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ON PROJECTIVE MODULES OVER DIRECTLY FINITE REGULAR RINGS SATISFYING THE COMPARABILITY AXIOM II

Dedicated to Professor Hisao Tominaga on his 60th birthday

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In [4], by observing the directly finiteness of projective modules, the first author classified directly finite (d.f. for short) regular rings satisfying the comparability axiom (c. axiom for short) into three types: Type A, Type B and Type C.

In the present paper, we give a more explicite criterion of the directly finiteness of projective modules over each types and show the following for a d.f. regular ring R satisfying the c. axiom: (a) R is Type A if and only if $Soc(R)=0$ and the intersection $I_0(R)$ of all nonzero ideals of R is nonzero. (b) R is Type B if and only if $Soc(R)=0$, $I_0(R)=0$ and the family $L(R)$ of all ideals of R has a cofinal subfamily. (c) R is Type C if and only if $Soc(R)\neq 0$, or $I_0(R)=0$ and $L(R)$ does not have any cofinal subfamilies. As an application we show the following for a projective module P over a d.f. regular ring satisfying the c. axiom: P is directly infinite (d.inf. for short) if and only if P contains a direct summand which is isomorphic to $\aleph_0 X$ for a suitable nonzero module X.

Throughout this paper we assume that R is a d.f. regular ring satisfying the c. axiom, and all R-modules considered are unital right R-modules.

1. Notations and definitions

For two R-modules X and Y, we use $X\lesssim Y$ (resp. $X\lesssim\oplus Y$) to mean that X is isomorphic to a submodule of Y (resp. a direct summand of Y). $X\precnsim Y$ means that $X\lesssim Y$ and $X\ncong Y$. For a submodule X of an R-module Y, $X<\oplus Y$ means that X is a direct summand of Y. For a cardinal number α and an R-module X, αX denotes a direct sum of α-copies of X. For a set I, we denote by $|I|$ the cardinal number of I. We denote by $L(R)$ the family of all ideals of R. Since R satisfies the c. axiom, $L(R)$ is a linearly ordered set under inclusion ([1, Proposition 8.5]). We put $I_0(R)=\cap\{I\mid 0\neq I\in L(R)\}$. We denote by $Soc(R)$ the socle of R. We note that if $Soc(R)\neq 0$ then it is homogeneous and coincides with $I_0(R)$.

The reader is reffered to K.R. Goodearl [1] for the following elementary properties on R;

i) Every finitely generated projective R-module is d.f..

ii) For any finitely generated projective R-modules P and Q, either $P \lesssim Q$ or $Q \lesssim P$ holds.

iii) For any finitely generated projective R-module P and any R-modules X and Y, $P \oplus X \cong P \oplus Y$ implies $X \cong Y$.

iv) For any projective R-module X and any finitely generated projective R-modules $Y_1, Y_2, \cdots$ such that $Y_1 \oplus \cdots \oplus Y_n \lesssim X$ for all n, we have that $\oplus Y_n \lesssim X$. In particular, for finitely generated projective R-modules P and Q, if $P \not\lesssim nQ$ for all n, then $\aleph_0 Q \lesssim P$.

v) R has a unique *dimension function* D, namely, D is a function from the family of all cyclic right ideals of R to $[0, 1]$ such that

a) $D(R)=1$,

b) if $J \lesssim K$, then $D(J) \leqq D(K)$ and

c) if $J \oplus K$ is a cyclic right ideal of R, then $D(J \oplus K)=D(J)+D(K)$. D is said to be *strictly positive* if $D(J)>0$ for all nonzero cyclic right ideals J of R. We note that D is strictly positive dimension function if and only if R is a simple ring, and that R is not simple if and only if there exists a nonzero r in R such that $\aleph_0(rR) \lesssim R$. Furthermore we note that $\{r \in R \mid D(rR)=0\}$ is the unique maximal ideal of R.

