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MINIMAL IMMERSIONS OF SOME CIRCLE BUNDLES OVER HOLOMORPHIC CURVES IN COMPLEX QUADRIC TO SPHERE

To the memory of Yuko

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0. Introduction

Minimal surfaces in a space of constant curvature has been studied by many mathematicians (cf. [3], [7], [8], [9]). In particular, for minimally immersed 2-sphere in the standard sphere, Calabi [4] and Barbosa [1] showed that: *There exists a canonical 1-1 correspondence between the set of generalized minimal immersions* $\chi : S^2 \to S^{2m}(1)$ *which are not contained in any lower dimensional subspace of* $\mathbb{R}^{2m+1}$, *and the set of totally isotropic holomorphic curves* $\Xi : S^2 \to \mathbb{CP}^{2m}$ *which are not contained in any complex hyperplane of* $\mathbb{CP}^{2m}$. *The correspondence is the one that associates with minimal immersion* χ *its directrix curve* (§2). Note that this fact is valid for *pseudo holomorphic map* [4] from a compact Riemann surface Σ^2 instead of S^2, and that the image of the directrix curve is contained in a complex quadric Q^{2m-1} of $\mathbb{CP}^{2m}$.

On the other hand one of the most interesting 3-dimensional minimal submanifolds in a sphere is the *minimal Cartan hypersurface* (MCH) of S^4, i.e. the minimal hypersurface with 3 distinct constant principal curvatures in a 4-sphere (cf. [5]). MCH is obtained from the directrix curve of the Veronese surface as follows: Let $\chi : S^2(1/3) \to S^4(1)$ be the Veronese immersion from the 2-sphere with constant Gauss curvature $1/3$ to the unit 4-sphere, and let $\Xi : S^2(1/3) \to \mathbb{CP}^4$ be the directrix curve of χ. Then χ is congruent to the fourth order Veronese embedding $\mathbb{CP}^1 \to \mathbb{CP}^4$ and the image $\Xi(S^2(1/3))$ is contained in a complex quadric Q^3 in $\mathbb{CP}^4$. Let $P = SO(5)/SO(3)$ be the set of ordered two orthonormal vectors in $\mathbb{R}^5$, and let $P(Q^3, S^1)$ be the circle bundle over Q^3, which is given as the pullback bundle of the Hopf fibration $S^9(\mathbb{CP}^4, S^1)$ with respect to the natural inclusion $Q^3 \subset \mathbb{CP}^4$. Then MCH is identified with the pullback bundle $\pi_\Xi : \Xi^* P \to S^2(1/3)$ such that each fiber $\pi_\Xi^{-1}(p)$ for $p \in S^2(1/3)$ is corresponding to the great circle which is determined by $\Xi(p) \in Q^3$

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in $S^4(1)$. In other words, MCH is realized as a tube of radius $\pi/2$ over the Veronese surface in S^4, so MCH is diffeomorphic to the unit normal bundle of the Veronese surface in the 4-sphere.

In this paper we will study, as a generalization of minimal Cartan hypersurface, minimal immersion of some circle bundle over a Riemann surface Σ^2 which is immersed in complex quadric $Q^{n-1} = SO(n+1)/SO(n-1) \times SO(2)$ to sphere S^n. More precisely let $P(Q^{n-1}, S^1)$ be the circle bundle over Q^{n-1}, where $P = SO(n+1)/SO(n-1)$ is the set of ordered two orthonormal vectors in $\mathbb{R}^{n+1}$. Here P is the pullback bundle of the Hopf fibration $S^{2n+1}(\mathbb{CP}^n, S^1)$ with respect to the natural inclusion $Q^{n-1} \subset \mathbb{CP}^n$. Let $\varphi : \Sigma^2 \to Q^{n-1}$ be a conformal immersion from a Riemann surface Σ^2 to the complex quadric, and let $\pi_\varphi : \varphi^* P \to \Sigma^2$ be the pullback bundle over Σ^2 with respect to φ. Then each fiber $\pi_\varphi^{-1}(p)$ for $p \in \Sigma^2$ is naturally identified with the great circle of S^n determined by the 2-plane $\varphi(p) \in Q^{n-1}$. We can define the map $\Phi : \varphi^* P \to S^n(1)$ by this identification.

