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ON PRIME RIGHT IDEALS OF INTERMEDIATE RINGS OF A FINITE NORMALIZING EXTENSION

TAICHI NAKAMOTO

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Introduction

Throughout this paper, S will represent a ring extension of a ring R with common identity 1. Let I be a right ideal of R, and $b_R(I)=\{r\in R \mid Rr\subset I\}$. I is called a prime right ideal, provided that if X, Y are right ideals of R with $XY\subset I$, then either $X\subset I$ or $Y\subset I$. It is clear that every maximal right ideal is a prime right ideal. If I is a prime right ideal, then $b_R(I)$ is a prime ideal. Next, let R' be a ring. An R-R'-bimodule M is called a *torsionfree* R-R'-bimodule if $r_M(X)=l_M(Y)=0$ for every essential ideal X of R and every essential ideal Y of R', where $r_M(X)$ (resp. $l_M(Y)$) is the right (resp. left) annihilator of X (resp. Y) in M, and M is called a *finite normalizing* R-R'-bimodule if there exist elements $a_1, a_2, \cdots, a_n$ of M such that $M=\sum_{i=1}^{n} Ra_i$ and $Ra_i=a_iR'$ for $i=1, 2, \cdots, n$. Such a system $\{a_i\}_i$ is called a normailzing generating system of M. Finally S is a *finite normalizing extension* of R if S is a finite normalizing R-R-bimodule.

In [1], [2], [3], [4] and [6], "cutting down" theorems for a prime ideals were studied. In the previous paper [7], we have obtained a "cutting down" theorem for a prime right ideal of a finite normalizing extension under the hypothesis that the finite normalizing extension considered is torsionfree. The present objective is to reprove the same without the hypothesis "torsionfree"; we shall prove the following theorem.

Theorem. *Let S be an arbitrary finite normalizing extension of R, T a ring with $R\subset T\subset S$. If J is a prime right ideal of T, then there exist prime right ideals $K_1, K_2, \cdots, K_s$ of R such that $\cap_{i=1}^{s} K_i=J\cap R$. In this case, $b_R(J\cap R)=\cap_{i=1}^{s} b_R(K_i)$.*

1. Preliminaries

Throughout this paper, S will represent a finite normalizing extension of R, and T a ring with $R\subset T\subset S$.

Let P be a prime ideal of T. In studying P and T/P, one can usually reduce problems to the case in which (1) S is a prime ring, and (2) $A\cap T\not\subset P$ for each non-zero ideal A of S; this case being described as a standard setting for P. Actually, in view of [3, Proposition 2.2], there exists a prime ideal Q of S with $Q\cap T\subset P$ such that, with identifications of subrings, $R/(Q\cap R)\subset T/(Q\cap T)\subset S/Q$ gives a standard setting for $P/(Q\cap T)$.

To our end, we quote the following results from [1], [2], and [3].

Proposition 1.1 ([1, Theorem 2.11] and [3, Theorem 2.13 and Proposition 2.14]). *Let P be a prime ideal of T for which the case is a standard setting. Then*

(1) *R is a semiprime ring,*

(2) *there exists a set $\{P_1, P_2, \cdots, P_m\}$ of at most n (=the number of normalizing generators of S over R) prime ideals of R such that $\cap_{i=1}^{m} P_i=0$ and the prime rings R/P_i are all isomorphic, and*

(3) *there exists a subset $\{P_{i_k}\}$ of $\{P_1, P_2, \cdots, P_m\}$ such that $P\cap R=\cap_k P_{i_k}$.*

Proposition 1.2 ([1, Propositions 3.3 and 5.3, and Lemma 5.2]). *Let S be a prime ring. Then*

(1) *S embeds in the right Martindale quotient ring $Q(S)$ of S,*

(2) *there exist orthogonal idempotents $f_1, f_2, \cdots, f_m$ in $V_{Q(S)}(R)$ such that $f_1+f_2+\cdots+f_m=1$ and $r_R(f_i)=P_i$ for all $i=1, 2, \cdots, m$, and,*

(3) *$f_iQ(S)f_j$ is a torsionfree f_iR-f_jR-bimodule and f_iSf_j is a torsionfree finite normalizing f_iR-f_jR-bimodule.*

