_j$$
$$= \sum_{i=1}^{j}\binom{j}{i}(-1)^{j-i}N(\mu(ki)^{pr})N(i^{pr})(k^r i^{pr} - \mu(ki)^{pr})N_v y_v$$
$$\equiv \sum_{i=1}^{j}\binom{j}{i}(-1)^{j-i}N(\mu(ki)^{pr})N(i^{pr})(k^r i^{pr} - (ki)^{pr})N_v y_v \pmod{p^{\nu_p(r)+2}y_v}$$
$$\equiv \sum_{i=1}^{j}\binom{j}{i}(-1)^{j-i}N(\mu(ki)^{pr})r(k - k^p)k^{r-1}N_v y_v \qquad \pmod{p^{\nu_p(r)+2}y_v}$$
$$\equiv \sum_{i=1}^{j}\binom{j}{i}(-1)^{j-i}i^v(1 - k^{p-1})rN_v y_v \qquad \pmod{p^{\nu_p(r)+2}y_v}\,.$$

Since $(1-k^{p-1})r \equiv 0 \pmod{p^{\nu_p(r)+1}}$, (5) is obtained by [10, Lemma 3.7].

Lemma 2.14. *Let u and j be integers with $1 \leqq u \leqq j \leqq p-1$. Then, for each k prime to p, we have*

$$\sum_{i=1}^{j} \binom{j}{i} (-1)^{j-i} \binom{\mu(ki)}{u} \equiv \begin{cases} 0 & (\mathrm{mod}\ p) \quad (u<j) \\ k^u & (\mathrm{mod}\ p) \quad (u=j) . \end{cases}$$

Proof. From [10, Lemma 3.7], we have

$$\begin{aligned} u! \times \sum_{i=1}^{j} \binom{j}{i} (-1)^{j-i} \binom{\mu(ki)}{u} &\equiv \sum_{i=1}^{j} \binom{j}{i} (-1)^{j-i} \mu(ki)^u \qquad (\mathrm{mod}\ p) \\ &\equiv k^u \sum_{i=1}^{j} \binom{j}{i} (-1)^{j-i} i^u \qquad (\mathrm{mod}\ p) \\ &= \begin{cases} 0 & (u<j) \\ k^u(u!) & (u=j) . \end{cases} \end{aligned}$$

q.e.d.

3. Proofs of parts (2) and (4) i) of Theorem

We begin with the method which is used in the proof of [7, Theorem 2].

Lemma 3.1. *Let X be a finite CW-complex and assume that $\widetilde{KO}(X)$ has an odd order. Then the real restriction*

$$\rho \colon \tilde{K}(X) \to \widetilde{KO}(X)$$

is an epimorphism. In particular, if $\tilde{K}(X)$ also has an odd order, then

$$\ker \rho = (1-\tau)\tilde{K}(X) ,$$

and

$$\ker J \circ \rho = \textstyle\sum_k (\bigcap_e k^e(\psi^k - 1)\tilde{K}(X))$$

where $\tau \colon \tilde{K}(X) \to \tilde{K}(X)$ is the conjugation and $J \colon \widetilde{KO}(X) \to \tilde{J}(X)$ is the natural projection.

Proof. Let $c \colon \widetilde{KO}(X) \to \tilde{K}(X)$ be the complexification. Since $\rho \circ c = 2 \colon \widetilde{KO}(X) \to \widetilde{KO}(X)$ is an isomorphism, c is a monomorphism and ρ is an epimorphism.

We now turn to the case in which $\tilde{K}(X)$ also has an odd order. Since $\rho = \rho \circ \tau$, $\rho((1-\tau)\tilde{K}(X)) \cong 0$. Conversely, assume $\rho(y)=0$ for some $y \in \tilde{K}(X)$, then $y + \tau(y) = c \circ \rho(y) = 0$. Since $\tilde{K}(X)$ has an odd order, $y = 2x$ for some $x \in \tilde{K}(X)$, and the equality $2y = y - \tau(y) = 2(1-\tau)x$ implies $y = (1-\tau)x$. Therefore

$$\ker \rho = (1-\tau)\tilde{K}(X) .$$

Since $\widetilde{KO}(X)$ has a finite order,

$$\ker J = \textstyle\sum_k (\bigcap_e k^e(\psi_R^k - 1)\widetilde{KO}(X)) .$$

It follows from the compatibility of the Adams operations with the real restriction (cf. [4, Lemma A 2]), that $\ker J\circ\rho$ coincides with the subgroup generated by the elements of $\ker\rho$ and $\sum_k(\bigcap_e k^e(\psi^k-1)\tilde{K}(X))$. Since $\tau=\psi^{-1}$: $\tilde{K}(X)\to\tilde{K}(X)$, we have

$$\ker\rho=(1-\tau)\tilde{K}(X)\subset\sum_k(\bigcap_e k^e(\psi^k-1)\tilde{K}(X)).$$

