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EXTENSION OF MEASURES TO INFINITE DIMENSIONAL SPACES OVER *P*-ADIC FIELD

Kumi YASUDA

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1. Introduction

In carrying out analysis on infinite dimensional spaces over p-adics, it is useful to give integral representations of functions. Satoh considered a normed vector space H over a local field K with orthonormal Schauder basis ([14]). He showed that any admissible probability measure on K is extended to a measure on the completion of H with respect to a measurable norm, applying Prokhorov's measure extension theorem to the projective limit of the images of orthogonal projections on H. This can be applied to a space of polynomials with coefficients in p-adics. On the other hand the present paper aims at extending probability measures to spaces including extension fields over p-adics of infinite degree, in which there exist no orthonormal basis in the sense of [14], except the case of unramified extensions. The spaces to which we extend measures are completions of infinite extension fields over p-adics with respect to specific seminorms induced by projections naturally related with traces on subextensions. We notice that our projections are not necessarily orthogonal in the sense of [14]. The subjects of our theorem include for instance the algebraic closure and the maximal unramified extension of the p-adic field. Kochubei proved independently that Gaussian measures on a local field can be extended to completion of an infinite extension and constructed a fractional differentiation operator relative to the measure ([9]).

Let p be a fixed prime integer. The p-adic field $\mathbb{Q}_p$ consists of formal power series

$$\sum_{i=m}^{\infty} \alpha_i p^i, \quad m \in \mathbb{Z}, \ \alpha_i \in \{0, 1, \dots, p-1\}.$$

With ordinary addition and multiplication as power series, $\mathbb{Q}_p$ becomes a field. The p-adic norm $\|\cdot\|$ is defined by

$$\left\| \sum_{i=m}^{\infty} \alpha_i p^i \right\| = p^{-m} \quad \text{if } \alpha_m \neq 0, \quad \text{and} \quad \|0\| = 0.$$

We denote by $\mathbb{Z}_p$ the valuation ring $\{x \in \mathbb{Q}_p \mid \|x\| \leq 1\}$.

If K is an extension field over $\mathbb{Q}_p$ of finite degree, the p-adic norm has a unique extension to K, which we denote by $\|\cdot\|$ again. The norm $\|\cdot\|$ is non-archimedean, i.e. satisfies the ultra-metric inequality:

$$\|x+y\| \leq \max\{\|x\|, \|y\|\}, \quad x, y \in K.$$

Let us denote by R_K the valuation ring $\{x \in K \mid \|x\| \leq 1\}$, then $P_K := \{x \in K \mid \|x\| < 1\}$ is a unique maximal ideal of R_K. The ramification index e_K of K is the positive integer such that

$$\{\|x\| \mid x \in K - \{0\}\} = \{p^{n/e_K} \mid n \in \mathbb{Z}\}.$$

If N_K is the extension degree of K over $\mathbb{Q}_p$ and f_K the degree of residue field R_K/P_K over $\mathbb{F}_p$, then it follows that $N_K = e_K f_K$. Put $r_K := p^{1/e_K}$ and $q_K := p^{f_K}$. If π_K is a prime element i.e. a generator of the ideal P_K of R_K, and if A_K is a complete system of representatives of the residue field, then K is interpreted as the set of formal power series

$$\sum_{i=m}^{\infty} \alpha_i \pi_K^i, \quad m \in \mathbb{Z}, \ \alpha_i \in A_K,$$

and the norm $\|\cdot\|$ is given by

$$\left\| \sum_{i=m}^{\infty} \alpha_i \pi_K^i \right\| = r_K^{-m} \quad \text{if } \alpha_m \neq 0.$$

The field K is a complete separable metric space with respect to the metric induced by the norm $\|\cdot\|$. The Haar measure $\mathfrak{m}_K$ on K is always assumed to be normalized so that $\mathfrak{m}_K(R_K) = 1$. Then it can be verified that $\mathfrak{m}_K(\|x\| \leq r_K^m) = q_K^m$. We will often write dx for $\mathfrak{m}_K(dx)$ and omit subscripts K (e.g., $R, \pi, \mathfrak{m}, \ldots$) if there is no fear of confusion.

For a topological space X, $\mathcal{B}(X)$ stands for the Borel field of X.

2. Extension of measures

Let $L \supset K$ be a field extension and the extension degree $[L : K]$ be finite. For $x \in L$, $K(x)$ denotes the subfield of L obtained by adjoining x to K. The trace map $\mathrm{Tr}_{L,K}$ is a K-linear map on L to K defined by

$$\mathrm{Tr}_{L,K}(x) = [L : K(x)] \sum_{i=1}^{k} x_i, \quad x \in L,$$

where $k = [K(x) : K]$, and $x = x_1, x_2, \ldots, x_k$ are all distinct conjugates of x over K. Any K-linear map f on L to K is of the form $f(\cdot) = \mathrm{Tr}_{L,K}(v\cdot)$ for a unique element

v of L. If $L \supset F \supset K$ then it can be verified that $\mathrm{Tr}_{F,K} \circ \mathrm{Tr}_{L,F} = \mathrm{Tr}_{L,K}$. For an unramified extension $L \supset K$, $\mathrm{Tr}_{L,K}$ maps R_L surjectively onto R_K (see [16]).

Now we introduce a map $T_K^L : L \to K$ for a finite extension $L \supset K$.

DEFINITION 2.1. For a finite extension $L \supset K$, we define a K-linear map T_K^L on L to K by

$$T_K^L(x) := \mathrm{Tr}_{L,K}([L:K]^{-1}x) = [L:K]^{-1}\,\mathrm{Tr}_{L,K}(x) = \frac{1}{k}\sum_{i=1}^{k} x_i, \quad x \in L.$$

Lemma 2.2. (i) *The map T_K^L of L to K is continuous and surjective.*
(ii) *If $L \supset F \supset K$ then $T_K^L = T_K^F \circ T_F^L$.*

Proof. (i) Since $\mathrm{Tr}_{L,K}$ is continuous, so is T_K^L. For surjectivity, take any $x \in K$ then $T_K^L(x) = x$.

(ii) $T_K^F \circ T_F^L(x) = [F:K]^{-1}[L:F]^{-1}\,\mathrm{Tr}_{F,K} \circ \mathrm{Tr}_{L,F}(x) = [L:K]^{-1}\,\mathrm{Tr}_{L,K}(x) = T_K^L(x)$. □

DEFINITION 2.3. Let $\mathbb{Q}_p^{\mathrm{alg}}$ stand for the algebraic closure of $\mathbb{Q}_p$. For each extension $K \supset \mathbb{Q}_p$ of finite degree, define a map T_K on $\mathbb{Q}_p^{\mathrm{alg}}$ to K by

$$T_K(x) = T_K^L(x) \quad \text{if } x \in L,\ L \supset K.$$

The map T_K is well-defined. Indeed, suppose that $x \in L$, $L \supset K$. Then

$$T_K^L(x) = T_K^{K(x)} \circ T_{K(x)}^L(x) = T_K^{K(x)}(x),$$

thus $T_K^L(x)$ is independent of the choice of L.

Put $K_1 = \mathbb{Q}_p$, and fix an increasing sequence $\mathcal{S} = \{K_n\}_{n=1}^{\infty}$ of extension fields over $\mathbb{Q}_p$ of finite degrees. Put $B = B_{\mathcal{S}} := \cup_{n=1}^{\infty} K_n \subset \mathbb{Q}_p^{\mathrm{alg}}$.