Proposition 1.3 ([2, Corollary 2.25 and Theorme 4.6]). *Let S be a prime ring, and $Q(S)$ the right Martindale quotient ring of S. Let f_i be as in Proposition 1.2 and put*

$$\begin{aligned} S_{ij} &= S\cap f_iQ(S)f_j = S\cap f_iSf_j, \\ T_{ij} &= T\cap f_iQ(S)f_j = T\cap f_iTf_j, \\ S_i &= S_{ij}+f_iR, \\ T_i &= T_{ii}+f_iR, \text{ and} \\ T^\# &= \textstyle\sum_{i,j=1}^{m} T_{ij} \quad (i, j = 1, 2, \cdots, m)\,. \end{aligned}$$

Then

(1) *T_i and S_i are rings,*

(2) *$f_iR\subset T_i\subset S_i\subset f_iSf_i\subset f_iQ(S)f_i$,*

(3) *$T_{ii}\subset S_{ii}\subset f_iSf_i$,*

(4) *$T_{ii}T^\#T\subset f_iT^\#T\subset T$, $RT^\#\subset T^\#$, $f_iRT^\#T\subset f_iT^\#T$ and $T_iT^\#T\subset f_iT^\#T$,*

(5) *$T^\#$ is an essential R-R-subbimodule of T, and*

(6) *there exists a non-zero ideal U of S such that $0\neq U\cap T\subset T^\#$.*

2. Proof of Theorem

Let J be a prime right ideal of T. Then $b_T(J)$ is a prime ideal of T. As was claimed at the opening of §1, in order to prove our cutting down theorem, we may assume that S is prime and the setting is standard for $b_T(J)$. Thus throughout this section, we keep the notations employed in §1. Furthermore, we set $h_i(J)=\{t_i\in T_i \mid t_i f_i T^* T\subset J\}$, which is a right ideal of T_i.

Lemma 2.1. *$h_i(J)=T_i$ if and only if $f_iT^*T\subset J$.*

Proof. If $h_i(J)=T_i$, then we have $f_iT^*T\subset T_i f_iT^*T\subset J$. Conversely, if $f_iT^*T\subset J$, then $T_iT^*T\subset f_iT^*T\subset J$ (by Proposition 1.3). Hence $T_i\subset h_i(J)$, and so $h_i(J)=T_i$.

Lemma 2.2. *The set $\{i \mid h_i(J)\neq T_i\}$ is not empty.*

Proof. If $h_i(J)=T_i$ for all $i=1, 2, \cdots, m$, then we have $T^*\subset f_1T^*T+f_2T^*T+\cdots+f_mT^*T\subset J$ (Lemma 2.1). But, by Proposition 1.3, there exists a non-zero ideal U of S such that $U\cap T\subset T^*\subset J$, which contradicts the setting being standard for $b_T(J)$.

We now reorder, if necessary, so that $f_1T^*T\not\subset J$ for $i=1, 2, \cdots, s$, and $f_iT^*T\subset J$ for $i=s+1, \cdots, m$.

Lemma 2.3. $b_T(J)\cap R\subset \cap_{i=1}^{s}P_i$.

Proof. By Proposition 1.3, there exists a non-zero ideal U of S such that $0\neq U\cap T\subset T^*$. Therefore, since the setting is standard for $b_T(J)$, for $i=1, 2, \cdots, s$, we have $(U\cap T)\cdot f_iT^*T\not\subset b_T(J)$, and so $TT^*f_i f_iT^*T\not\subset b_T(J)$. Let us set $Q'_{(i)}=\{t_i\in T_i \mid TT^*f_i\, t_i f_iT^*T\subset b_T(J)\}$ for each $i=1, 2, \cdots, s$. Then, as is well known, by the correspondence of prime ideals in a Morita context

$$C_i=\begin{pmatrix} T & TT^*f_i \\ f_iT^*T & T_i \end{pmatrix},$$

$Q'_{(i)}$ is a prime ideal of T_i corresponding to the prime ideal $b_T(J)$ of T. By [3, Proposition 2.11], we have $Q'_{(i)}\cap f_iR=0$. Since $TT^*f_i\cdot(b_T(J)\cap R)\cdot f_iT^*T\subset Tb_T(J)T\subset b_T(J)$, we obtain $f_i(b_T(J)\cap R)f_i\subset f_iR\cap Q'_{(i)}=0$, and hence $b_T(J)\cap R\subset r_R(f_i)=P_i$. Hence $b_T(J)\cap R\subset \cap_{i=1}^{s}P_i$.