Therefore,

$$\ker J\circ\rho=\sum_k(\bigcap_e k^e(\psi^k-1)\tilde{K}(X)).$$

This completes the proof.

From Lemma 2.5, we have

$$\tilde{K}^i(L_0^t(p)/L_0^n(p))\cong\tilde{K}^i(L_0^n(p))\cong 0$$

for each odd integer i. Therefore we have a short exact sequence

$$0\to\tilde{K}(S^{2r}(L_0^t(p)/L_0^n(p)))\xrightarrow{j_n}\tilde{K}(S^{2r}(L_0^t(p)))\xrightarrow{i_n}\tilde{K}(S^{2r}(L_0^n(p)))\to 0. \tag{3.2}$$

We now put

$$\begin{aligned} V_n &= \ker i_n \\ U_n &= \sum_{k\not\equiv 0(\mathrm{mod}\ p)}(\psi^k-1)V_n. \end{aligned} \tag{3.3}$$

Then we have the following property.

(3.4) *The group V_n is the direct sum of cyclic groups generated by*

$$p^{[(n-i)/(p-1)]+1}y_i\quad(i=1,\cdots,p-1).$$

The order of $p^{[(n-i)/(p-1)]+1}y_i$ is $p^{[(t-i)/(p-1)]-[(n-i)/(p-1)]}$.

Moreover we have the following.

Lemma 3.5. *Assume $r>0$. Then U_n is the subgroup of V_n generated by*

$$p^{[(n-i)/(p-1)]+1}Y_i\quad(i=1,\cdots,p-1)\quad\text{and}\quad p^{[(n-v)/(p-1)]+\nu_p(r)+2}y_v.$$

In the case $r=0$, U_n is the subgroup of V_n generated by

$$p^{[(n-i)/(p-1)]+1}Y_i\quad(i=1,\cdots,p-1).$$

Proof. Put $s=[n/(p-1)]$ and $j=n-s(p-1)$, and consider the case $r>0$. The lemma is true for U_t by (2.12), Lemma 2.13 and Proposition 2.6. Assume that the lemma is true for U_{n+1}, that is

$$\begin{aligned} U_{n+1}=\langle\{p^{s+1}Y_i\mid 1\leqq i\leqq j+1\}&\cup\{p^sY_i\mid j+2\leqq i\leqq p-1\}\\ &\cup\{p^{\nu_p(r)+[(n+1-v)/(p-1)]+2}y_v\}\rangle. \end{aligned}$$

Then, by Lemmas 2.13 and 2.14, we have

$$(\psi^k-1)(p^s y_{j+1}) \equiv \begin{cases} (k^{r+j+1}-1)p^s Y_{j+1} & (\mathrm{mod}\ U_{n+1}) \quad (j+1 \neq v) \\ -(k^{p-1}-1)rp^s y_v & (\mathrm{mod}\ U_{n+1}) \quad (j+1=v)\,. \end{cases}$$

Since $U_n = \langle U_{n+1} \cup \{(\psi^k-1)(p^s y_{j+1}) \mid k \not\equiv 0 \ (\mathrm{mod}\ p)\}\rangle$, the lemma is true for U_n. The proof of the case $r=0$ is similar to the above proof. q.e.d.

We now turn to the proof of the part (2) of Theorem. From (3.4), Lemmas 2.13, 3.1 and 3.5 we obtain

$$\begin{aligned} \tilde{J}(S^{2r}(L_0^t(p)/L_0^n(p))) &\cong V_n/U_n \\ &\cong \begin{cases} \langle p^{[(n-v)/(p-1)]+1} y_v \rangle & (r=0) \\ \langle p^{[(n-v)/(p-1)]+1} y_v \rangle / \langle p^{\nu_p(r)+[(n-v)/(p-1)]+2} y_v \rangle & (r>0)\,. \end{cases} \end{aligned}$$

Then, the equality

$$([(t-v)/(p-1)]+1)-([(n-v)/(p-1)]+1) = [(t+r)/(p-1)]-[(n+r)/(p-1)]$$

establishes the part (2) of Theorem.