EXAMPLES. [E 2.1] K_n = the smallest field containing all extensions of degrees less than n. $B = \mathbb{Q}_p^{\mathrm{alg}}$.
[E 2.2] K_n = the unramified extension of degree $n!$. B is the maximal unramified extension of $\mathbb{Q}_p$.

We will often abbreviate subscripts and superscripts K_n to n, e.g. $R_n := R_{K_n}$, $T_n^m := T_{K_n}^{K_m}$, and we put $\mathcal{B}_n := \mathcal{B}(K_n)$. For each n, we denote by T_n the restriction of T_{K_n} to B. We put on B the topology induced by T_n, $n \geq 1$, i.e. the weakest topology relative to which T_n are continuous for all n. Let $\overline{B}$ be the completion of B, and we denote by T_n again the continuation of T_n to $\overline{B}$. Our aim is to extend measures to $\overline{B}$.

Suppose that we are given a sequence $\{X_n\}_{n=1}^{\infty}$ of topological spaces and measurable maps f_n^m of X_m onto X_n for $m \geq n$. We say that $\{X_n\}_{n=1}^{\infty}$ is projective with respect to f_n^m, if $f_n^m = f_n^l \circ f_l^m$ holds for $m \geq l \geq n$. We denote by p_m the canonical map on $\operatorname{proj\,lim} X_n$ to X_m;

$$p_m((x_n)_{n=1}^{\infty}) = x_m,$$

and put on $\operatorname{proj\,lim} X_n$ the topology induced by p_m, $m \geq 1$. If each X_n is a separable metric space and if the maps f_n^m are continuous, then the Borel field $\mathcal{B}(\operatorname{proj\,lim} X_n)$ is generated by the sets $p_m^{-1}(A_m)$ ($m \geq 1$, $A_m \in \mathcal{B}(X_m)$). Assume furthermore that the spaces X_n are complete. If we are given a probability measure μ_n on $(X_n, \mathcal{B}(X_n))$ for each n such that

$$\mu_n(A_n) = \mu_{n+1}\left(\left(f_n^{n+1}\right)^{-1}(A_n)\right)$$

for any $A_n \in \mathcal{B}(X_n)$, then there exists a unique Borel probability measure μ_∞ on $\operatorname{proj\,lim} X_n$ such that

$$\mu_\infty\left(p_n^{-1}(A_n)\right) = \mu(A_n)$$

for every n and $A_n \in \mathcal{B}(X_n)$. For these results refer to [12].

Let us come back to the sequence $\mathcal{S} = \{K_n\}_{n=1}^{\infty}$ of finite extensions of $\mathbb{Q}_p$. Lemma 2.2 implies that $\mathcal{S}$ is projective with respect to T_n^m.

DEFINITION 2.4. We say $\{\mu_n\}_{n=1}^{\infty}$ is a consistent sequence of probability measures (associated with $\mathcal{S} = \{K_n\}_{n=1}^{\infty}$), if μ_n is a probability measure on K_n such that

$$\mu_n(A_n) = \mu_{n+1}\left(\left(T_n^{n+1}\right)^{-1}(A_n)\right)$$

for all n and $A_n \in \mathcal{B}_n$.

If we are given a consistent sequence $\{\mu_n\}$ of probability measures, then it can be uniquely extended to a Borel probability measure $\tilde{\mu}_\infty$ on $\operatorname{proj\,lim} K_n$. Whereas we have:

Proposition 2.5. *Topological $\mathbb{Q}_p$-vector spaces $\overline{B}$ and* $\operatorname{proj\,lim} K_n$ *are isomorphic.*

Proof. Let us show that

$$\iota(w) = (T_n(w)) : \overline{B} \to \operatorname{proj\,lim} K_n$$

gives an isomorphism of $\overline{B}$ onto $\operatorname{proj\,lim} K_n$. If $w \in B$ and $m \geq n$, then Lemma 2.2 (ii) implies $T_n^m \circ T_m(w) = T_n(w)$. By taking limit we can see that this is valid for all $w \in \overline{B}$, and hence $\iota(w) \in \operatorname{proj\,lim} K_n$. For injectivity, suppose w, $w' \in \overline{B}$ satisfy $\iota(w) = \iota(w')$. Take a sequence $\{x_k\}$ in B such that $\lim_{k\to\infty} x_k = w$. Then for every n,

$$T_n(w') = T_n(w) = \lim_{k\to\infty} T_n(x_k),$$

which implies, by the definition of topology of $\overline{B}$, $w' = \lim_{k\to\infty} x_k = w$ in $\overline{B}$. Let us prove that ι is surjective. If we take any element $\omega = (x_n)_{n=1}^{\infty}$ of $\operatorname{proj\,lim} K_n$, then for any $m \geq n$ we have

$$p_n(\iota(x_m)) = T_n(x_m) = x_n = p_n(\omega).$$

Therefore for every n, $\lim_{m\to\infty} p_n(\iota(x_m)) = p_n(\omega)$ in K_n, which shows $\lim_{m\to\infty} \iota(x_m) = \omega$ in $\operatorname{proj\,lim} K_n$. Since $\overline{B}$ is complete, we have $\omega \in \iota(\overline{B})$. Taking it into account that $p_n \circ \iota = T_n$, we can see that ι is homeomorphic. The $\mathbb{Q}_p$-linearity of ι follows immediately from the linearity of T_n, and thus ι gives an isomorphism of $\overline{B}$ onto $\operatorname{proj\,lim} B$.

Thus putting $\mu_\infty := \tilde{\mu}_\infty \circ \iota$, we derive the following measure extension to the space $\overline{B}$.

Theorem 2.6. *Assume that we are given a consistent sequence $\{\mu_n\}_{n=1}^{\infty}$ of Borel probability measures. Then there exists a unique Borel probability measure μ_∞ on $\overline{B}$ such that*

$$\mu_\infty\left(T_n^{-1}(A_n)\right) = \mu_n(A_n)$$

for any n and $A_n \in \mathcal{B}_n$.

REMARK. Consider the case that $B = \mathbb{Q}_p^{\mathrm{alg}}$. If we write $\mathbb{C}_p$ for the completion of $B = \mathbb{Q}_p^{\mathrm{alg}}$ with respect to the p-adic norm, then neither $\mathbb{C}_p$ nor $\overline{B}$ contains the other. Indeed, for each fixed n, let $L_k^{(n)}$ $(k = 1, 2, \ldots)$ be the unramified extension of K_n of degree p^k. We can take $a_k^{(n)} \in R_{L_k^{(n)}}$ such that $\mathrm{Tr}_{L_k^{(n)}, K_n}(a_k^{(n)}) = 1$. Put $b_k^{(n)} = p^k a_k^{(n)}$, then we have $T_n(b_k^{(n)}) = 1$ for all k, whereas $\|b_k^{(n)}\| \to 0$ as $k \to \infty$. This implies that T_n is not continuous with respect to the p-adic norm. Conversely, if we put $c_k = 1 - p^k a_k^{(k)}$, then we have $\|c_k\| = 1$, and $\lim_{k\to\infty} T_n(c_k) = 0$ for every n. Thus the p-adic norm is not continuous with respect to the topology induced by T_n, $n \geq 1$.

In the next section we shall give some examples of symmetric probability measures on K_n which can be extended to $\overline{B}$. On the other hand, the following lemma shows that there exists no non-trivial symmetric probability measure on $\mathbb{C}_p$.