Let $\{P_i\}_{i=1}^{\infty}$ be a subfamily of the family of all cyclic projective R-modules. We say that $\{P_i\}_{i=1}^{\infty}$ is a *cofinal* subfamily if all P_i are nonzero, $P_1 \gtrsim P_2 \gtrsim \cdots$ and for any nonzero cyclic projective R-module X there exists a positive integer n satisfying $X \gtrsim\!\!\!\!\!\!\not\;\; P_n$. Similarly a subfamily $\{I_i\}_{i=1}^{\infty}$ of $L(R)$ is said to be a *cofinal* subfamily of $L(R)$ if all I_i are nonzero, $I_1 \geqq I_2 \geqq \cdots$ and for any nonzero X in $L(R)$ there exists a positive integer n satisfying $X \gneqq I_n$.

2. Directly finiteness and directly infiniteness

We start the following

Theorem 2.1. (*a*) *For countably generated projective R-modules P and Q, either $P \lesssim Q$ or $Q \lesssim P$ holds.*

(*b*) *If P and Q are countably generated projective R-modules such that $P \lesssim Q$ and $Q \lesssim P$, then $P \cong Q$.*

Proof. We show the theorem by modifying the proof of [3, Lemma 2.5]. Let $P=\oplus_{i=1}^{\infty} P_i$ and $Q=\oplus_{i=1}^{\infty} Q_i$ be cyclic decompositions of P and Q. (a) Assume $P \not\lesssim Q$. Then there exists a positive integer n such that $P_1 \oplus \cdots \oplus P_n \not\lesssim Q$ and so $Q_1 \oplus \cdots \oplus Q_m \lesssim P_1 \oplus \cdots \oplus P_n$ for all m. Therefore $Q \lesssim P_1 \oplus \cdots \oplus P_n < \oplus P$. (b) It is sufficient to assume that P and Q are non-finitely generated projective and that $Q \not\lesssim P_1 \oplus \cdots \oplus P_n$ for all n from the assumption. Since $Q \lesssim P$, there exists a positive integer n_1 such that $Q_1 \lesssim P_1 \oplus \cdots \oplus P_{n_1}$ and $Q_1 \not\lesssim P_1 \oplus \cdots \oplus P_{n_1-1}$, and there exists a positive integer m_1 such that $Q_1 \oplus \cdots \oplus Q_{m_1} \lesssim P_1 \oplus \cdots \oplus P_{n_1}$ and $Q_1 \oplus \cdots \oplus Q_{m_1+1} \not\lesssim P_1 \oplus \cdots \oplus P_{n_1}$. Then we have a decomposition $Q_{m_1+1} = X_{m_1+1} \oplus Y_{m_1+1}$ such that $Y_{m_1+1} \neq 0$ and

$$P_1 \oplus \cdots \oplus P_{n_1} \cong Q_1 \oplus \cdots \oplus Q_{m_1} \oplus X_{m_1+1} .$$

Since $Q \lesssim P$, $Y_{m_1+1} \oplus Q_{m_1+2} \oplus \cdots \lesssim P_{n_1+1} \oplus P_{n_1+2} \oplus \cdots$. Then there exists a positive integer $n_2\,(>n_1)$ such that $Y_{m_1+1} \lesssim P_{n_1+1} \oplus \cdots \oplus P_{n_2}$ and $Y_{m_1+1} \not\lesssim P_{n_1+1} \oplus \cdots \oplus P_{n_2-1}$. So, there exists a positive integer $m_2(>m_1)$ such that $Y_{m_1+1} \oplus Q_{m_1+2} \oplus \cdots \oplus Q_{m_2} \lesssim P_{n_1+1} \oplus \cdots \oplus P_{n_2}$ and $Y_{m_1+1} \oplus Q_{m_1+2} \oplus \cdots \oplus Q_{m_2+1} \not\lesssim P_{n_1+1} \oplus \cdots \oplus P_{n_2}$. Then we have a decomposition $Q_{m_2+1} = X_{m_2+1} \oplus Y_{m_2+1}$ such that $Y_{m_2+1} \neq 0$ and

$$P_{n_1+1} \oplus \cdots \oplus P_{n_2} \cong Y_{m_1+1} \oplus Q_{m_1+2} \oplus \cdots \oplus Q_{m_2} \oplus X_{m_2+1} .$$

Continuing this procedure, we have that $P \cong Q$.