In §1 we review complex quadric Q^{n-1} and construct the circle bundle P over Q^{n-1}, and in §2 we see some surfaces and holomorphic curves in Q^{n-1}. In §3 we show that if a three dimensional submanifold M in a sphere S^n is foliated by great circles of S^n, then there is an associated surface Σ^2 in Q^{n-1}. Conversely we construct the map $\Phi : \varphi^* P \to S^n(1)$ from the surface $\varphi : \Sigma^2 \to Q^{n-1}$ explicitly and, on the set of regular points of Φ, we determine the condition with respect to φ for which the pullback bundle $\varphi^* P$ is minimal in $S^n(1)$ (Proposition 3.9 and Corollary 3.10). In §4 we show that if the immersion $\varphi : \Sigma^2 \to Q^{n-1}$ is holomorphic, then the corresponding map $\Phi : \varphi^* P \to S^n(1)$ is regular at each point in $\pi_\varphi^{-1}(x)$ for $x \in \Sigma^2$ if and only if x is not a real point (Definition 2.7) of φ. Moreover we can see that Φ is minimal if and only if either Φ is totally geodesic or the corresponding holomorphic curve $\varphi(\Sigma^2)$ in Q^{n-1} is first order isotropic (Theorem 1). As a consequence, we can construct full and minimal immersion $\Phi : \Xi^* P \to S^{2m}(1)$ from the directrix curve $\Xi : S^2 \to Q^{2m-1}$ of fully immersed minimal 2-sphere $\chi : S^2 \to S^{2m}(1)$ (Theorem 2). We also discover relations of the curvatures between holomorphic curve $\varphi : \Sigma^2 \to Q^{n-1}$ and the immersion $\Phi : \varphi^* P \to S^n(1)$.

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1. Preliminaries

First of all, we recall the Fubini-Study metric on the complex projective space $\mathbb{CP}^n$. The Euclidean metric $\langle\ ,\ \rangle$ on $\mathbb{C}^{n+1}$ is given by

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{x} \cdot \mathbf{u} + \mathbf{y} \cdot \mathbf{v},$$

where $\mathbf{z} = \mathbf{x} + i\mathbf{y}$, $\mathbf{w} = \mathbf{u} + i\mathbf{v} \in \mathbb{C}^{n+1}$ ($i = \sqrt{-1}$), $\mathbf{x}$, $\mathbf{y}$, $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^{n+1}$ and $\mathbf{x} \cdot \mathbf{y}$ denotes the standard inner product on $\mathbb{R}^{n+1}$. The sphere $S^{2n+1}(1/c)$ of radius $\sqrt{c}$ ($c > 0$) in $\mathbb{C}^{n+1}$ is the principal fiber bundle over $\mathbb{CP}^n$ with the structure group S^1 and the projection map π (*the Hopf fibration*). The tangent space of S^{2n+1} at a point $\mathbf{z}$ is

$$T_{\mathbf{z}}S^{2n+1} = \{\mathbf{w} \in \mathbb{C}^{n+1};\, \langle \mathbf{z}, \mathbf{w}\rangle = 0\}.$$

Let

$$T'_{\mathbf{z}} = \{\mathbf{w} \in \mathbb{C}^{n+1};\, \langle \mathbf{z}, \mathbf{w}\rangle = \langle \mathbf{z}, i\mathbf{w}\rangle = 0\}.$$

Then the distribution $T'_{\mathbf{z}}$ defines a connection in the principal fiber bundle $S^{2n+1}(\mathbb{CP}^n, S^1)$, because $T'_{\mathbf{z}}$ is complementary to the subspace $\{i\mathbf{z}\}$ tangent to the fiber through $\mathbf{z}$, and invariant under the S^1-action. The Fubini-Study metric $\overline{g}$ of constant holomorphic sectional curvature $4/c$ is then given by $\overline{g}(X, Y) = \langle X^*, Y^*\rangle$, where X, $Y \in T_x\mathbb{CP}^n$ and X^*, Y^* are respectively their horizontal lifts at a point $\mathbf{z}$ with $\pi(\mathbf{z}) = x$. The complex structure on T'_z defined by multiplication of $i = \sqrt{-1}$ induces a canonical complex structure J on $\mathbb{CP}^n$ through π_*.