Proposition 2.7. *Let μ be a probability measure on $\mathbb{C}_p$ and suppose that $\mu(u\cdot) = \mu(\cdot)$ for all $u \in \mathbb{C}_p$ with norm* 1. *Then $\mu(\{0\}) = 1$.*

Proof. For each pair (a_0, a_1) of rational numbers such that $a_0 > a_1$, let $\mathcal{R}(a_0, a_1)$ be the collection of all sets of the form $B(z, p^{a_1}) := \{y \in \mathbb{C}_p \mid \|y - z\| \leq p^{a_1}\}$ for

$z \in \mathbb{C}_p$, $\|z\| = p^{a_0}$. Let $\mathcal{S} = \{K_n\}$ be such that $B = \mathbb{Q}_p^{\mathrm{alg}}$. Take N such that p^{a_0}, $p^{a_1} \in \{\|x\| \mid x \in K_N - \{0\}\} = \{r_N^k \mid k \in \mathbb{Z}\}$, and for each $n \geq N$, let $\mathcal{R}_n(a_0, a_1)$ be the collection of all sets of the form $B(x, p^{a_1})$ for $x \in K_n$, $\|x\| = p^{a_0}$. Then we have

$$\mathcal{R}(a_0, a_1) = \cup_{n \geq N} \mathcal{R}_n(a_0, a_1). \tag{2.1}$$

Indeed, take any element $B(z, p^{a_1})$ in $\mathcal{R}(a_0, a_1)$. Since $\mathbb{Q}_p^{\mathrm{alg}} = \cup_{n \geq N} K_n$ is dense in $\mathbb{C}_p$, we can take $n \geq N$ and $x \in K_n$ such that $\|z - x\| < p^{a_1}$. Then the ultra-metric inequality implies that $\|x\| = p^{a_0}$ and $B(z, p^{a_1}) = B(x, p^{a_1})$.

Fix $n \geq N$ and let $k_0 = e_n a_0$, $k_1 = e_n a_1$. For $x = \sum_{i=-k_0}^{\infty} \alpha_i \pi_n^i$ and $x' = \sum_{i=-k_0}^{\infty} \alpha'_i \pi_n^i$ in K_n, the set $B(x, p^{a_1})$ coincides with $B(x', p^{a_1})$ if and only if $\alpha_i = \alpha'_i$ for $i = -k_0, \ldots, -k_1 - 1$. Hence $\mathcal{R}_n(a_0, a_1)$ consists of $(q_n - 1)q_n^{k_0 - k_1 - 1} = (1 - q_n^{-1})p^{N_n(a_0 - a_1)}$ elements, which shows by (2.1) that $\mathcal{R}(a_0, a_1)$ is a countable set. Notice that for any two elements $B(z, p^{a_1})$ and $B(z', p^{a_1})$ of $\mathcal{R}(a_0, a_1)$, we have $B(z', p^{a_1}) = z^{-1}z'B(z, p^{a_1})$ and $\|z^{-1}z'\| = 1$, and therefore $\mu(B(z, p^{a_1})) = \mu(B(z', p^{a_1}))$ by the assumption. Since the set $A(a_0) := \{z \in \mathbb{C}_p \mid \|z\| = p^{a_0}\}$ is disjoint union of countable sets in $\mathcal{R}(a_0, a_1)$, its measure $\mu(A(a_0))$ must be 0. Thus we obtain

$$\mu(\mathbb{C}_p - \{0\}) = \sum_{a_0 \in \mathbb{Q}} \mu(A(a_0)) = 0. \qquad \square$$

3. Characteristic functions and Consistent measures

Let $K \supset \mathbb{Q}_p$ be an extension of finite degree. A character of K is a continuous homomorphism on additive group K to multiplicative group of complex numbers of absolute value 1. We denote by K^* the group consisting of all characters of K.

Let φ_0 be the element of $\mathbb{Q}_p^*$ defined by

$$\varphi_0\left(\sum_{i=m}^{\infty} \alpha_i p^i\right) = \begin{cases} \exp\left(2\pi\sqrt{-1}\sum_{i=m}^{-1} \alpha_i p^i\right), & \text{if } m \leq -1, \\ 1, & \text{otherwise,} \end{cases}$$

then $\varphi_0(\mathbb{Z}_p) = \{1\}$ and $\varphi_0(p^{-1}\mathbb{Z}_p) \neq \{1\}$. For each extension K over $\mathbb{Q}_p$ of finite degree, $\psi_K^1 := \varphi_0 \circ T_{\mathbb{Q}_p}^K$ belongs to K^*. Put $l = l_K := \mathrm{ord}\left(\psi_K^1\right)$, i.e. l is the integer such that $\psi_K^1(xR) = \{1\}$ if and only if $\|x\| \leq r^l$. If $\mathcal{D}$ is the different of K over $\mathbb{Q}_p$, then $\mathcal{D} = \{\|x\| \leq \|N\|r^{-l}\}$. If K is tamely ramified (i.e. $(p, e) = 1$), then $r^l = \|N\|r^{e-1} = \|f\|r^{e-1}$. In particular, for unramified K (i.e. $e = 1$) we have $r^l = p^l = \|N\| = \|f\|$. If K is strongly ramified (i.e. $(p, e) \neq 1$), then $\|N\|r^e \leq r^l \leq \|f\|r^{e-1}$. For these results concerning with $\mathrm{ord}(\psi_K^1)$, we can refer to [11], [15], and [16].

We can identify K^* with K by means of the correspondence

$$x \in K \leftrightarrow \psi_K^x(\cdot) := \psi_K^1(x\cdot) \in K^*,$$

(Theorem 3 and following Corollary in II of [16]).

Lemma 3.1.

$$\int_{\|y\|=r^m} \psi_K^x(y)dy = \begin{cases} (q-1)q^{m-1}, & \text{if } \|x\| \leq r^{l-m}, \\ -q^{m-1}, & \text{if } \|x\| = r^{l-m+1}, \\ 0, & \text{if } \|x\| \geq r^{l-m+2}. \end{cases}$$

Proof. If $\|x\| \leq r^{l-m}$, then $\psi_K^x(y) \equiv 1$ on $\{\|y\| \leq r^m\}$. Hence

$$\int_{\|y\|\leq r^m} \psi_K^x(y)dy = \mathfrak{m}(\|y\| \leq r^m) = q^m. \tag{3.1}$$

If $\|x\| \geq r^{l-m+1}$, then there exists y_0 such that $\|y_0\| \leq r^m$ and $\psi_K^x(y_0) \neq 1$. The ultrametric inequality implies that $\|y + y_0\| \leq r^m$ if and only if $\|y\| \leq r^m$, and therefore

$$\int_{\|y\|\leq r^m} \psi_K^x(y)dy = \int_{\|y\|\leq r^m} \psi_K^x(y + y_0)dy = \psi_K^x(y_0) \int_{\|y\|\leq r^m} \psi_K^x(y)dy.$$

Since $\psi_K^x(y_0) \neq 1$, we have

$$\int_{\|y\|\leq r^m} \psi_K^x(y)dy = 0, \tag{3.2}$$

and our assertion follows immediately from (3.1) and (3.2). □

For a probability measure μ_K on K, we interpret the characteristic function $\widehat{\mu_K}$ as the function on K by

$$\widehat{\mu_K}(x) = \int_K \psi_K^x(y)\mu_K(dy).$$

A function g on K is the characteristic function of a probability measure on K, if and only if it is positive definite, continuous, and $g(0) = 1$, and the correspondence between such functions and probability measures is one-to-one (see Theorems 3.1 and 3.2 in IV of [12]).