We turn now to the proof of the part (4) i) of Theorem. By the above proof, we have the following lemma.

Lemma 3.6. *If* $n+r+1 \not\equiv 0$ (mod $p-1$), *then the quotient map*

$$q_n \colon S^{2r}(L_0^t(p)/L_0^n(p)) \to S^{2r}(L_0^t(p)/L_0^{n+1}(p))$$

induces the isomorphism

$$J(q_n^!) \colon \tilde{J}(S^{2r}(L_0^t(p)/L_0^{n+1}(p))) \to \tilde{J}(S^{2r}(L_0^t(p)/L_0^n(p)))\,.$$

In the exact sequence of triple $(L_0^t(p), L_0^{n+1}(p), L^n(p))$, we have

$$\widetilde{KO}^{-2r+1}(L_0^t(p)/L_0^{n+1}(p)) \cong 0$$

by Lemma 2.5. Hence, we have an exact sequence,

$$\widetilde{KO}(S^{2r}(L_0^t(p)/L_0^{n+1}(p))) \xrightarrow{f} \widetilde{KO}(S^{2r}(L_0^t(p)/L^n(p))) \xrightarrow{g} \widetilde{KO}(S^{2n+2r+2}) \to 0\,.$$

Therefore, the row of the commutative diagram

$$\begin{tikzcd} & \tilde{J}(S^{2r}(L_0^t(p)/L_0^n(p))) & \\ \tilde{J}(S^{2r}(L_0^t(p)/L_0^{n+1}(p))) \arrow[ur, "J(q_n^!)"] \arrow[r, "J(f)"] & \tilde{J}(S^{2r}(L_0^t(p)/L^n(p))) \arrow[u] \arrow[r, "J(g)"] & \tilde{J}(S^{2n+2r+2}) \arrow[r] & 0 \end{tikzcd}$$

is exact by [3, Theorem (3.12)]. Since the map $J(q_n^!)$ is the isomorphism by Lemma 3.6, we have a split short exact sequence

$$0 \to \tilde{J}(S^{2r}(L_0^t(p)/L_0^{n+1}(p))) \to \tilde{J}(S^{2r}(L_0^t(p)/L^n(p))) \to \tilde{J}(S^{2n+2r+2}) \to 0\,.$$

This completes the proof.

4. Proof of the part (4) ii) of Theorem

In this section we assume that $n+r+1\equiv 0 \pmod{(p-1)}$. We set $(n+1-v)/(p-1)=s$ where v denotes the integer defined in section 2. By making use of the isomorphisms

$$\tilde{K}(S^{2r}(L_0^t(p)/L_0^n(p)))\cong V_n$$

and

$$\tilde{K}(S^{2r}(L_0^t(p)/L_0^{n+1}(p)))\cong V_{n+1},$$

we obtain the following commutative diagram, in which the row is exact:

$$(4.1)\qquad \begin{tikzcd} & & \tilde{K}(S^{2r}(CP^t/CP^n)) \arrow[r] \arrow[d] & \tilde{K}(S^{2r}(CP^{n+1}/CP^n)) \arrow[d, "\cong"] & \\ 0 \arrow[r] & V_{n+1} \arrow[r, "f_1"] \arrow[d, equal] & \tilde{K}(S^{2r}(L_0^t(p)/L^n(p))) \arrow[r, "f_2"] \arrow[d, "f_3"] & \tilde{K}(S^{2n+2r+2}) \arrow[r] & 0 \\ & V_{n+1} \arrow[r] & V_n . & & \end{tikzcd}$$

It follows from Corollary 2.7 that we have $x\in\tilde{K}(S^{2r}(L_0^t(p)/L^n(p)))$ such that $f_3(x)=p^s y_v$ and $f_2(x)$ is a generator of $\tilde{K}(S^{2n+2r+2})$. Since $\tilde{K}(S^{2n+2r+2})$ is isomorphic to Z, we have a direct sum decomposition

$$\tilde{K}(S^{2r}(L_0^t(p)/L^n(p)))\cong f_1(V_{n+1})\oplus Z\{x\}$$

where $Z\{x\}$ means the infinite cyclic group generated by x.