Given a vector field X on $\mathbb{CP}^n$, there is a corresponding *basic vector field* X' on S^{2n+1} such that at $z \in S^{2n+1}$, $X'_z \in T'_z$ and $(\pi_*)_z X'_z = X_{\pi(z)}$. If X, Y are vector fields on $\mathbb{CP}^n$, the Kählerian covariant derivative takes the form

$$\overline{\nabla}^{\mathbb{CP}}_X Y = (\pi_*)\nabla'_{X'}Y',$$

where X', Y' are the basic vector fields corresponding to X, Y and ∇' is the Levi-Civita connection on S^{2n+1}.

Next we recall a description of a complex quadric Q^{n-1} in $\mathbb{CP}^n$ (cf. [13]). Let P be the space of ordered two orthonormal vectors in $\mathbb{R}^{n+1}$, i.e.,

$$P = \{Z \in M(n+1, 2, \mathbb{R});\ {}^tZZ = E_2\}. \tag{1.1}$$

As a homogeneous space, P is isomorphic to $SO(n+1)/SO(n-1)$ (Stiefel manifold) with $\dim_{\mathbb{R}} P = 2n - 1$. Denote $Z = (\mathbf{e}, \mathbf{f}) \in P$, where $\mathbf{e}$ and $\mathbf{f}$ are column vectors of Z. Then the tangent space of P at the point Z is

$$\begin{aligned} T_Z P &= \{X \in M(n+1, 2, \mathbb{R});\ {}^tXZ + {}^tZX = 0\}, \\ &= \mathbb{R}(-\mathbf{f}, \mathbf{e}) \oplus \{(\mathbf{x}, \mathbf{y});\ \mathbf{x}, \mathbf{y} \perp \operatorname{span}\{\mathbf{e}, \mathbf{f}\}\}, \end{aligned}$$

and the Riemannian metric $\widetilde{g}$ on P is given by

$$\widetilde{g}(X, Y) = \operatorname{trace}({}^tXY), \qquad X, Y \in T_Z P \subset M(n+1, 2, \mathbb{R}).$$

Let Q^{n-1} be the space of oriented 2-planes in $\mathbb{R}^{n+1}$. Then P is the principal fiber bundle over Q^{n-1} with the structure group S^1 and the projection map $\pi' : P \to Q^{n-1}$

defined by

$$\pi'((\mathbf{e}, \mathbf{f})) = \operatorname{span}\{\mathbf{e}, \mathbf{f}\}. \tag{1.2}$$

Let

$$T'(\mathbf{e}, \mathbf{f}) = \{(\mathbf{x}, \mathbf{y}) \in M(n+1, 2, \mathbb{R});\ \mathbf{x}, \mathbf{y} \perp \operatorname{span}\{\mathbf{e}, \mathbf{f}\}\}.$$

Then the distribution $T'_{(\mathbf{e},\mathbf{f})}$ defines a connection in the principal fiber bundle $P(Q^{n-1}, S^1)$, because $T'_{(\mathbf{e},\mathbf{f})}$ is complementary to the subspace $\mathbb{R}(-\mathbf{f}, \mathbf{e})$ tangent to the fiber through $(\mathbf{e}, \mathbf{f})$, and invariant under the S^1-action.

The metric g is then given by $g(X, Y) = \widetilde{g}(X^*, Y^*)$, where X, $Y \in T_z Q^{n-1}$ and X^*, Y^* are respectively their horizontal lifts at a point $Z = (\mathbf{e}, \mathbf{f})$ with $\pi'(Z) = z$. The complex structure on $T'_{(\mathbf{e},\mathbf{f})}$ defined by

$$(\mathbf{x}, \mathbf{y}) \mapsto (-\mathbf{y}, \mathbf{x}) \tag{1.3}$$

induces a canonical complex structure J' on Q^{n-1} through π_*. Given a vector field X on Q^{n-1}, there is a corresponding *basic vector field* X' on P such that at $Z = (\mathbf{e}, \mathbf{f}) \in P$, $X'_Z \in T'_Z$ and $(\pi'_*)_Z X'_Z = X_{\pi'(Z)}$. If X, Y are vector fields on Q^{n-1}, the Kählerian covariant derivative takes the form

$$\overline{\nabla}^Q_X Y = (\pi'_*)\nabla^P_{X'} Y' \tag{1.4}$$

where X', Y' are the basic vector fields corresponding to X, Y and ∇^P is the Levi-Civita connection on P.