Lemma 2.4. *If $i\leqq s$, then $h_i(J)$ is a prime right ideal of T_i and $A\cap T_i\not\subset h_i(J)$ for each non-zero ideal A of S_i.*

Proof. Let a, b be elements of T_i with $aT_ib\subset h_i(J)$ and $b\notin h_i(J)$. By Proposition 1.3, there exists a non-zero ideal U of S such that $0\neq U\cap T\subset T^*$. Then, since $a\cdot f_i T^*T(U\cap T)f_i\cdot bf_i T^*T\subset af_i T^*TT^*f_i bf_i T^*T\subset aT_i bf_i T^*T\subset J$, we have either $af_i T^*T\subset J$ or $(U\cap T)bf_i T^*T\subset J$. But, noting that $U\cap T\not\subset J$ and $bf_i T^*T\not\subset J$, we get $(U\cap T)f_ibf_i T^*T\not\subset J$. Hence $af_i T^*T\subset J$, and so $a\in h_i(J)$. We have thus seen that $h_i(J)$ is a prime right ideal of T_i. Next we claim that

$b_{T_i}(h_i(J))\cap f_iR=0$. Let $f_ir\in b_{T_i}(h_i(J))\cap f_iR\,(r\in R)$. Then $f_iT^*TT^*f_irf_iT^*T\subset T_if_irf_iT^*T\subset J$, and so, $f_iT^*T\not\subset J$ implies $TT^*f_irf_iT^*T\subset J$, and therefore $TT^*f_irf_iT^*T\subset b_T(J)$. Hence $f_ir\in Q'_{(i)}\cap f_iR=0$, by the proof of Lemma 2.3 ([3, Prop. 2.11]). We have thus seen that $b_{T_i}(h_i(J))\cap f_iR=0$. Finally, if A is a non-zero ideal of S_i such that $A\cap T_i\subset h_i(J)$, then $0\neq A\cap f_iR=(A\cap T_i)\cap f_iR\subset b_{T_i}(h_i(J))\cap f_iR$ by [2, Proposition 2.20], But this contradicts $b_{T_i}(h_i(J))\cap f_iR=0$. This proves that $A\cap T_i\not\subset h_i(J)$ for each non-zero ideal A of S_i.

Lemma 2.5. *If I is a non-zero ideal of f_iR, then there exists a non-zero ideal A of S such that $A\cap f_iAf_i\cap T_i\subset IT_i$.*

Proof. If M is an R-R-subbimodule of T_i with $IT_i\cap M=0$, then $IM\subset IT_i\cap M=0$. Since f_iR is a prime ring and T_i is f_iR-f_iR-torsionfree (Proposition 1.1 and 1.2), we have $M=0$, and therefore IT_i is an essential R-R-subbimodule of T_i. Now, choose a relative complement T_i^* of T_i in the R-R-bimodule $Q(S)$. Noting that IT_i is R-R-essential in T_i, we see that $IT_i\oplus T_i^*$ is R-R-essential in $Q(S)$, so that $(IT_i\oplus T_i^*)\cap S$ is R-R-essential in S. Then, by [3, Corollary 2.25], there exists a non-zero ideal A of S such that $A\subset(IT_i\oplus T_i^*)\cap S\,(\subset IT_i\oplus T_i^*)$. Now, it is easy to see that $A\cap f_iAf_i\cap T_i\subset IT_i$.