For the Adams operation, we have the following lemma.

Lemma 4.2. (1) *For each integer k prime to p, we have*

$$\psi^k(x)\equiv k^{n+r+1}x-(((k^{n+r+1}-1)+r(k^{p-1}-1))/p)f_1(p^{s+1}y_v) \pmod{f_1(U_{n+1})}.$$

(2) *If $k\equiv 0 \pmod p$, then we have*

$$\psi^k(x)\equiv k^{n+r+1}x-(k^{n+r+1}/p)f_1(p^{s+1}y_v) \pmod{f_1(U_{n+1})}.$$

Proof. (1) We necessarily have

$$\psi^k(x)\equiv \alpha f_1(p^{s+1}y_v)+\beta x \pmod{f_1(U_{n+1})}$$

for some integers α and β by (2.12), (3.4), Lemmas 2.13 and 3.5. By using the ψ-map f_2, we see that $\beta=k^{n+r+1}$. Now project into V_n; $f_1(p^{s+1}y_v)$ maps into $p^{s+1}y_v$ and x into $p^s y_v$, and we see that

$$(k^{n+r+1}+p\alpha)(p^s y_v)\equiv\psi^k(p^s y_v) \pmod{U_{n+1}}.$$

It follows from Lemma 2.13 that

$$(k^{n+r+1}+p\alpha)(p^s y_v) \equiv ((1-k^{p-1})r+1)p^s y_v \pmod{U_{n+1}}.$$

This implies that

$$\alpha f_1(p^{s+1}y_v) \equiv -(((k^{n+r+1}-1)+r(k^{p-1}-1))/p)f_1(p^{s+1}y_v) \pmod{f_1(U_{n+1})}$$

and

$$\begin{aligned}\psi^k(x) &\equiv \alpha f_1(p^{s+1}y_v)+k^{n+r+1}x \pmod{f_1(U_{n+1})}\\ &\equiv k^{n+r+1}x-(((k^{n+r+1}-1)+r(k^{p-1}-1))/p)f_1(p^{s+1}y_v) \pmod{f_1(U_{n+1})}.\end{aligned}$$

(2) If $k \equiv 0 \pmod p$, then

$$\psi^k(\sigma^i) = \psi^k(\eta-1)^i = (\psi^k(\eta)-1)^i = (\eta^k-1)^i = (1-1)^i = 0,$$

and

$$\psi^k(y_i) = 0.$$

Hence, the desired result is obtained by using the similar method used in the proof of (1). q.e.d.

We now recall some definition in [3]. Set $Y=\tilde{K}(S^{2r}(L_0^t(p)/L^n(p)))$ and let f be a function which assigns to each integer k a non-negative integer $f(k)$. Given such a function f, we define Y_f to be the subgroup of Y generated by $\{k^{f(k)}(\psi^k-1)(y) \mid k\in Z, y\in Y\}$, that is

$$Y_f = \langle\{k^{f(k)}(\psi^k-1)(y) \mid k\in Z, y\in Y\}\rangle.$$

Then the kernel of the homomorphism $J''\colon Y\to J''(Y)$ coincides with $\bigcap_f Y_f$ where the intersection runs over all functions f.

Suppose that f satisfies

(4.3) $f(k) \geqq t+\max\{v_q(\mathfrak{m}(n+r+1)) \mid q \text{ is a prime divisor of } k\}$

for every $k\in Z$. In the following calculation we put $n+r+1=u$ for the sake of simplicity. From Lemmas 4.2 and 2.2, we have

$$\begin{aligned}&k^{f(k)}(\psi^k-1)(x)\\ &\equiv k^{f(x)}(k^u-1)x-k^{f(k)}(((k^u-1)+r(k^{p-1}-1))/p)f_1(p^{s+1}y_v) \pmod{f_1(U_{n+1})}\\ &= k^{f(k)}(k^u-1)x-k^{f(k)}N(u/p^{\nu_p(u)})((u(k^u-1)+ur(k^{p-1}-1))/p^{\nu_p(u)+1})f_1(p^{s+1}y_v)\\ &\equiv k^{f(k)}(k^u-1)x-k^{f(k)}N(u/p^{\nu_p(u)})((u(k^u-1)-r(k^u-1))/p^{\nu_p(u)+1})f_1(p^{s+1}y_v)\\ &\qquad\qquad \pmod{f_1(U_{n+1})}\\ &= (k^{f(k)}(k^u-1)/p^{\nu_p(u)+1})(p^{\nu_p(u)+1}x-N(u/p^{\nu_p(u)})(n+1)f_1(p^{s+1}y_v)).\end{aligned}$$