REMARK. When R is simple, we can drop the assumption 'countably generated' from above Theorem 2.1 (see [3, Theorems 2.4 and 2.6]). But the assumption can not be removed in general. For, if R is a non-simple d.f. regular ring satisfying the c. axiom, then there exists a nonzero r in R such that $\aleph_0(rR) \lesssim R$. So, $R \not\lesssim \alpha(rR)$ and $\alpha(rR) \not\lesssim R$, where $|R| < \alpha$. Next, if we take R as in [1, Example 5.15], then there exists a simple right ideal S of R such that $\aleph S \lesssim R$. Then $\aleph_0 R \lesssim \aleph S \oplus \aleph_0 R$ and $\aleph S \oplus \aleph_0 R \lesssim \aleph_0 R$, but $\aleph_0 R \not\cong \aleph S \oplus \aleph_0 R$.

Corollary 2.2. *Let P and Q be countably generated projective R-modules and let n be a positive integer.*

(*a*) *If $nP \cong nQ$, then $P \cong Q$.*

(*b*) *If $nP \lesssim nQ$, then $P \lesssim Q$.*

Proof. (a) We prove the statement by induction on n. So, assume that this holds for n and $(n+1)P \cong (n+1)Q$. Then we have decompositions $nP = X_1 \oplus X_2$ and $P = Y_1 \oplus Y_2$ such that $X_1 \oplus Y_1 \cong nQ$ and $X_2 \oplus Y_2 \cong Q$. By Theorem 2.1 (a), either $Y_1 \lesssim X_2$ or $X_2 \lesssim Y_1$ holds. If $Y_1 \lesssim X_2$, then $nQ \cong X_1 \oplus Y_1 \lesssim X_1 \oplus X_2 = nP$ and $P = Y_1 \oplus Y_2 \lesssim X_2 \oplus Y_2 \cong Q$; so $nQ \lesssim nP$ and $nP \lesssim nQ$. Hence the induction hypothesis says that $P \cong Q$. If $X_2 \lesssim Y_1$, similarly, we have that $P \cong Q$. (b) Assume that $nP \lesssim nQ$ and $P \not\lesssim Q$. Then $Q \lesssim P$, and so $nQ \lesssim nP$. Therefore $nP \cong nQ$ by Theorem 2.1 (b); whence $P \cong Q$ by (a), a contradiction.

Now, for our purpose we define a relation "$\sim$" on the family of all cyclic projective R-modules $CP(R)$ by the rule: For any P and Q in $CP(R)$, $P \sim Q$ if and only if $P \lesssim mQ$ and $Q \lesssim nP$ for some positive integers m and n. For P in $CP(R)$ we put $[P] = \{Q \in CP(QR) \mid Q \sim P\}$. Then the relation "$\sim$" is a congruence relation.

Proposition 2.3. *Let P be a non-finitely generated, countably generated projective R-module with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ such that $P_1\gtrsim P_2\gtrsim\cdots$ and $[P_1]=[P_2]=\cdots$. If $\{P_i\}_{i=1}^{\infty}$ is cofinal, then P is d.inf. if and only if $P\cong\aleph_0 P_i$ for all i.*

Proof. "If" part is clear. "Only if" part. Let P be a d.inf. projective R-module. From [4, Theorem 6], $tP_i\not\gtrsim P_{i+1}\oplus P_{i+2}\oplus\cdots$ for all t and hence $\aleph_0 P_i\lesssim P_{i+1}\oplus P_{i+2}\oplus\cdots\leqq P$. Since $P\lesssim\aleph_0 P_i$, it follows from Theorem 2.1 that $P\cong\aleph_0 P_i$ for all i.

REMARK. Let P be a non-finitely generated, countably generated projective R-module with a cyclic decomposition $P=\oplus_{i\in I}P_i$ such that $|I|=\aleph_0$ and $[P_i]=[P_j]$ for any $i,j\in I$. If there exists $i\in I$ such that $|\{j\in I\,|\,P_i\lesssim P_j\}|=\aleph_0$, then $P\cong\aleph_0 P_i$ for all $i\in I$ by Theorem 2.1.