Corollary 2.6. *If $i\leqq s$, then $h_i(J)\cap f_iR$ is a prime right ideal of f_iR and $b_{f_iR}(h_i(J)\cap f_iR)=0$.*

Proof. Let X, Y be right ideals of f_iR such that $XY\subset h_i(J)\cap f_iR$ and $Y\not\subset h_i(J)\cap f_iR$. Then f_iRY is a non-zero ideal of f_iR, and so there exists a non-zero ideal A of S such that $A\cap f_iAf_i\cap T_i\subset f_iRYT_i$ (Lemma 2.5). Since $A\cap f_iAf_i$ is a non-zero ideal of S_i ([2, Proposition 2.22]), $A\cap f_iAf_i\cap T_i\not\subset h_i(J)$ by Lemma 2.4. Therefore, since $XT_i(A\cap T_{ii})\subset X(f_iRYT_i\cap T_{ii})\subset Xf_iRYT_i\subset h_i(J)$, we see that $XT_i\subset h_i(J)$, and therefore $X\subset h_i(J)\cap f_iR$. This proves that $h_i(J)\cap f_iR$ is a prime right ideal of f_iR. Next, suppose, to the contrary, that $b_{f_iR}(h_i(J)\cap f_iR)\neq 0$. Then, again by Lemma 2.5 and [2, Proposition 2.22], there exists a non-zero dideal B of S such that $B\cap f_iBf_i\cap T_i\subset b_{f_i}(h_i(J)\cap f_iR)\,T_i\subset h_i(J)$ and $B\cap f_iBf_i$ is a non-zero ideal of S_i. But this contradicts Lemma 2.4.

Now, we shall prove the final Lemma which implies our Theorem.

Lemma 2.7. *There exist prime right ideals $K_1, K_2, \cdots, K_s$ of R such that $\cap_{i=1}^s K_i=J\cap R$ and $b_R(K_i)=P_i$. In this case, $b_R(J\cap R)=\cap_{i=1}^s P_i$.*

Proof. We now set $K_i=\{r\in R\mid f_ir\in h_i(J)\cap f_iR\}$ $(i=1, 2, \cdots, s)$. Then, by Corollary 2.6, we can easy seen that K_i is a prime right ideal of R and $b_R(K_i)=r_R(f_i)=P_i$. If $r\in J\cap R$, then $f_irf_iT^*T\subset rT^*T\subset J$, and so $f_ir\in h_i(J)\cap f_iR$. This implies that $J\cap R\subset\cap_{i=1}^s K_i$. Conversely, let r be arbitrary element of $\cap_{i=1}^s K_i$. Then, for each $i\leqq s$, $f_ir\in h_i(J)\cap f_iR$, and so $f_irf_iT^*T\subset J$. On the other hand, noting that $f_iT^*T\subset J$ for $i\geqq s+1$, it follows that $rT^*T\subset$

$\sum_{i=1}^{s} f_i r f_i T^{\sharp} T + \sum_{i=s+1}^{m} f_i r f_i T^{\sharp} T \subset J + \sum_{i=s+1}^{m} f_i T^{\sharp} T \subset J$. In view of Proposition 1.3, there exists a non-zero ideal U of S such that $U \cap T \subset T^{\sharp}$. Then $r\cdot(U \cap T) \subset rT^{\sharp}T \subset J$ and $U \cap T \not\subset J$. (Note that the setting is standard for $b_T(J)$.) Hence $r \in J \cap R$. We have thus seen that $J \cap R = \bigcap_{i=1}^{s} K_i$. Furthermore $b_R(J \cap R) \subset \bigcap_{i=1}^{s} b_R(K_i) \subset \bigcap_{i=1}^{s} K_i = J \cap R$. This implies that $b_R(J \cap R) = \bigcap_{i=1}^{k} b_R(K_i) = \bigcap_{i=1}^{s} P_i$, and our proof is complete.

3. Examples

In §2, we have proved a "cutting down" theorem for a prime right ideal of a finite normalizing extension. On the other hand, from Heinicke and Robson [3, Theorem 2.12], we obtain another "cutting down" theorem for a prime right ideal. That is, if J is a prime right ideal of T, then there exist right ideals $H_1, H_2, \cdots, H_h$ of R such that $\bigcap_{i=1}^{h} H_i = J \cap R$ and each $H_i/(J \cap R)$ is a prime right R-module, where a right R-module N is called prime, provided that if $uI=0$ for $0 \neq u \in N$ and an ideal I of R then $NI=0$. In this section, we give some examples which show that these two expressions are essentially different.