By virtue of [3, Theorem (2.7) and Lemma (2.12)],

$$\begin{aligned}&\langle f_1(U_{n+1}) \cup \{k^{f(k)}(\psi^k-1)(x) \mid k\in Z\}\rangle \\ &\quad = \langle f_1(U_{n+1}) \cup \{(\mathfrak{m}(u)/p^{\nu_p(u)+1})(p^{\nu_p(u)+1}x - N(u/p^{\nu_p(u)})(n+1)f_1(p^{s+1}y_v))\}\rangle \\ &\quad = \langle f_1(U_{n+1}) \cup \{\mathfrak{m}(u)x - (\mathfrak{m}(u)/p^{\nu_p(u)+1})N(u/p^{\nu_p(u)})(n+1)f_1(p^{s+1}y_v)\}\rangle .\end{aligned}$$

Therefore,

$$Y_f = \langle f_1(U_{n+1}) \cup \{\mathfrak{m}(n+r+1)x - Mf_1(p^{s+1}y_v)\}\rangle$$

where $M=(\mathfrak{m}(n+r+1)/p^{\nu_p(n+r+1)+1})N((n+r+1)/p^{\nu_p(n+r+1)})(n+1)$. Since this is true for every function f which satisfies (4.3), we have

$$J''(Y) = Y/\langle f_1(U_{n+1}) \cup \{\mathfrak{m}(n+r+1)x - Mf_1(p^{s+1}y_v)\}\rangle . \tag{4.4}$$

We now recall the notation of $h(r, t, n+1)$:

$$h(r,t,n+1) = \begin{cases} \min\{\nu_p(r)+1, [(t+r)/(p-1)]-(n+r+1)/(p-1)\} & (r>0) \\ [t/(p-1)]-(n+1)/(p-1) & (r=0) . \end{cases}$$

Then we have the following lemma.

Lemma 4.5.

$$J''(\tilde{K}(S^{2r}(L_0^t(p)/L^n(p)))) \cong Z_{p^i} \oplus Z_{\mathfrak{m}(n+r+1)p^{h(r,t,n+1)-i}}$$

where $i=\min\{\nu_p(n+1), h(r, t, n+1)\}$.

Proof. By (4.4), we have

$$\begin{aligned}&J''(\tilde{K}(S^{2r}(L_0^t(p)/L^n(p)))) \\ &\quad = \langle\{x, f_1(p^{s+1}y_v)\}\rangle / \langle\{\mathfrak{m}(n+r+1)x - Mf_1(p^{s+1}y_v), p^{h(r,t,n+1)}f_1(p^{s+1}y_v)\}\rangle .\end{aligned}$$

Since $\nu_p(M)=\nu_p(n+1)$, the greatest common divisor of M and $p^{h(r,t,n+1)}$ equals to p^i. Choose integers a and b with

$$aM + bp^{h(r,t,n+1)} = p^i .$$

Then, it is easily seen that

$$\begin{aligned}&\langle\{\mathfrak{m}(n+r+1)x - Mf_1(p^{s+1}y_v), p^{h(r,t,n+1)}f_1(p^{s+1}y_v)\}\rangle \\ &\quad = \langle\{\mathfrak{m}(n+r+1)p^{h(r,t,n+1)-i}x, p^i(a(\mathfrak{m}(n+r+1)/p^i)x - f_1(p^{s+1}y_v))\}\rangle\end{aligned}$$

and

$$\langle\{x, f_1(p^{s+1}y_v)\}\rangle = \langle\{x, a(\mathfrak{m}(n+r+1)/p^i)x - f_1(p^{s+1}y_v)\}\rangle .$$

This proves the lemma.

Now, by virtue of the above lemma, the proof of Theorem (4) ii) is completed by the following lemma.

Lemma 4.6. $\tilde{J}(S^{2r}(L_0^t(p)/L^n(p)))\cong J''(\tilde{K}(S^{2r}(L_0^t(p)/L^n(p))))$.