Proposition 2.4. *Let P be a non-countably generated (d.inf.) projective R-module with a cyclic decomposition $P=\oplus_{\alpha\in I}P_\alpha$ such that $[P_\alpha]=[P_\beta]$ for any α, $\beta\in I$. Then there exists an infinite cardihal τ ($>\aleph_0$) such that $P\cong\tau P_\gamma$ for any $\gamma\in I$.*

Proof (cf. the proof of [3, Theorem 2.6]). Let B be the set of all countably infinite subsets of I, and let $\gamma\in I$. We consider the family consisting of all subsets F of B satisfying the following properties:

(1) each member of F is pairwise disjoint, and

(2) for each member I' of F, $P_{I'}=\oplus_{\alpha\in I'}P_\alpha\cong\aleph_0 P_\gamma$.

Then this family is non-empty set from the proof of [4, Theorem 6] and Remark of Proposition 2.3. Since this family is inductively ordered set under inclusion, there exists a maximal member F by Zorn's Lemma. Put $I^*=\cup_{K\in F}K$. If $I^*=I$, then our proof is complete. Next, consider the case that $I^*\neq I$. Let I^{**} be a complement of I^* in I. From the proof of [4, Theorem 6], Remark of Proposition 2.3 and the maximality of F, I^{**} is a countable set. Choose one member K' of F and put $F'=F-\{K'\}$ and $K''=K'\cup I^{**}$. Then K'' is a countably infinite set and $P_{K''}\cong\aleph_0 P_\gamma$ since $[P_\alpha]=[P_\beta]$ for any $\alpha,\beta\in I$. Therefore $P=(\oplus_{K\in F'}(\oplus_{\alpha\in K}P_\alpha))\oplus(\oplus_{\alpha\in K''}P_\alpha)\cong\tau P_\gamma$ for some infinite cardinal $\tau>\aleph_0$.

Corollary 2.5 ([3, Theorem 2.6]). *Assume that R is simple. Then every d.inf. projective R-module is a free R-module.*

Proof. Let P be a d.inf. projective R-module with a cyclic decomposition $P=\oplus_{\alpha\in I}P_\alpha$ and we consider $R\oplus P=R\oplus(\oplus_{\alpha\in I}P_\alpha)$. Noting that R is simple, we see that $[X]=[R]$ for a nonzero cyclic projective R-module X; whence $[P_\alpha]=[R]$ for all $\alpha\in I$. By Proposition 2.3, its Remark and Proposition 2.4, $R\oplus P\cong\tau R$ for some infinite cardinal τ. Therefore $P\cong\tau R$ by the cancellation property of R.

Theorem 2.6. *Let P and Q be d.inf. projective R-modules with cyclic decom-*

positions $P=\oplus_{\alpha\in I}P_\alpha$ *and* $Q=\oplus_{\beta\in J}Q_\beta$ *such that* $[P_\alpha]=[P_{\alpha'}]$ *and* $[Q_\beta]=[Q_{\beta'}]$ *for any* α, $\alpha'\in I$ *and* β, $\beta'\in J$. *If* $P\lesssim Q$ *and* $Q\lesssim P$, *then* $P\cong Q$.

Proof. Since $P\lesssim Q$ and $Q\lesssim P$, we note that $[P_\alpha]=[Q_\beta]$ for any P_α and Q_β. If $|I|\leqq\aleph_0$ and $|J|\leqq\aleph_0$, then $P\cong Q$ from Theorem 2.1. Therefore we may consider the following cases:

1) $|I|\leqq\aleph_0$ and $|J|>\aleph_0$.
2) $|I|>\aleph_0$ and $|J|>\aleph_0$.

In order to prove for these cases, we show the following for any nonzero cyclic projective R-module T and cardinal numbers σ and ρ:

(#) If $\rho T\lesssim\sigma T$, then $\rho\leqq\sigma$.