We consider an injective map $\widetilde{\imath}$ from P to a $2n+1$-dimensional sphere S^{2n+1} of radius $\sqrt{2}$ in $\mathbb{C}^{n+1}$, defined by

$$\widetilde{\imath}((\mathbf{e}, \mathbf{f})) = \mathbf{e} + i\mathbf{f}.$$

For tangent vectors $(-\mathbf{f}, \mathbf{e})$ and $(\mathbf{x}, \mathbf{y})$ $(\mathbf{x}, \mathbf{y} \perp \operatorname{span}\{\mathbf{e}, \mathbf{f}\})$ in $T_{(\mathbf{e},\mathbf{f})}P$, the differential map of $\widetilde{\imath}$ is

$$\begin{aligned}(\widetilde{\imath}_*)_{(\mathbf{e},\mathbf{f})}(-\mathbf{f}, \mathbf{e}) &= -\mathbf{f} + i\mathbf{e},\\ (\widetilde{\imath}_*)_{(\mathbf{e},\mathbf{f})}(\mathbf{x}, \mathbf{y}) &= \mathbf{x} + i\mathbf{y},\end{aligned}$$

so $\widetilde{\imath}$ is an embedding. Now we can define a holomorphic embedding $\imath : Q^{n-1} \to \mathbb{CP}^n$ as

$$\imath(\operatorname{span}\{\mathbf{e}, \mathbf{f}\}) = \pi(\mathbf{e} + i\mathbf{f}).$$

Hence we have the following commutative diagram:

$$\begin{array}{ccc} P & \xrightarrow{\tilde{\imath}} & S^{2n+1} \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ Q^{n-1} & \xrightarrow[\imath]{} & \mathbb{CP}^n \end{array}$$

Q^{n-1} is also defined by the quadratic equation $z_0^2+z_1^2+\cdots+z_n^2=0$, where $z_0, z_1, \ldots, z_n$ is a homogeneous coordinate of $\mathbb{CP}^n$.

REMARK 1.1. Note that $P(Q^{n-1}, S^1)$ is nothing but the pullback bundle of the Hopf fibration $S^{2n+1}(\mathbb{CP}^n, S^1)$ with respect to $\imath$. Clearly we have the following identification:

$$\begin{array}{ccc} \operatorname{span}\{\mathbf{e},\mathbf{f}\} & \mapsto & \{\cos\theta\mathbf{e}+\sin\theta\mathbf{f} \mid \theta \in S^1\} \\ \Cap & & \Cap \\ Q^{n-1} & & \{\text{oriented great circles } S^1 \subset S^n\}. \end{array}$$

Then for each oriented great circle $C \in Q^{n-1}$, the fiber of C with respect to π' is identified with C itself as

$$\begin{array}{ccc} (\cos\theta\mathbf{e}+\sin\theta\mathbf{f}, -\sin\theta\mathbf{f}+\cos\theta\mathbf{e}) & \mapsto & (\cos\theta\mathbf{e}+\sin\theta\mathbf{f}). \\ \Cap & & \Cap \\ (\pi')^{-1}(C) & & C \end{array}$$

With respect to the metric induced by $\tilde{\imath}$ and $\imath$, P and Q^n become Riemannian manifolds, respectively, and the projection $\pi' : P \to Q^{n-1}$ becomes a Riemannian submersion. The normal space of P in S^{2n+1} (resp. Q^{n-1} in $\mathbb{CP}^n$) at the point $(\mathbf{e}, \mathbf{f})$ is spanned by the following orthonormal vectors:

$$N_1' = \frac{\mathbf{e}-i\mathbf{f}}{\sqrt{2}}, \qquad N_2' = \frac{\mathbf{f}+i\mathbf{e}}{\sqrt{2}},$$