Proof. In this proof we put $A=S^{2r}(L_0^t(p)/(L^n(p))$, $B=S^{2r}(L_0^t(p)/L_0^{n+1}(p))$ and $C=S^{2n+2r+2}$ for the sake of simplicity.

Consider the homomorphisms

$$J''(\tilde{K}(B)) \underset{J''(c_1)}{\overset{J''(\rho_1)}{\rightleftarrows}} J''(\widetilde{KO}(B))$$

where ρ_1 and c_1 are the real restriction and the complexification. Then, by making use of Lemma 2.5, Lemma 3.1 and [3, Lemma (3.8) and Theorem (3.12)], we see that $J''(\ker \rho_1)\cong 0$ and so $J''(\rho_1)$ and $J''(c_1)$ are the isomorphisms. Since p is an odd prime and $n+r+1\equiv 0 \pmod{p-1}$, there are following cases.

i) If $n+r+1\equiv 0 \pmod 4$, then we have the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \widetilde{KO}(B) & \longrightarrow & \widetilde{KO}(A) & \longrightarrow & \widetilde{KO}(C) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle c_1} & & \downarrow{\scriptstyle c_2} & & \downarrow{\scriptstyle c_3} & & \\
0 & \longrightarrow & \tilde{K}(B) & \xrightarrow{f} & \tilde{K}(A) & \longrightarrow & \tilde{K}(C) & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

of exact sequences. By making use of [3, Lemma (3.8) and Theorem (3.12)], we have the following commutative and exact diagram:

$$\begin{array}{ccccccc}
0 & & & & 0 & & \\
\downarrow & & & & \downarrow & & \\
J''(\widetilde{KO}(B)) & \longrightarrow & J''(\widetilde{KO}(A)) & \longrightarrow & J''(\widetilde{KO}(C)) & \longrightarrow & 0 \\
\downarrow{\scriptstyle J''(c_1)} & & \downarrow{\scriptstyle J''(c_2)} & & \downarrow{\scriptstyle J''(c_3)} & & \\
J''(\tilde{K}(B)) & \xrightarrow{J''(f)} & J''(\tilde{K}(A)) & \longrightarrow & J''(\tilde{K}(C)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

Since $J''(f)$ is a monomorphism by (4.4), $J''(c_2)$ is also an isomorphism. Therefore we obtain

$$\tilde{J}(A)\cong J''(\widetilde{KO}(A))\cong J''(\tilde{K}(A)).$$

ii) If $n+r+1\equiv 2 \pmod 4$, then we have a commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & \tilde{K}(B) & \xrightarrow{f} & \tilde{K}(A) & \longrightarrow & \tilde{K}(C) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle \rho_3} & & \\
0 & \longrightarrow & \widetilde{KO}(B) & \longrightarrow & \widetilde{KO}(A) & \longrightarrow & \widetilde{KO}(C) & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}
$$

of exact sequences. Inspecting the diagram, we see that ρ_2 is an epimorphism and

$$f|_{\ker\rho_1}\colon \ker\rho_1 \to \ker\rho_2$$

is an isomorphism. By making use of [3, Lemma (3.8) and Theorem (3.12)] we have the following commutative and exact diagram:

$$
\begin{array}{ccccccc}
0 \longrightarrow & J''(\ker\rho_1) & \longrightarrow & J''(\ker\rho_2) & \longrightarrow & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & J''(\tilde{K}(B)) & \longrightarrow & J''(\tilde{K}(A)) & \longrightarrow & J''(\tilde{K}(C)) & \longrightarrow 0 \\
 & \downarrow{\scriptstyle J''(\rho_1)} & & \downarrow{\scriptstyle J''(\rho_2)} & & \downarrow{\scriptstyle J''(\rho_3)} & \\
 & J''(\widetilde{KO}(B)) & \longrightarrow & J''(\widetilde{KO}(A)) & \longrightarrow & J''(\widetilde{KO}(C)) & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 & .
\end{array}
$$

It follows from the first part of the proof that $J''(\ker\rho_1)\cong 0$. Thus we have

$$\tilde{J}(A)\cong J''(\widetilde{KO}(A))\cong J''(\tilde{K}(A)).$$

This completes the proof of the lemma.