Let τ be a cardinal number. We regard τ an initial ordinal; so $|\{\text{ordinal } \alpha\,|\,\alpha<\tau\}|=\tau$. Put $\Lambda(\tau)=\{\text{ordinal } \alpha\,|\,\alpha<\tau\}$. We shall prove (#) by the transfinite induction on σ. First assume that $\sigma=\aleph_0$ and let f be a monomorphism from ρT to $\sigma T=\aleph_0 T$. Putting $\Gamma_m=\{\alpha\in\Lambda(\rho)\,|\,f(T_\alpha)\leqq\oplus_{i=1}^m T_i\}$, we see that $f(\oplus_{\alpha\in\Gamma_m}T_\alpha)\leqq\oplus_{i=1}^m T_i$, $|\Gamma_m|\leqq m$ and $\cup_m\Gamma_m=\Lambda(\rho)$. Therefore $\rho=|\Lambda(\rho)|=|\cup_m\Gamma_m|\leqq\aleph_0=\sigma$. Next assume that (#) holds for any cardinal number $\sigma'<\sigma$, and let f be a monomorphism from ρT to σT. For any x in $\Lambda(\sigma)$, put $\Gamma_x=\{\alpha\in\Lambda(\rho)\,|\,f(T_\alpha)\leqq\oplus_{\beta\leqq x}T_\beta\}$. Then $f(\oplus_{\alpha\in\Gamma_x}T_\alpha)\leqq\oplus_{\beta\leqq x}T_\beta$, $\cup_{x\in\Lambda(\sigma)}\Gamma_x=\Lambda(\rho)$ and $|\{\beta\,|\,\beta\leqq x\}|<\sigma$ because σ is an initial ordinal. From the induction hypothesis, $|\Gamma_x|<\sigma$. Therefore we see that $\rho=|\Lambda(\rho)|=|\cup_{x\in\Lambda(\sigma)}\Gamma_x|\leqq\sigma^2=\sigma$ as desired.

Case 1) Let $P_\beta\in\{P_\alpha\}_{\alpha\in I}$. Since $|J|>\aleph_0$ and $[P_\alpha]=[Q_\beta]$ for all Q_β, $Q\cong\tau P_\alpha$ for a suitable cardinal number τ. Since $P\lesssim Q\cong\tau P_\alpha$ and $Q\lesssim P$, we see that $\aleph_0P_\alpha\lesssim P$ and $P\lesssim\aleph_0P_\alpha$, whence $\aleph_0P_\alpha\cong P$ by Theorem 2.1. As $\tau P_\alpha\cong Q\lesssim\aleph_0P_\alpha$, $\tau\leqq\aleph_0$ by (#), a contradiction.

Case 2) By Proposition 2.4 and (#), we immediately have that $P\cong Q$.

Corollary 2.7 ([3, Proposition 2.7]). *Assume that* R *is simple. If* P *and* Q *are d.inf. projective* R*-modules such that* $P\lesssim Q$ *and* $Q\lesssim P$, *then* $P\cong Q$.

3. Types A, B and C

In [4] we showed the following result, which already used in Proposition 2.3: A non-finitely generated projective R-module P is d.f. if and only if P is countably generated with a cyclic decomposition $P=\oplus_{i=1}^\infty P_i$ satisfying the conditions (*) and (A), or (*) and (B) below:

(*) $P_i\gtrsim P_{i+1}$ for all i, and there exists no nonzero R-module X such that $X\lesssim P_i$ for all i.

(A) There exists a positive integer m such that

(1) For each $i\geqq m$, $P_i\lesssim t_iP_{i+1}$ for some positive integer t_i, and

(2) $\oplus_{i=m}^\infty P_i\lesssim tP_m$ for some positive integer t.

(B) There exists an increasing sequence $1=i_1<i_2<\cdots$, of positive integers such that $P_{i_n}\gtrsim\aleph_0P_{i_{n+1}}$ for $n=1, 2, \cdots$.

And, from this result, we classified d.f. regular rings R satisfying the c. axiom into three types:

Type A: There exists a non-finitely generated d.f. projective R-module P with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ satisfying (*) and (A).

Type B: There exists a non-finitely generated d.f. projective R-module P with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ satisfying (*) and (B).

Type C: All d.f. projective R-modules are finitely generated.

REMARK. If a ring R is Type A (resp. Type B), then all non-finitely generated d.f. projective R-module P have a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ satisfying (*) and (A) (resp. (*) and (B)) by [4, Theorem 6 and Remark 2]. We note that (*) holds then $Soc(R)=0$. So, if $Soc(R)\neq 0$ then R is type C.