We have seen in the previous section that a consistent sequence of probability measures can be extended to a probability measure on $\overline{B}$. In order to find consistent sequences of measures we shall give a correspondence between probability measures on $\overline{B}$ and functions on B. Let $\mathcal{G}$ be the set of positive definite functions g on B such that $g(0) = 1$ and the restriction to K_n is continuous for every n. We shall particularly observe the case that the measure μ_n is symmetric, i.e. $\mu_n(u_n \cdot) = \mu_n(\cdot)$ for all $u_n \in K_n$ of norm 1. We say a function $g \in \mathcal{G}$ is symmetric if $g(u\cdot) = g(\cdot)$ for any $u \in B$ of norm 1.

Proposition 3.2. (i) *Probability measures on $\overline{B}$ correspond in one-to-one way to consistent sequences $\{\mu_n\}_{n=1}^{\infty}$.*
(ii) *Consistent sequences $\{\mu_n\}_{n=1}^{\infty}$ correspond in one-to-one way to functions belonging to $\mathcal{G}$. Every measure μ_n ($n = 1, 2, \ldots$) is symmetric if and only if the corresponding function in $\mathcal{G}$ is symmetric.*

Proof. (i) Assume that we are given a probability measure μ on $\overline{B}$. Then it can be easily verified that the sequence $\{\mu_n\}_{n=1}^{\infty}$ given by

$$\mu_n(A_n) = \mu\left(T_n^{-1}(A_n)\right), \quad A_n \in \mathcal{B}_n \tag{3.3}$$

is consistent. Let μ_∞ be the unique extension of $\{\mu_n\}_{n=1}^{\infty}$, then $\mu_\infty\left(T_n^{-1}(A_n)\right) = \mu_n(A_n) = \mu\left(T_n^{-1}(A_n)\right)$ for every n and $A_n \in \mathcal{B}_n$. If we take notice of the identification between $\overline{B}$ and $\operatorname{proj\,lim} K_n$ established in Proposition 2.5, then we can see that $\mathcal{B}(\overline{B})$ is generated by the sets $T_n^{-1}(A_n)$ ($n \geq 1$, $A_n \in \mathcal{B}_n$). Hence μ_∞ coincides with μ, and thus (3.3) gives a one-to-one correspondence of probability measures on $\overline{B}$ to consistent sequences.

(ii) For a consistent sequence $\{\mu_n\}_{n=1}^{\infty}$, define a function g on B by

$$g(x) = \widehat{\mu_n}(x), \text{ if } x \in K_n.$$

The function $g(x)$ is defined independently of the choice of n. Indeed, if $x \in K_n \subset K_m$ then

$$\begin{aligned}\widehat{\mu_m}(x) &= \int_{K_m} \varphi_0 \circ T_1^m(xy)\mu_m(dy) \\ &= \int_{K_m} \varphi_0 \circ T_1^n(xT_n^m(y))\mu_m(dy) \\ &= \left(\mu_m \circ (T_n^m)^{-1}\right)^{\wedge}(x) \\ &= \widehat{\mu_n}(x).\end{aligned}$$

Since $g|_{K_n} = \widehat{\mu_n}$ is positive definite and continuous for each n, we see immediately that g belongs to $\mathcal{G}$. Conversely if g is any element of $\mathcal{G}$, then $g|_{K_n}$ is the characteristic function of a probability measure on K_n, say μ_n^g. If $x \in K_n \subset K_m$ then

$$\int_{K_n} \psi_n^x(y)\left(\mu_m^g \circ (T_n^m)^{-1}\right)(dy) = \int_{K_m} \psi_m^x(y)\mu_m^g(dy) = g(x) = \int_{K_n} \psi_n^x(y)\mu_n^g(dy),$$

thus $\{\mu_n^g\}_{n=1}^{\infty}$ is consistent. Obviously these correspondences $\{\mu_n\}_{n=1}^{\infty}$ to g and g to $\{\mu_n^g\}_{n=1}^{\infty}$ give the inverse of each other.

Let $\{\mu_n\}_{n=1}^{\infty}$ be consistent and $g \in \mathcal{G}$ the corresponding function. For x, $u \in B$,

$\|u\| = 1$, take n such that $x, u \in K_n$, then

$$g(x) = \int_{K_n} \psi_n^x(y)\mu_n(dy),$$
$$g(ux) = \int_{K_n} \psi_n^x(y)\mu_n(u^{-1}dy).$$

Hence g is symmetric if and only if μ_n is symmetric for every n. □

By the above proposition, every function g in $\mathcal{G}$ corresponds to a probability measure μ_∞ on $\overline{B}$. The correspondence is given by

(3.4) $$g(x) = \int_{\overline{B}} \varphi_0 \circ T_1^n(xT_n(w))\mu_\infty(dw), \quad \text{if } x \in K_n.$$

Here let us give some examples of symmetric functions g in $\mathcal{G}$ and the corresponding consistent sequence of symmetric probability measures.

EXAMPLES. [E 3.1] For $\lambda > 0$, put

$$g^{(1)}(x) = \begin{cases} 1, & \text{if } \|x\| \le \lambda, \\ 0, & \text{otherwise.} \end{cases}$$

The corresponding sequence $\{\mu_n^{(1)}\}_{n=1}^\infty = \{\mu_n^{(1)}(\lambda)\}_{n=1}^\infty$ is given by

$$\frac{d\mu_n^{(1)}}{dx}(x) = \begin{cases} q_n^{-l_n + \lfloor \log\lambda/\log r_n \rfloor}, & \text{if } \|x\| \le r_n^{l_n - \lfloor \log\lambda/\log r_n \rfloor}, \\ 0, & \text{otherwise,} \end{cases}$$

where $\lfloor a \rfloor$ stands for the integer part of a. The measure $\mu_n^{(1)}$ is a Gaussian measure on K_n.

[E 3.2] For α, $\beta > 0$, put

$$g^{(2)}(x) = \exp\left(-\alpha\|x\|^\beta\right).$$

The corresponding sequence $\{\mu_n^{(2)}\}_{n=1}^\infty = \{\mu_n^{(2)}(\alpha, \beta)\}_{n=1}^\infty$ is given by

$$\frac{d\mu_n^{(2)}}{dx}(x) = \|x\|^{-N_n}\sum_{i=0}^{\infty} q_n^{-i}\left\{\exp\left(-\alpha r_n^{\beta(l_n-i)}\|x\|^{-\beta}\right) - \exp\left(-\alpha r_n^{\beta(l_n-i+1)}\|x\|^{-\beta}\right)\right\}.$$

The measure $\mu_n^{(2)}$ is a stable law on K_n ([19]).