$$(\text{resp. } N_1 = (\pi_*)N_1', \qquad N_2 = (\pi_*)N_2').$$

The shape operators $A^{\imath}_{N_1}$ and $A^{\imath}_{N_2}$ of Q^{n-1} in $\mathbb{CP}^n$ with respect to unit normal vectors N_1 and N_2 at $\pi(\mathbf{e}, \mathbf{f})$ are given by

$$\text{(1.5)} \qquad \begin{aligned} \langle A^{\imath}_{N_1}\pi'_*(\mathbf{x},\mathbf{y}), \pi'_*(\mathbf{u},\mathbf{v})\rangle &= \frac{-\mathbf{x}\cdot\mathbf{u}+\mathbf{y}\cdot\mathbf{v}}{\sqrt{2}}, \\ \langle A^{\imath}_{N_2}\pi'_*(\mathbf{x},\mathbf{y}), \pi'_*(\mathbf{u},\mathbf{v})\rangle &= -\frac{\mathbf{x}\cdot\mathbf{v}+\mathbf{y}\cdot\mathbf{u}}{\sqrt{2}}, \end{aligned}$$

where $(\mathbf{x}, \mathbf{y})$ and $(\mathbf{u}, \mathbf{v}) \in T'_{(\mathbf{e},\mathbf{f})}$.

2. Surfaces and holomorphic curves of complex quadric

Let Σ^2 be a Riemann surface and let $\varphi : \Sigma^2 \to Q^{n-1}$ be a conformal immersion to the complex quadric. Then there exists a local lift $\widetilde{\varphi} : U \to P$ (U is an open set in Σ^2) of φ, i.e., $\widetilde{\varphi}(p) = (\mathbf{e}(p), \mathbf{f}(p)) \in P$ for $p \in U$ (cf. §1), where $(\mathbf{e}(p), \mathbf{f}(p))$ is an ordered orthonormal frame of the 2-plane $\varphi(p) \in Q^{n-1}$. Put $\widetilde{\psi} = \widetilde{\imath} \circ \widetilde{\varphi} : \Sigma^2 \to S^{2n+1} \subset \mathbb{C}^{n+1}$ and $\psi = \imath \circ \varphi : \Sigma^2 \to \mathbb{CP}^n$, respectively. Then $\widetilde{\psi}$ is written as

$$\widetilde{\psi}(p) = \mathbf{e}(p) + i\mathbf{f}(p), \tag{2.1}$$

where $\mathbf{e}$ and $\mathbf{f}$ are both $\mathbb{R}^{n+1}$-valued function on some open set of Σ^2.

Let (t_1, t_2) be an isothermal coordinate on some coordinate neighborhood U of Σ^2. We put the differential of $\mathbf{e}$ and $\mathbf{f}$ with respect to (t_1, t_2) as

$$\mathbf{e}_j := \partial \mathbf{e}/\partial t_j = \lambda_j \mathbf{f} + \mathbf{p}_j, \quad \mathbf{f}_j := \partial \mathbf{e}/\partial t_j = -\lambda_j \mathbf{e} + \mathbf{q}_j \quad (j = 1, 2), \tag{2.2}$$

where $\lambda_j : \Sigma^2 \to \mathbb{R}$ $(j = 1, 2)$ is a function, and $\mathbf{p}_j$, $\mathbf{q}_j \perp \operatorname{span}\{\mathbf{e}, \mathbf{f}\}$. Then the differential map $(\widetilde{\varphi}_*)_p : T_p(\Sigma^2) \to T_{\widetilde{\varphi}(p)}(P)$ is

$$(\widetilde{\varphi}_*)_p(\partial/\partial t_j) = (\lambda_j \mathbf{f} + \mathbf{p}_j, -\lambda_j \mathbf{e} + \mathbf{q}_j), \tag{2.3}$$

and the horizontal part with respect to $\pi' : P \to Q^{n-1}$ is

$$\mathcal{H}(\widetilde{\varphi}_*)_p(\partial/\partial t_j) = (\mathbf{p}_j, \mathbf{q}_j). \tag{2.4}$$