In this section, as is mentioned in the introduction, we shall give ideal theoritic characterizations of each types.

Lemma 3.1 ([1, Corollary 2.23]). *Let H and J be right ideals of R, and assume that H is finitely generated. Then $H\leqq RJ$ if and only if $H\lesssim nJ$ for some positive integer n.*

For an element a of a ring R, we put

$$\Sigma_a = \Sigma\{xR \mid x\in R \text{ and } xR\lesssim aR\} .$$

Lemma 3.2. *(a) For each $a\in R$, Σ_a is the smallest ideal of R containing a, and hence $\Sigma_a=RaR$.*

(b) For each $a, b\in R$, $\Sigma_a\leqq\Sigma_b$ if and only if $aR\lesssim n(bR)$ for some positive integer n.

(c) For $a, b\in R$, $\Sigma_a\lneqq\Sigma_b$ if and only if $\aleph_0(aR)\lesssim bR$.

Proof. (a) Let $r\in R$ and $\sum_{i=1}^{n}x_ir_i\in\Sigma_a$ such that $r_i\in R$ and $x_iR\lesssim aR$ for each i. Then $(rx_ir_i)R\lesssim\oplus(x_ir_i)R\leqq x_iR\lesssim aR$ and $rx_ir_i\in\Sigma_a$ for each i, and so $r(\sum_{i=1}^{n}x_ir_i)\in\Sigma_a$. Thus Σ_a is an ideal of R containing a. Let I be an ideal of R containing a. If $xR\lesssim aR$ and $x\in R$, then $xR\leqq RaR\leqq I$ from Lemma 3.1. Therefore $\Sigma_a\leqq I$ and hence $\Sigma_a=RaR$. (b) is clear from (a) and Lemma 3.1, and (c) follows from (b).

Theorem 3.3. *The following are equivalent:*

(a) R is Type A.

(b) $Soc(R)=0$ and $I_0(R)\neq 0$.

(c) There exists a non-finitely generated d.f. projective R-module P with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ such that $\{P_i\}_{i=1}^{\infty}$ is cofinal, $[P_1]=[P_2]=\cdots$ and $tP_1\gtrsim P_2\oplus P_3\oplus\cdots$ for some positive integer t.

Proof. (a)$\rightarrow$(b). Assume that Type A. Then of course $Soc(R)=0$. Now assume that $I_0(R)=0$. Since R is Type A, we have a non-finitely generated d.f. projective R-module P with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ which satisfies (*) and $[P_m]=[P_{m+1}]=\cdots$ for some positive integer m. Let $P_i\cong x_iR$ for some $x_i\in R$. Noting that $I_0(R)=0$, we have a nonzero ideal X of R such that $X\lneqq\Sigma_{x_m}=\Sigma_{x_{m+1}}=\cdots$, which contradicts that $P=\oplus_{i=1}^{\infty}P_i$ satisfying (*). Therefore we see that $I_0(R)\neq 0$. (b)$\to$(c). Take a nonzero element x_1 in $I_0(R)$; then $\Sigma_{x_1}=I_0(R)$. Since $Soc(R)=0$, there exist nonzero cyclic right ideals $x_{i+1}R$ and $y_{i+1}R$ of R such that $x_iR=x_{i+1}R\oplus y_{i+1}R$ and $x_{i+1}R\lesssim y_{i+1}R$ for each i; so $2(x_{i+1}R)\lesssim x_iR$. Put $P=\oplus_{i=1}^{\infty}x_iR$. If there exists a nonzero element y of R such that $yR\lesssim x_iR$ for all i, then $\Sigma_{x_1}=\Sigma_{x_i}=\Sigma_y$ for all i by the smallity of Σ_{x_1}. Hence there exist positive integers t and m such that $2t(x_mR)\lesssim x_1R\lesssim t(yR)\lesssim t(x_mR)$ from Lemma 3.2; whence $2t(x_mR)\lesssim t(x_mR)$ which contradicts the directly finiteness of $t(x_mR)$. Therefore $\{x_iR\}_{i=1}^{\infty}$ is cofini cofinal. By the smallity of Σ_{x_1}, we see that $[x_1R]=[x_iR]$ for all i and $\oplus_{i=1}^{\infty}x_iR\lesssim 2(x_1R)$, and hence $\oplus_{i=1}^{\infty}x_iR$ is d.f.. (c)$\to$(a) is clear.