5. Proofs of the other parts of Theorem

In this section we complete the proof of Theorem. We begin with the part (1). Since $\widetilde{KO}(S^{2r+1}(L_0^t(p)/L_0^n(p)))\cong 0$, we have

$$\tilde{J}(S^{2r+1}(L_0^t(p)/L_0^n(p)))\cong 0.$$

This proves the part (1).

Since $2\widetilde{KO}(S^{2n+2r+3})\cong 0$ and $\widetilde{KO}(S^{2r}(L_0^t(p)/L_0^{n+1}(p))$ has an odd order by Lemma 2.5, by making use of the exact sequences of the triple $(L_0^t(p), L_0^{n+1}(p),$

$L^n(p)$), we have an isomorphism

$$\widetilde{KO}(S^{2r+1}(L_0^t(p)/L^n(p)))\cong\widetilde{KO}(S^{2n+2r+3}). \tag{5.1}$$

Hence we obtain

$$\tilde{J}(S^{2r+1}(L_0^t(p)/L^n(p)))\cong\tilde{J}(S^{2n+2r+3}).$$

This proves the part (3).

Similarly, by making use of the exact sequences of the triple ($L^t(p)$, $L_0^t(P)$, $L_0^n(p)$), we have an isomorphism

$$\tilde{J}(S^{2r+1}(L^t(p)/L_0^n(p)))\cong\tilde{J}(S^{2t+2r+2}).$$

This proves the part (5).

We now turn to the part (7). Put $A=S^{2r+1}(L^t(p)/L^n(p))$, $B=S^{2r+1}(L_0^t(p)/L^n(p))$, $C=S^{2t+2r+2}$, and $D=S^{2r+1}(L_0^{t+1}(p)/L^n(p))$. Then, we have the commutative diagram

$$\begin{tikzcd}
 & & \widetilde{KO}(D) \arrow[r, "i"] \arrow[d] & \widetilde{KO}(B) \arrow[d, equal] & \\
0 \arrow[r] & \widetilde{KO}(C) \arrow[r] & \widetilde{KO}(A) \arrow[r] & \widetilde{KO}(B) \arrow[r] & 0
\end{tikzcd}$$

where the row is exact. It follows from (5.1) that i is an isomorphism and $\widetilde{KO}(B)\cong\widetilde{KO}(S^{2n+2r+3})$. Hence the row sequence splits as an exact sequence of ψ-groups. Thus we have

$$\begin{aligned}\tilde{J}(S^{2r+1}(L^t(p)/L^n(p)))&\cong\tilde{J}(C)\oplus\tilde{J}(B)\\&\cong\tilde{J}(S^{2t+2r+2})\oplus\tilde{J}(S^{2n+2r+3})\end{aligned}$$

This proves the part (7).

We now turn to the part (8). Put $A=S^{2r}(L^t(p)/L^n(p))$, $B=S^{2r}(L_0^t(p)/L^n(p))$, $C=S^{2t+2r+1}$, $D=S^{2r}(L_0^{t+1}(p)/L^n(p))$, and $E=S^{2r}(L_0^{t+1}(p)/L_0^t(p))$. Then we have the commutative diagram

$$\begin{tikzcd}
 & \widetilde{KO}(E) \arrow[r] \arrow[d, "j"] & \widetilde{KO}(D) \arrow[r] \arrow[d] & \widetilde{KO}(B) \arrow[d, equal] & \\
0 \arrow[r] & \widetilde{KO}(C) \arrow[r] \arrow[d, "\delta_1"] & \widetilde{KO}(A) \arrow[r] \arrow[d, "\delta_2"] & \widetilde{KO}(B) \arrow[r] & 0 \\
 & \widetilde{KO}(C) \arrow[r, equal] & \widetilde{KO}(C) & &
\end{tikzcd}$$

of exact sequences. Since $\widetilde{KO}(E)$ has an odd order and $2\widetilde{KO}(C)\cong 0$, we have

$j(\widetilde{KO}(E))\cong 0$. This implies that δ_1 is an isomorphism. Hence the middle row sequence of the diagram splits as an exact sequence of ψ-groups. Therefore we have

$$\tilde{J}(S^{2r}(L^t(p)/L^n(p)))\cong\tilde{J}(S^{2r}(L_0^t(p)/L^n(p)))\oplus\tilde{J}(S^{2t+2r+1}).$$

This proves the part (8).

Finally we note that the proof of part (6) is similar to that of part (8). q.e.d.

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