In advance of giving examples, we claim the following: Let k be a field, $U=k[x_1, x_2, \cdots]$ a polynomial ring over k in countable many indeterminate x_i, and $B=(x_1, x_2, \cdots)$ the maximal two-sided ideal of U generated by $x_1, x_2, \cdots$. Let us set $V=U/B^2$, $W=B/B^2$, $D=\begin{pmatrix} V & V \\ V & V \end{pmatrix}$ and $A=\begin{pmatrix} W & W \\ W & W \end{pmatrix}$. Then the unique maximal non-zero two-sided ideal A of D is neither left nor right D-finitely generated.

EXAMPLE 3.1. Let D be a ring containing a non-zero unique maximal two-sided ideal A of D which is neither left nor right D-finitely generated. Let M be a maximal right ideal of D with $b_D(M)=A$. Let

$$S=\begin{pmatrix} D & D & D \\ D & D & D \\ D & D & D \end{pmatrix},\ R=\begin{pmatrix} D & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{pmatrix},\ T=\begin{pmatrix} D & D & A \\ D & D & A \\ D & D & D \end{pmatrix} \text{ and } J=\begin{pmatrix} M & M & A \\ M & M & A \\ D & D & D \end{pmatrix}.$$

Then S is a finite normalizing extension of R, T is not a finite normalizing extension of R, and J is a prime right ideal of T. Let us set

$K_1=\begin{pmatrix} M & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{pmatrix}$ and $K_2=\begin{pmatrix} D & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & D \end{pmatrix}$. Then K_1 and K_2 are prime right ideal of R with $J \cap R = K_1 \cap K_2$, and $K_i/(J \cap R)$ are prime right R-modules.

EXAMPLE 3.2. Let D, A and M be as in Example 3.1. Let us set

$$S=\begin{pmatrix} D & D & D & D \\ D & D & D & D \\ D & D & D & D \\ D & D & D & D \end{pmatrix},\ R=\begin{pmatrix} D & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & D & 0 \\ 0 & 0 & 0 & D \end{pmatrix},\ T=\begin{pmatrix} D & D & D & A \\ D & D & D & A \\ D & D & D & A \\ D & D & D & D \end{pmatrix} \text{ and } J=\begin{pmatrix} M & M & M & A \\ M & M & M & A \\ M & M & M & A \\ D & D & D & D \end{pmatrix}.$$

Then S is a finite normalizing extension of R, T is not a finite normalizing extension of R, and J is a prime right ideal of T. Let us put

$$K_1=\begin{pmatrix} M & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & D & 0 \\ 0 & 0 & 0 & D \end{pmatrix}, \quad K_2=\begin{pmatrix} D & 0 & 0 & 0 \\ 0 & M & 0 & 0 \\ 0 & 0 & D & 0 \\ 0 & 0 & 0 & D \end{pmatrix} \quad \text{and} \quad K_3=\begin{pmatrix} D & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & M & 0 \\ 0 & 0 & 0 & D \end{pmatrix}.$$

Then K_1, K_2 and K_3 are the prime right ideals of R such that $J\cap R=K_1\cap K_2\cap K_3$. But the $K_i/(J\cap R)$ are not prime right R-modules. Next, we consider the following ideals:

$$H_1=\begin{pmatrix} D & 0 & 0 & 0 \\ 0 & M & 0 & 0 \\ 0 & 0 & M & 0 \\ 0 & 0 & 0 & D \end{pmatrix}, \quad H_2=\begin{pmatrix} M & 0 & 0 & 0 \\ 0 & D & 0 & 0 \\ 0 & 0 & M & 0 \\ 0 & 0 & 0 & D \end{pmatrix} \quad \text{and} \quad H_3=\begin{pmatrix} M & 0 & 0 & 0 \\ 0 & M & 0 & 0 \\ 0 & 0 & D & 0 \\ 0 & 0 & 0 & D \end{pmatrix}.$$

Then $H_1\cap H_2\cap H_3=J\cap R$ and $H_i/(J\cap R)$ are prime right R-modules. But it is easy to see that none of H_i is a prime right ideal.

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Department of Applied Mathematics
Okayama University of Science
Ridaicho, Okayama-Shi
Okayama 700, Japan