Since (t_1, t_2) is an isothermal coordinate of Σ^2, we have

$$\rho := \|\mathbf{p}_1\|^2 + \|\mathbf{q}_1\|^2 = \|\mathbf{p}_2\|^2 + \|\mathbf{q}_2\|^2, \quad \mathbf{p}_1 \cdot \mathbf{p}_2 + \mathbf{q}_1 \cdot \mathbf{q}_2 = 0. \tag{2.5}$$

(1.3) and (2.4) imply

Proposition 2.1. *Let $\varphi : \Sigma^2 \to Q^{n-1}$ be an immersion from a Riemann surface to a complex quadric. Then*

$$\begin{aligned} &\varphi \text{ is holomorphic} \iff \mathbf{p}_1 = \mathbf{q}_2 \text{ and } \mathbf{p}_2 = -\mathbf{q}_1, \\ &\varphi \text{ is anti-holomorphic} \iff \mathbf{p}_1 = -\mathbf{q}_2 \text{ and } \mathbf{p}_2 = \mathbf{q}_1. \end{aligned} \tag{2.6}$$

Note that the Kähler angle α of the immersion $\varphi : \Sigma^2 \to Q^{n-1}$ is given by

$$\begin{aligned} \cos\alpha &= \rho^{-1}\langle J\varphi_*(\partial/\partial t_1), \varphi_*(\partial/\partial t_2)\rangle \\ &= \rho^{-1}(\mathbf{p}_1 \cdot \mathbf{q}_2 - \mathbf{p}_2 \cdot \mathbf{q}_1). \end{aligned}$$

Suppose that the immersion $\varphi : \Sigma^2 \to Q^{n-1}$ is holomorphic, i.e., (2.6) holds. Then (2.3) is written as

$$(\widetilde{\varphi}_*)_p(\partial/\partial t_1) = (\lambda_1 \mathbf{f} + \mathbf{p}_1, -\lambda_1 \mathbf{e} - \mathbf{p}_2), \tag{2.7}$$

$$(\widetilde{\varphi}_*)_p(\partial/\partial t_2) = (\lambda_2\mathbf{f} + \mathbf{p}_2, -\lambda_2\mathbf{e} + \mathbf{p}_1),$$

and the horizontal part of these vectors are

$$\mathcal{H}(\widetilde{\varphi}_*)_p(\partial/\partial t_1) = (\mathbf{p}_1, -\mathbf{p}_2),$$
$$\mathcal{H}(\widetilde{\varphi}_*)_p(\partial/\partial t_2) = (\mathbf{p}_2, \mathbf{p}_1).$$

EXAMPLE 2.2. Let $\psi_n : \mathbb{CP}^1 \to \mathbb{CP}^n$ be the Veronese embedding of order n given by

$$\psi_n(z) = \left[1, \sqrt{\binom{n}{1}}z, \sqrt{\binom{n}{2}}z^2, \ldots, \sqrt{\binom{n}{k}}z^k, \ldots, \sqrt{\binom{n}{n-1}}z^{n-1}, z^n\right],$$

where z is an inhomogeneous coordinate of $\mathbb{CP}^1$. Then $\psi_n(\mathbb{CP}^1)$ is contained in some Q^{n-1} in $\mathbb{CP}^n$ if and only if n is even. When $n = 4m$ $(m \geq 1)$, $\psi_{4m} : \mathbb{CP}^1 \to Q^{4m-1} \subset \mathbb{CP}^{4m}$ is represented as:

$$\begin{aligned}\psi_{4m}(z) = \Bigg[& 1 + z^{4m}, i(1 - z^{4m}), \\ & \sqrt{\binom{4m}{1}}(z - z^{4m-1}), i\sqrt{\binom{4m}{1}}(z + z^{4m-1}), \\ & \sqrt{\binom{4m}{2}}(z^2 + z^{4m-2}), i\sqrt{\binom{4m}{2}}(z^2 - z^{4m-2}), \\ & \sqrt{\binom{4m}{3}}(z^3 - z^{4m-3}), i\sqrt{\binom{4m}{3}}(z^3 + z^{4m-3}), \\ & \ldots, \\ & \sqrt{\binom{4m}{2m-1}}(z^{2m-1} - z^{2m+1}), i\sqrt{\binom{4m}{2m-1}}(z^{2m-1} + z^{2m+1}), \\ & \sqrt{2\binom{4m}{2m}}z^{2m}\Bigg],\end{aligned}$$