[E 3.3] For ρ, $\sigma > 0$ and $0 < \kappa < \rho^{-\sigma}$, put

$$g^{(3)}(x) = \begin{cases} -\kappa\|x\|^\sigma + 1, & \text{if } \|x\| \le \rho, \\ 0, & \text{otherwise.} \end{cases}$$

The corresponding sequence $\{\mu_n^{(3)}\}_{n=1}^{\infty} = \{\mu_n^{(3)}(\rho, \sigma, \kappa)\}_{n=1}^{\infty}$ is given by

$$\frac{d\mu_n^{(3)}}{dx}(x) = \begin{cases} \left(1 - \dfrac{(q_n - 1)r_n^{\sigma\lfloor \log\rho/\log r_n\rfloor}\kappa}{q_n - r_n^{-\sigma}}\right) q_n^{-l_n + \lfloor \log\rho/\log r_n \rfloor}, & \text{if } \|x\| \le r_n^{l_n - \lfloor \log\rho/\log r_n\rfloor}, \\ \dfrac{q_n(r_n^{\sigma} - 1)r_n^{\sigma l_n}\kappa}{q_n - r_n^{-\sigma}} \|x\|^{-\sigma - N_n}, & \text{otherwise.} \end{cases}$$

Now consider the case that for every n, $K_n \supset \mathbb{Q}_p$ is an abelian extension with Galois group G_n. Then $B \supset \mathbb{Q}_p$ is an abelian extension and its Galois group G consists of sequences $\sigma = (\sigma_1, \sigma_2, \dots)$ of $\sigma_n \in G_n$ satisfying $\sigma_{n+1}|_{K_n} = \sigma_n$, whose action being defined by $\sigma x = \sigma_n x$ provided $x \in K_n$. Every element $\sigma \in G$ defines a continuous map $x \in B \mapsto \sigma x \in B$. Indeed for every n and $x \in B$, take $N \ge n$ such that $x \in K_N$. Then for any $\sigma = (\sigma_1, \sigma_2, \dots)$ in G we have

$$\begin{aligned} T_n(\sigma x) &= [K_N : K_n]^{-1} \sum_{\tau \in \mathrm{Gal}(K_N/K_n)} \tau\sigma_N x \\ &= \sigma_N \left([K_N : K_n]^{-1} \sum_{\tau \in \mathrm{Gal}(K_N/K_n)} \tau x \right) \\ &= \sigma_n T_n(x). \end{aligned}$$

Hence if $\{x_k\}_{k=1,2,\dots}$ is a sequence in B converging to $x \in B$, then for every n and $\sigma \in G$,

$$T_n(\sigma x_k) = \sigma_n T_n(x_k) \to \sigma_n T_n(x) = T_n(\sigma x),$$

as $k \to \infty$. Thus σx_k converges to σx. Hence the map $x \mapsto \sigma x$ can be uniquely extended to a continuous map on $\overline{B}$ to itself.

We shall show results concerning with G-invariance of probability measures on $\overline{B}$.

Proposition 3.3. *A probability measure μ_∞ on $\overline{B}$ is G-invariant if and only if the corresponding function $g \in \mathcal{G}$ satisfies $g \circ \sigma = g$ for any $\sigma \in G$.*

Proof. Let $w \in \overline{B}$, $x \in K_n$, and $\sigma \in G$. Since σ^{-1} is continuous and $T_k(w) \to w$ as $k \to \infty$, we apply G_k-invariance of T_1^k: $T_1^k\left(x(\sigma_k^{-1}y)\right) = T_1^k((\sigma_k x)y)$, $x, y \in K_k, \sigma_k \in G_k$, to obtain

$$\begin{aligned} T_1^n(xT_n(\sigma^{-1}w)) &= \lim_{k\to\infty} T_1^k(x(\sigma^{-1}T_k(w))) \\ &= \lim_{k\to\infty} T_1^k((\sigma x)T_k(w)) \\ &= T_1^n((\sigma x)T_n(w)). \end{aligned} \tag{3.5}$$

Let μ_∞^σ be the probability measure on $\overline{B}$ defined by $\mu_\infty^\sigma(\cdot) = \mu_\infty(\sigma\cdot)$, and $g^\sigma \in \mathcal{G}$ be the corresponding function. If $x \in K_n$ then by (3.5),

$$\begin{aligned} g^\sigma(x) &= \int_{\overline{B}} \varphi_0 \circ T_1^n(xT_n(\sigma^{-1}w))\mu_\infty(dw) \\ &= \int_{\overline{B}} \varphi_0 \circ T_1^n((\sigma x)T_n(w))\mu_\infty(dw) = g(\sigma x). \end{aligned}$$

Therefore $\mu_\infty^\sigma = \mu_\infty$ if and only if $g = g \circ \sigma$. □

Corollary 3.4. (i) *If $\{\mu_n\}_{n=1}^\infty$ is a consistent sequence of symmetric probability measures, then the extension μ_∞ is G-invariant.*
(ii) *If ν is a probability measure on $\mathbb{Q}_p$, then the function $g_\nu := \hat{\nu} \circ T_1$ belongs to $\mathcal{G}$, and the corresponding measure on $\overline{B}$ is G-invariant.*

Proof. (i) By Proposition 3.2 (ii), the function $g \in \mathcal{G}$ corresponding to μ_∞ is symmetric. For $\sigma = (\sigma_1, \sigma_2, \dots) \in G$ and $x \in B - \{0\}$, taking n such that $x \in K_n$ we have $\|\sigma x\| = \|\sigma_n x\| = \|x\|$, since G_n acts on K_n isometrically. Therefore we obtain $g(\sigma x) = g((\sigma x/x)x) = g(x)$.

(ii) Since $\hat{\nu}$ is positive definite and continuous on $\mathbb{Q}_p$, and since T_1 is $\mathbb{Q}_p$-linear and continuous on each K_n, it is immediately checked that g_ν belongs to $\mathcal{G}$. For $x \in B$ take n such that $x \in K_n$. Then G_n-invariance of T_1^n implies

$$g_\nu(\sigma x) = \hat{\nu} \circ T_1^n(\sigma x) = \hat{\nu} \circ T_1^n(x) = g_\nu(x).$$

□

4. Subspaces of measure 1

For each example in [E 3.1] to [E 3.3] we shall find a non-archimedean norm of the form $\sup_n \varepsilon_n \|T_n(\cdot)\|$ ($\varepsilon_n > 0$), on a subspace of $\overline{B}$ in which the extended measure μ_∞ is concentrated. Let us prove firstly that the support of the extended measure in [E 3.1] is included in a bounded set with respect to a certain norm.

DEFINITION 4.1. Put $\|w\|_* := \sup_n r_n^{-l_n-1}\|T_n(w)\|$ for $w \in \overline{B}$, and $B_* := \{w \in \overline{B} \mid \|w\|_* < \infty\}$.

We see that $\|\cdot\|_*$ defines a non-archimedean norm on B_*. Indeed it is easily seen that $\|\cdot\|_*$ is a norm. This is non-archimedean since

$$\begin{aligned} \|w+v\|_* &= \sup_n r_n^{-l_n-1}\|T_n(w) + T_n(v)\| \\ &\le \sup_n r_n^{-l_n-1} \max\{\|T_n(w)\|, \|T_n(v)\|\} \end{aligned}$$

$$= \max\left\{\sup_n r_n^{-l_n-1}\|T_n(w)\|, \sup_n r_n^{-l_n-1}\|T_n(v)\|\right\}.$$

Proposition 4.2. *For $\lambda > 0$, let $\mu_n^{(1)} = \mu_n^{(1)}(\lambda)$ be as in* [E 3.1] *and $\mu_\infty^{(1)}$ the extended measure on $\overline{B}$. Then*

$$\mu_\infty^{(1)}\left\{\|w\|_* \le \lambda^{-1}\right\} = 1.$$

Proof. Note that

$$\mu_\infty^{(1)}\left(\left\{w \in \overline{B} \mid \|T_n(w)\| \le \lambda^{-1} r_n^{l_n+1}\right\}\right) = \mu_n^{(1)}\left(\left\{x \in K_n \mid \|x\| \le \lambda^{-1} r_n^{l_n+1}\right\}\right) = 1,$$

for every n. Then

$$\mu_\infty^{(1)}\left(\|w\|_* \le \lambda^{-1}\right) = \mu_\infty^{(1)}\left(\bigcap_n \left\{w \in \overline{B} \mid \|T_n(w)\| \le \lambda^{-1} r_n^{l_n+1}\right\}\right) = 1. \qquad \square$$

In order to investigate the cases [E 3.2] and [E 3.3], we shall give a lemma.