Proposition 3.4. *Assume that $Soc(R)=0$ and $I_0(R)=0$. Then a non-finitely generated projective R-module P is d.f. (if and only) if P has a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ such that $\{P_i\}_{i=1}^{\infty}$ is cofinal.*

Proof. Assume that P is a countably generated projective R-module with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ such that $\{P_i\}_{i=1}^{\infty}$ is cofinal. We express each P_i as $P_i\cong x_iR$, where $x_i\in R$. Then $\Sigma_{x_1}\geqq\Sigma_{x_2}\geqq\cdots$. If there exists a positive integer j such that $\Sigma_{x_j}=\Sigma_{x_{j+1}}=\cdots$, we have a nonzero ideal RxR such that $RxR\lneqq\Sigma_{x_j}$ since $I_0(R)=0$. By Lemma 3.2, we have $xR\lesssim x_iR$ for all i, which contradicts that $\{P_i\}_{i=1}^{\infty}$ is cofinal. Therefore we have an increasing sequence $i_1<i_2<\cdots$, of positive integers such that $\Sigma_{x_{i_1}}\gneqq\Sigma_{x_{i_2}}\gneqq\cdots$. Then $P_{i_n}\cong x_{i_n}R\gtrsim\aleph_0(x_{i_{n+1}}R)\cong P_{i_{n+1}}$ by Lemma 3.2. Thus above (B) holds and hence P is d.f..

Theorem 3.5. *The following are equivalent:*

(*a*) *R is type B.*

(*b*) *$Soc(R)=0$, $I_0(R)=0$ and $L(R)$ has a cofinal subfamily.*

(*c*) *There exists a non-finitely generated d.f. projective R-module P with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ such that $\{P_i\}_{i=1}^{\infty}$ is cofinal and $[P_1]\neq[P_2]\neq\cdots$.*

Proof. (a)$\to$(b). Assume that R is Type B. Then it must hold that $Soc(R)=0$. We have a countably generated d.f. projective R-module P with a cyclic decomposition $P=\oplus_{i=1}^{\infty}P_i$ satisfying (*) and (B). Let $P_i\cong x_iR$ for $x_i\in R$. Then $\cap_{i=1}^{\infty}\Sigma_{x_i}=0$ and $\{\Sigma_{x_i}\}_{i=1}^{\infty}$ is a cofinal subfamily of $L(R)$. (b)$\to$(c). From the assumption, we have a cofinal subfamily $\{I_i\}_{i=1}^{\infty}$ of $L(R)$ such that $I_1\gneqq I_2\gneqq\cdots$. Take $x_i\in I_i-I_{i+1}$. Since $L(R)$ is a linearly ordered set under inclusion, we see that $I_i\geqq\Sigma_{x_i}\gneqq I_{i+1}$; so $\Sigma_{x_1}\gneqq\Sigma_{x_2}\gneqq\cdots$. Putting that $P=\oplus_{i=1}^{\infty}x_iR$, we see that $\{x_iR\}_{i=1}^{\infty}$ is cofinal and $[x_1R]\neq[x_2R]\neq\cdots$; and hence P is d.f. from Proposition 3.4. (c)$\to$(a) is clear, since (B) follows from (c).

Theorem 3.6. *The following are equivalent:*
(*a*) *R is Type C.*
(*b*) $Soc(R) \neq 0$, *or* $I_0(R)=0$ *and* $L(R)$ *does not have any cofinal subfamilies.*

Proof. This is immediate from Theorems 3.3 and 3.5.

REMARK. By theorems above, we see that Types A, B and C are right-left symmetric.