and when $n = 4m - 2$ $(m \geq 1)$, $\psi_{4m-2} : \mathbb{CP}^1 \to Q^{4m-3} \subset \mathbb{CP}^{4m-2}$ is represented as:

$$\begin{aligned}\psi_{4m-2}(z) = \Bigg[& 1 + z^{4m-2}, i(1 - z^{4m-2}), \\ & \sqrt{\binom{4m-2}{1}}(z - z^{4m-3}), i\sqrt{\binom{4m-2}{1}}(z + z^{4m-3}),\end{aligned}$$

$$\sqrt{\binom{4m-2}{2}}(z^2+z^{4m-4}),\ i\sqrt{\binom{4m-2}{2}}(z^2-z^{4m-4}),$$
$$\sqrt{\binom{4m-2}{3}}(z^3-z^{4m-5}),\ i\sqrt{\binom{4m-2}{3}}(z^3+z^{4m-5}),$$
$$\dots,$$
$$\sqrt{\binom{4m-2}{2m-2}}(z^{2m-2}+z^{2m}),\ i\sqrt{\binom{4m-2}{2m-2}}(z^{2m-2}-z^{2m}),$$
$$\left. i\sqrt{2\binom{4m-2}{2m-1}}z^{2m-1}\right].$$

EXAMPLE 2.3. Let f be a holomorphic immersion from a Riemann surface Σ^2 to $\mathbb{CP}^m$, and let $\imath'$ be the inclusion of $\mathbb{CP}^m$ to Q^{2m} defined by $\pi(\mathbf{z}) \mapsto \pi'((\mathbf{z}, i\mathbf{z}))$. Then the composition $\imath' \circ f$ gives a holomorphic curve of Q^{2m}.

EXAMPLE 2.4. Let f be an immersion from a Riemann surface Σ^2 to $\mathbb{R}^{n+1}$. Then the Gauss map $G : \Sigma^2 \to Q^{n-1}$ of f is anti-holomorphic if and only if the immersion f is minimal (cf. [8]). So from a (non-flat) minimal surface in $\mathbb{R}^{n+1}$, we can find a holomorphic curve $\overline{G} : \Sigma^2 \to Q^{n-1}$ by taking the complex conjugate of G.

Theorem 2.5 ([1, 4]). *There exists a canonical* 1-1 *correspondence between the set of generalized minimal immersions* $\chi : S^2 \to S^{2m}(1)$ *which are not contained in any lower dimensional subspace of* $\mathbb{R}^{2m+1}$ *and the set of totally isotropic holomorphic curves* $\Xi : S^2 \to \mathbb{CP}^{2m}$ *which are not contained in any complex hyperplane of* $\mathbb{CP}^{2m}$. *The correspondence is the one that associates with minimal immersion* χ *its directrix curve.*

This theorem holds for *pseudo-holomorphic maps* [4] χ from a Riemann surface Σ^2 to $\to S^{2m}(1)$, i.e.

$$((\partial^j\chi, \partial^k\chi)) = 0, \quad j + k > 0,$$

where $\partial^j\chi = \partial^j\chi/\partial z^j$, z is a local isothermal parameter of Σ^2, and ((,)) denotes the symmetric product of $\mathbb{C}^{2m+1}$.

A holomorphic curve $\Xi : \Sigma^2 \to \mathbb{CP}^{2m}$ is *totally isotropic* if and only if $\Xi(\Sigma^2)$ is not contained in any complex hyperplane of $\mathbb{CP}^{2m}$ and for a local expression ξ of Ξ,

$$((\xi, \xi)) = ((\xi', \xi')) = \cdots = ((\xi^{m-1}, \xi^{m-1})) = 0,$$

where $\xi^k = \partial^k\xi$. In particular, the image of a totally isotropic holomorphic curve Ξ : $S^2 \to \mathbb{P}^{2m}(\mathbb{C})$ is contained in Q^{2m-1}, for $((\xi, \xi)) = 0$. So Ξ gives a holomorphic curve of the complex quadric.