Lemma 4.3. (i) *For $u \ge 1$ and $v \ge 0$,* $\exp(-v) - \exp(-uv) \le (u-1)v$.
(ii) *For $0 < s < s_0$, put $C_{s,s_0} := \sup_{1 < a \le p^{s_0}} (a-1)/(1 - a^{s/s_0 - 1})$. Then $0 < C_{s,s_0} < \infty$.*

Proof. (i) Put $f_u(v) = \exp(-v) - \exp(-uv)$ and $g_u(v) = (u-1)v$. Then we have

$$\frac{d}{dv}(f_u - g_u)(v) \le -(u-1)\big(1 - \exp(-v)\big) \le 0,$$

and $f_u(0) - g_u(0) = 0$. This implies that $f_u(v) \le g_u(v)$ for $v \ge 0$.
(ii) The assertion is clear if we notice that

$$\lim_{a \to 1} \frac{a-1}{1 - a^{s/s_0 - 1}} = -\left(\frac{d}{da} a^{s/s_0 - 1}\bigg|_{a=1}\right)^{-1} = \left(1 - \frac{s}{s_0}\right)^{-1} < \infty. \qquad \square$$

DEFINITION 4.4. For a sequence $\varepsilon = \{\varepsilon_n\}_{n=1}^\infty$ of positive numbers, put $\|w\|_\varepsilon := \sup_n \varepsilon_n \|T_n(w)\|$ for $w \in \overline{B}$, and $B_\varepsilon := \{w \in \overline{B} \mid \|w\|_\varepsilon < \infty\}$.

We can verify that $\|\cdot\|_\varepsilon$ defines a non-archimedean norm on B_ε similarly as $\|\cdot\|_*$.

Proposition 4.5. (i) *For α, $\beta > 0$, let $\mu_n^{(2)} = \mu_n^{(2)}(\alpha, \beta)$ be as in* [E 3.2] *and $\mu_\infty^{(2)}$ the extended measures on $\overline{B}$. If there exists $0 < s < \beta$ such that $\sum_n \varepsilon_n^s < \infty$ and $\sum_n \varepsilon_n^s r_n^{\beta l_n} < \infty$, then $\mu_\infty^{(2)}(B_\varepsilon) = 1$.*

(ii) *For ρ, $\sigma > 0$ and $0 < \kappa < \rho^{-\sigma}$, let $\mu_n^{(3)} = \mu_n^{(3)}(\rho, \sigma, \kappa)$ be as in* [E 3.3] *and $\mu_\infty^{(3)}$ the extended measure on $\overline{B}$. If there exists $0 < s < \sigma$ such that $\sum_n \left(\varepsilon_n r_n^{l_n+1}\right)^s < \infty$, then $\mu_\infty^{(3)}(B_\varepsilon) = 1$.*

Proof. (i) For each n,

$$\begin{aligned}
&\int_{\overline{B}} \|T_n(w)\|^s \mu_\infty^{(2)}(dw) \\
&= \int_{K_n} \|x\|^s \mu_n^{(2)}(dx) \\
&\le 1 + \sum_{m=0}^{\infty} \int_{\|x\|=r_n^m} \|x\|^s \mu_n^{(2)}(dx) \\
&= 1 + \left(1 - q_n^{-1}\right) \sum_{m=0}^{\infty} r_n^{ms} \sum_{i=0}^{\infty} q_n^{-i} \left(\exp\left(-\alpha r_n^{\beta(l_n-m-i)}\right) - \exp\left(-\alpha r_n^{\beta(l_n-m-i+1)}\right)\right).
\end{aligned}$$

Apply Lemma 4.3 to $u = r_n^\beta$, $v = \alpha r_n^{\beta(l_n-m-i)}$, and $s_0 = \beta$, noticing that $1 < r_n^\beta \le p^\beta$, then

$$\begin{aligned}
\int_{\overline{B}} \|T_n(w)\|^s \mu_\infty^{(2)}(dw) &\le 1 + \left(1 - q_n^{-1}\right) \sum_{m=0}^{\infty} r_n^{ms} \sum_{i=0}^{\infty} q_n^{-i} \left(r_n^\beta - 1\right) \alpha r_n^{\beta(l_n-m-i)} \\
&= 1 + \left(1 - q_n^{-1}\right)\left(r_n^\beta - 1\right) \alpha r_n^{\beta l_n} \left(1 - r_n^{s-\beta}\right)^{-1} \left(1 - q_n^{-1} r_n^{-\beta}\right)^{-1} \\
&\le 1 + \left(r_n^\beta - 1\right)\left(1 - r_n^{s-\beta}\right)^{-1} \alpha r_n^{\beta l_n} \\
&\le 1 + C_{s,\beta} \alpha r_n^{\beta l_n}.
\end{aligned}$$

Therefore we have

$$\int_{\overline{B}} \left(\sum_n (\varepsilon_n \|T_n(w)\|)^s\right) \mu_\infty^{(2)}(dw) \le \sum_n \varepsilon_n^s + C_{s,\beta}\alpha \sum_n \varepsilon_n^s r_n^{\beta l_n} < \infty,$$

which implies that $\|w\|_\varepsilon \le (\sum_n (\varepsilon_n \|T_n(w)\|)^s)^{1/s} < \infty$, $\mu_\infty^{(2)}$-a.s..

(ii) For each n,

$$\begin{aligned}
&\int_{\overline{B}} \|T_n(w)\|^s \mu_\infty^{(3)}(dw) \\
&= \int_{K_n} \|x\|^s \mu_n^{(3)}(dx) \\
&= \left(1 - \frac{(q_n - 1) r_n^{\sigma \lfloor \log\rho / \log r_n \rfloor} \kappa}{q_n - r_n^{-\sigma}}\right) q_n^{-l_n + \lfloor \log\rho/\log r_n \rfloor} \sum_{m=-\infty}^{l_n - \lfloor \log\rho/\log r_n \rfloor} r_n^{ms} \left(q_n^m - q_n^{m-1}\right)
\end{aligned}$$

$$+\frac{q_n(r_n^\sigma - 1)r_n^{\sigma l_n}\kappa}{q_n - r_n^{-\sigma}} \sum_{m=l_n-\lfloor \log\rho/\log r_n\rfloor+1}^{\infty} r_n^{m(s-\sigma-N_n)}\left(q_n^m - q_n^{m-1}\right)$$
$$= \frac{q_n - 1}{q_n - r_n^{-s}} r_n^{s(l_n - \lfloor \log\rho/\log r_n\rfloor)}\left(1 + \frac{r_n^s - 1}{1 - r_n^{s-\sigma}} r_n^{\sigma\lfloor \log\rho/\log r_n\rfloor}\kappa\right).$$

Apply Lemma 4.3 (ii) to $s_0 = \sigma$ noticing that $r_n^s \le r_n^\sigma$, then we obtain

$$\int_{\overline{B}}\left(\sum_n \varepsilon_n\left(\|T_n(w)\|\right)^s\right)\mu_\infty^{(3)}(dw) \le \rho^{-s}\left(1 + C_{s,\sigma}\rho^\sigma\kappa\right)\sum_n\left(\varepsilon_n r_n^{l_n+1}\right)^s < \infty.$$

Hence $\sum_n \varepsilon_n\left(\|T_n(w)\|\right)^s$ is finite $\mu_\infty^{(3)}$-a.s., and so is $\|w\|_\varepsilon$. □

5. Extension of semigroups

We shall apply our extension theorem to extend Markov processes. In what follows we always assume that a semigroup $\{\mu^t\}_{t\ge 0}$ of probability measures on a field K is such that μ^t converges to the $\delta-$measure at the origin as $t \to 0$.