As an application we show the following

Theorem 3.7. *A projective R-module P is d.inf. if and only if there exists a nonzero R-module X such that* $\aleph_0 X$ *is isomorphic to a direct summand of P.*

Proof. "If" part is clear. "Only if" part. Let $P=\oplus_{\alpha\in I}P_\alpha$ be a d.inf. projective R-module where each P_α is nonzero cyclic. If $Soc(R)\neq 0$, then for any nonzero simple right ideal $X\leqq Soc(R)$, clearly $|I|X\lesssim\oplus P=\oplus_{\alpha\in I}P_\alpha$, whence $\aleph_0 X\lesssim\oplus P$. So, we may consider the case $Soc(R)=0$. If $|I|>\aleph_0$, by the proof of [4, Theorem 6], there exists $P_\beta\in\{P_\alpha\}_{\alpha\in I}$ such that $|\{P_\alpha\in\{P_\alpha\}_{\alpha\in I}\,|\,P_\beta\lesssim P_\alpha\}|\geqq\aleph_0$; so $\aleph_0 P_\beta\lesssim\oplus P$. Hence we may further assume that $|I|=\aleph_0$, so say $P=\oplus_{i=1}^{\infty}P_i$. If $\{P_i\}_{i=i}^{\infty}$ is not cofinal, then clearly there exists a desired X. Hence assume that $\{P_i\}_{i=1}^{\infty}$ is cofinal. Since P is d.inf., we see from Proposition 3.4 that $I_0(R)\neq 0$. Noting that P is d.inf., together with Theorem 3.3, we see that $[P_m]=[P_{m+1}]=\cdots$ for positive integer m and $tP_m\not\gtrsim P_{m+1}\oplus P_{m+2}\oplus\cdots$ for all t. Then there exists an ascending chain $m=m_1<m_2<\cdots$, of positive integers such that $P_m\lesssim\oplus P_{m_i+1}\oplus\cdots\oplus P_{m_{i+1}}$ for $i=1, 2, \cdots$, and so $\aleph_0 P_m\lesssim\oplus P_{m_1+1}\oplus P_{m_1+2}\oplus\cdots<\oplus P$ as desired.

Finally we give an example of Type A which has infinitely many ideals.

EXAMPLE (cf. [2, p. 486–p. 489]). Choose a field F and set $R_0=F$. For each positive integer n, let R_n be the ring of all $\aleph_0\times\aleph_0$ matrices over R_{n-1} of the form

$$x=\begin{pmatrix} x_{11} & \cdots & x_{1n} & & & 0 \\ \vdots & & \vdots & & & \\ x_{n1} & \cdots & x_{nn} & & & \\ & & & a & & \\ & & & & a & \\ 0 & & & & & \ddots \end{pmatrix}$$

,where $x_{ij}\in R_{n-1}$ and $a\in F$, and put $\alpha_n=\begin{pmatrix} 1_{n-1} & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix}\in R_n$, where 1_{n-1} is the identity element in R_{n-1}. We define a ring homomorphism $p_n\colon R_n\to F$ by the rule $p_n(x)=a$ for x above, and define a ring homomorphism $f_{n+1,n}\colon R_n\to R_{n+1}$ by the rule

$$f_{n+1,n}(y)=\begin{bmatrix} y & & & 0 \\ & p_n(y) & & \\ & & p_n(y) & \\ 0 & & & \ddots \end{bmatrix}$$

for all $y\in R_n$. Then each R_n is a non-simple unit-regular ring satisfying the c. axiom. Put $R=\varinjlim_{n\geqq 1} R_n$ and let ϕ_n: $R_n\rightarrow R$ be the canonical map. Then we see that R is a non-simple unit-regular ring satisfying the c. axiom with a nonzero socle of R. Now set $S_n=M_{2^n}(R)$ for $n=1, 2, \cdots$. Map each $R_n\rightarrow R_{n+1}$ along the diagonal, i.e., map $x\rightarrow\begin{bmatrix} x & 0 \\ 0 & x\end{bmatrix}$, and set $S=\varinjlim S_n$, and let ψ_n: $S_n\rightarrow S$ be the canonical map. Then S is a non-simple d.f. regular ring satisfying the c. axiom which is Type A and has an ascending chain $S\psi_1(\phi_1(\alpha_1))S\lneqq S\psi_1(\phi_2(\alpha_2))S\lneqq\cdots$ of ideals of S.

Unfortunately we do not have any examples of d.f. regular rings R satisfying the c. axiom such that $I_0(R)=0$; so we do not have any examples of Type B and non-trivial Type C.

References

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