Proposition 5.1. *Assume that for every n, $\{\mu_n^t\}_{t\ge0}$ is a semigroup of probability measures on K_n, and that for every $t \ge 0$, $\{\mu_n^t\}_{n=1}^\infty$ is a consistent sequence. If we let μ_∞^t be the extension of $\{\mu_n^t\}_{n=1}^\infty$ for each t, then $\{\mu_\infty^t\}_{t\ge0}$ is a semigroup on $\overline{B}$.*

Proof. Since $\mu_\infty^t\left(T_n^{-1}(A_n)\right) = \mu_n^t(A_n)$ for $n \ge 1$ and $A_n \in \mathcal{B}_n$, we have for s, $t \ge 0$,

$$\begin{aligned}
\mu_\infty^s * \mu_\infty^t\left(T_n^{-1}(A_n)\right) &= \int_{\overline{B}} \mu_\infty^s\left(T_n^{-1}(A_n) - w\right)\mu_\infty^t(dw)\\
&= \int_{\overline{B}} \mu_\infty^s\left(T_n^{-1}(A_n - T_n(w))\right)\mu_\infty^t(dw)\\
&= \int_{K_n} \mu_n^s(A_n - x)\mu_n^t(dx)\\
&= \mu_n^{s+t}(A_n)\\
&= \mu_\infty^{s+t}\left(T_n^{-1}(A_n)\right).
\end{aligned}$$

Since the sets $T_n^{-1}(A_n)$ $(n \ge 1, A_n \in \mathcal{B}_n)$ generate $\mathcal{B}(\overline{B})$, we obtain $\mu_\infty^s * \mu_\infty^t = \mu_\infty^{s+t}$. □

Thus it can be seen that if we are given a temporally and spatially homogeneous Markov process X_n on each K_n whose transition function $\mu_n^t(\cdot) = P(X_n(t) \in \cdot \mid X_0 = 0)$ is consistent, then we can construct a Markov process on $\overline{B}$.

In order to find semigroups which can be extended, let us characterize them by means of characteristic functions. Let K be an extension of $\mathbb{Q}_p$ of finite degree. If F

is a σ-finite measure on K satisfying

$$F(N^c) < \infty \tag{5.1}$$

for any neighborhood N of the origin, and

$$\int_K \left(1 - \operatorname{Re}\psi_K^x(y)\right) F(dy) < \infty \tag{5.2}$$

for every $x \in K$, then the function

$$f(x) = \exp\left[\int_K \left(\psi_K^x(y) - 1\right) F(dy)\right]$$

gives characteristic function of a probability measure on K.

Let $\{\mu^t\}_{t\geq 0}$ be a semigroup on K. Then $\widehat{\mu^t}(x)$ has a unique representation

$$\widehat{\mu^t}(x) = \exp\left[t\left(\int_K \psi_K^x(y) - 1\right) F(dy)\right], \tag{5.3}$$

where $F = F(\{\mu^t\}_{t\geq 0})$ is a σ-finite measure on K uniquely determined by $\{\mu^t\}_{t\geq 0}$, which satisfies (5.1) and (5.2). For these results concerning the representation of characteristic functions, refer to [12].

Lemma 5.2. *Let $\{\mu^t\}_{t\geq 0}$ be a semigroup on K and assume that μ^t is symmetric for every t. Then the measure F in the representation* (5.3) *is symmetric.*

Proof. Let u be any element of K of norm 1. Then

$$\exp\left[t\int_K \left(\psi_K^x(y) - 1\right) F(udy)\right] = \widehat{\mu^t}(u^{-1}x) = \widehat{\mu^t}(x) = \exp\left[t\int_K \left(\psi_K^x(y) - 1\right) F(dy)\right].$$

By the uniqueness of the representation, we obtain $F(dy) = F(udy)$. □

Lemma 5.3. *If* (5.3) *is the representation of a semigroup $\{\mu^t\}_{t\geq 0}$ of symmetric probability measures on K, then for $x \neq 0$,*

$$\widehat{\mu^t}(x) = \exp\left[-t(q-1)^{-1}\left(qF\left(\|y\| \geq r^{-k+l+1}\right) - F\left(\|y\| \geq r^{-k+l+2}\right)\right)\right],$$

where $\|x\| = r^k$.

Proof. Let $\|x\| = r^k$ and $m \geq -k + l + 1$. For $\alpha = (\alpha_{-m-k}, \dots, \alpha_{-l-1}) \in A_K^{m+k-l}$, $\alpha_{-m-k} \neq 0$, define a set $D(\alpha)$ by

$$D(\alpha) := \left\{ y \in K \mid \left\| y - \sum_{i=-m-k}^{-l-1} \alpha_i \pi^i \right\| \leq r^l \right\}.$$

Since F is symmetric by Lemma 5.2, and since for any α and α' there exists $u \in K$ of norm 1 such that $x^{-1}D(\alpha') = ux^{-1}D(\alpha)$, $F\left(x^{-1}D(\alpha)\right)$ take the same value for all α. Notice that the set $\{y \in K \mid \|y\| = r^m\}$ is disjoint union of $x^{-1}D(\alpha)$'s for $(q-1)q^{m+k-l-1}$ distinct α's, then we have for each α,

$$F\left(x^{-1}D(\alpha)\right) = (q-1)^{-1}q^{-m-k+l+1}F\left(\|y\| = r^m\right).$$

If $y \in x^{-1}D(\alpha)$ then $\psi_K^x(y) = \psi_K^1(\sum_{i=-m-k}^{-l-1}\alpha_i\pi^i)$. Therefore we have

$$\begin{aligned}
&\int_{\|y\|=r^m} \left(\psi_K^x(y) - 1\right) F(dy) \\
&= \sum_\alpha \int_{x^{-1}D(\alpha)} \psi_K^x(y)F(dy) - F\left(\|y\| = r^m\right) \\
&= F\left(\|y\| = r^m\right)\left\{(q-1)^{-1}q^{-m-k+l+1}\sum_\alpha \psi_K^1\left(\sum_{i=-m-k}^{-l-1}\alpha_i\pi^i\right) - 1\right\}.
\end{aligned}$$

Here by Lemma 3.1,

$$\begin{aligned}
\sum_\alpha \psi_K^1\left(\sum_{i=-m-k}^{-l-1}\alpha_i\pi^i\right) &= \sum_\alpha \left(\mathfrak{m}\left(x^{-1}D(\alpha)\right)\right)^{-1}\int_{x^{-1}D(\alpha)} \psi_K^x(y)dy \\
&= q^{k-l}\int_{\|y\|=r^m} \psi_K^x(y)dy \\
&= \begin{cases} -1, & \text{if } m = -k+l+1, \\ 0, & \text{if } m \geq -k+l+2. \end{cases}
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{\|y\|=r^m} \left(\psi_K^x(y) - 1\right) F(dy) \\
&= \begin{cases} -(q-1)^{-1}qF(\|y\| = r^{-k+l+1}), & \text{if } m = -k+l+1, \\ -F(\|y\| = r^m), & \text{if } m \geq -k+l+2. \end{cases}
\end{aligned}$$

Since $\int_{\|y\|=r^m}(\psi_K^x(y) - 1)F(dy) = 0$ for $m \leq -k+l$, we obtain

$$\begin{aligned}
\int_K \left(\psi_K^x(y) - 1\right) F(dy) &= \sum_{m=-k+l+1}^{\infty} \int_{\|y\|=r^m} \left(\psi_K^x(y) - 1\right) F(dy) \\
&= -(q-1)^{-1}qF\left(\|y\| = r^{-k+l+1}\right) - \sum_{m=-k+l+2}^{\infty} F\left(\|y\| = r^m\right) \\
&= -(q-1)^{-1}\left(qF\left(\|y\| \geq r^{-k+l+1}\right) - F\left(\|y\| \geq r^{-k+l+2}\right)\right). \quad \square
\end{aligned}$$

Now we can give a characterization of consistent sequences of semigroups of symmetric probability measures;

Proposition 5.4. *A sequence* $\left\{\{\mu_n^t\}_{t\geq 0}\right\}_{n=1}^{\infty}$ *of semigroups of symmetric probability measures such that* $\{\mu_n^t\}_{n=1}^{\infty}$ *is consistent for each* t, *corresponds in one-to-one way to a non-negative function* h *on* $\|B\| := \{\|x\| \mid x \in B\}$ *satisfying the followings.*

$$(5.4) \qquad h\left(r_n^k\right) \geq (q_n - 1)\sum_{i=1}^{\infty} q_n^{-i} h\left(r_n^{k-i}\right), \quad \textit{for every integer } k \textit{ and } n \geq 1,$$

$$(5.5) \qquad \lim_{k\to -\infty} h\left(r_n^k\right) = 0, \quad \textit{for every } n \geq 1.$$

The correspondence is given by the formula

$$\widehat{\mu_n^t}(x) = \exp\left[-th(\|x\|)\right].$$

Proof. Assume that $\{\mu_n^t\}_{t\geq 0}$ is a semigroup of symmetric probability measures on K_n and that $\{\mu_n^t\}_{n=1}^{\infty}$ is consistent for every t. Let g be the element of $\mathcal{G}$ corresponding to the consistent sequence $\{\mu_n^1\}_{n=1}^{\infty}$. Since μ_n^1 is symmetric, g is real and symmetric, and hence g is of the form $g(x) = \exp[-h(\|x\|)]$, where h is a function on $\|B\|$ to $[0, +\infty]$. Notice that h is uniquely determined by the sequence $\left\{\{\mu_n^t\}_{t\geq 0}\right\}_{n=1}^{\infty}$. By Lemma 5.3, for each n there exists a unique σ-finite measure F_n on K_n such that $F_n\left(\|y\| \geq r_n^m\right) < \infty$ for every integer m, and

$$h\left(r_n^k\right) = (q_n - 1)^{-1}\left(q_n F_n\left(\|y\| \geq r_n^{-k+l_n+1}\right) - F_n\left(\|y\| \geq r_n^{-k+l_n+2}\right)\right), \quad k \in \mathbb{Z}.$$

Then we can easily derive that

$$(5.6) \qquad F_n\left(\|y\| \geq r_n^m\right) = (q_n - 1)\sum_{i=1}^{\infty} q_n^{-i} h\left(r_n^{-m+l_n-i+2}\right), \quad m \in \mathbb{Z}.$$

Since $F_n\left(\|y\| \geq r_n^m\right) < \infty$ for $m \in \mathbb{Z}$, $h\left(r_n^k\right)$ must be finite for any integer k. The formula (5.6) also implies

$$h\left(r_n^k\right) \leq q_n \sum_{i=1}^{\infty} q_n^{-i} h\left(r_n^{k-i+1}\right) = q_n(q_n - 1)^{-1} F_n\left(\|y\| \geq r_n^{-k+l_n+1}\right) \to 0$$

as $k \to -\infty$, thus (5.5) holds. We obtain (5.4) by applying (5.6) to the inequality

$$F_n\left(\|y\| \geq r_n^{-k+l_n+1}\right) - F_n\left(\|y\| \geq r_n^{-k+l_n+2}\right) \geq 0.$$

Conversely for a given non-negative function h on $\|B\|$ satisfying (5.4) and (5.5), define a symmetric measure F_n on K_n by the formula (5.6) and

$$\begin{aligned} & F_n\left(\{y \in K_n \mid \|y-x\| \le r_n^k\}\right) \\ &= (q_n-1)^{-1} q_n^{-(m-k+1)}\left(F_n\left(\|y\| \ge r_n^m\right) - F_n\left(\|y\| \ge r_n^{m+1}\right)\right), \quad \text{if } \|x\| = r_n^m > r_n^k. \end{aligned}$$

Here (5.4) and (5.5) imply

$$\begin{aligned} & F_n\left(\|y\| \ge r_n^m\right) \le h\left(r_n^{-m+l_n+2}\right) < \infty, \\ & \lim_{m\to\infty} F_n\left(\|y\| \ge r_n^m\right) = 0, \end{aligned}$$

and

$$\begin{aligned} & F_n\left(\|y\| \ge r_n^m\right) - F_n\left(\|y\| \ge r_n^{m+1}\right) \\ &= q_n^{-1}(q_n-1)\left(h\left(r_n^{-m+l_n+1}\right) - (q_n-1)\sum_{i=1}^{\infty} q_n^{-i} h\left(r_n^{-m+l_n+1-i}\right)\right) \ge 0. \end{aligned}$$

Therefore F_n is a σ-finite measure with finite mass on complement of any neighborhood of the origin. For $0 \ne x \in K_n$, let $\|x\| = r_n^{k_n}$. Since $\psi_n^x(y) = 1$ if $\|y\| \le r_n^{-k_n+l_n}$, we have

$$\int_{K_n} \left(1 - \operatorname{Re}\psi_n^x(y)\right) F_n(dy) \le 2F_n\left(\|y\| > r_n^{-k_n+l_n}\right) < \infty.$$

Thus for every $t \ge 0$,

$$f_n^t(x) := \exp\left[t\int_{K_n} \left(\psi_n^x(y) - 1\right) F_n(dy)\right]$$

gives the characteristic function of a probability measure on K_n, say μ_n^t, and it can be seen that $\{\mu_n^t\}_{t\ge 0}$ is a semigroup. Furthermore if $0 \ne x \in K_n$ then by Lemma 5.3 and the formula (5.6),

$$\begin{aligned} \widehat{\mu_n^t}(x) &= \exp\left[-t(q_n-1)^{-1}\left(q_n F_n\left(\|y\| \ge r_n^{-k_n+l_n+1}\right) - F_n\left(\|y\| \ge r_n^{-k_n+l_n+2}\right)\right)\right] \\ &= \exp\left[-th(\|x\|)\right], \quad \text{where } \|x\| = r_n^{k_n}, \end{aligned}$$

which is independent of the choice of n such that $x \in K_n$. Hence $\{\mu_n^t\}_{n=1}^{\infty}$ is consistent for every t. □

EXAMPLE. We can see that for $\alpha, \beta > 0$, $h(\|y\|) = \alpha\|y\|^{\beta}$ satisfies (5.4) and (5.5). If $\mu_n^{(2)} = \mu_n^{(2)}(\alpha, \beta)$ is the probability measure on K_n defined in [E 3.2], then there exists a consistent sequence of semigroups $\{\mu_n^t\}_{t\ge 0}$, such that $\mu_n^1 = \mu_n^{(2)}$. For each n the semigroup $\{\mu_n^t\}_{t\ge 0}$ is associated with a stable process on K_n ([18]). Thus stable processes can be extended to $\overline{B}$.

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Graduate School of Mathematics
Kyushu University
6-10-1 Hakozaki
Higashi-ku, Fukuoka
812-8581 Japan