The directrix curve of a minimal immersion $\chi : S^2 \to S^{2m}(1)$ is nothing but the map $\Xi : \Sigma^2 \to Q^{2m-1}$ defined by $\Xi(p)$ = the $(m-1)$-th normal space at p with respect to χ.

EXAMPLE 2.6. Let $\chi : S^2(1/3) \to S^4(1)$ be the Veronese immersion from the sphere of constant Gaussian curvature $1/3$ to the unit 4-sphere. Then the directrix curve of χ is congruent to $\psi_4 : S^2 \to Q^3 \subset \mathbb{CP}^4$ of Example 2.2.

DEFINITION 2.7. For a holomorphic curve $\varphi : \Sigma^2 \to Q^{n-1}$, $x \in \Sigma^2$ is called a *real point* [10, p. 131] if

$$\mathbf{p}_1 \wedge \mathbf{p}_2 = 0,$$

at x, and $x \in \Sigma^2$ is called an *isotropic point* [10, p. 130], if

$$\|\mathbf{p}_1\|^2 = \|\mathbf{p}_2\|^2 \neq 0, \text{ and } \mathbf{p}_1 \cdot \mathbf{p}_2 = 0. \tag{2.8}$$

at x, respectively. φ is called *first order isotropic* [10, p. 134]) if every point $x \in \Sigma^2$ is isotropic.

With respect to the above notation, a holomorphic curve $\varphi : \Sigma^2 \to Q^{n-1}$ is first order isotropic if and only if $(\xi', \xi') = 0$. On the other hand, if every point of Σ^2 is real, then $\varphi(\Sigma^2)$ is contained in a totally geodesic Q^1 in Q^{n-1} [10, Theorem 3.1]. These definitions do not depend on the choice of the section $(\mathbf{e}, \mathbf{f})$.

For a holomorphic curve $\varphi : \Sigma^2 \to Q^{n-1}$ (which in not necessary first order isotropic), put

$$\mathbf{p}^*_{j,k} = \text{ orthogonal component of } \frac{\partial \mathbf{p}_j}{\partial t_k} \text{ to span}\{\mathbf{e}, \mathbf{f}\} \text{ in } \mathbb{R}^{n+1}, \tag{2.9}$$
$$\mathbf{p}^{**}_{j,k} = \text{ orthogonal component of } \frac{\partial \mathbf{p}_j}{\partial t_k} \text{ to span}\{\mathbf{e}, \mathbf{f}, \mathbf{p}_1, \mathbf{p}_2\} \text{ in } \mathbb{R}^{n+1}.$$

Since

$$\overline{\nabla}^Q_{\varphi_*(\partial/\partial t_2)}\varphi_*(\partial/\partial t_1) = \pi'_*(\mathbf{p}^*_{1,2} + \lambda_1\mathbf{p}_1, -\mathbf{p}^*_{2,2} - \lambda_1\mathbf{p}_2)$$

and

$$\overline{\nabla}^Q_{\varphi_*(\partial/\partial t_1)}\varphi_*(\partial/\partial t_2) = \pi'_*(\mathbf{p}^*_{2,1} - \lambda_2\mathbf{p}_2, \mathbf{p}^*_{1,1} - \lambda_2\mathbf{p}_1)$$

are equal (cf. (1.4)), we have

$$\mathbf{p}^{**}_{1,1} + \mathbf{p}^{**}_{2,2} = 0, \quad \mathbf{p}^{**}_{1,2} = \mathbf{p}^{**}_{2,1}. \tag{2.10}$$

Suppose the holomorphic curve $\varphi : \Sigma^2 \to Q^{n-1}$ is *first order isotropic*. Then the second fundamental form σ^φ of φ is written as

$$\sigma^\varphi(\partial/\partial t_1, \partial/\partial t_1) = -\sigma^\varphi(\partial/\partial t_2, \partial/\partial t_2) = \pi'_*(\mathbf{p}^{**}_{1,1}, -\mathbf{p}^{**}_{1,2}),$$
$$\sigma^\varphi(\partial/\partial t_1, \partial/\partial t_2) = \pi'_*(\mathbf{p}^{**}_{1,2}, \mathbf{p}^{**}_{1,1}).$$

Using (1.5) and the Gauss equation, the Gauss curvature K^{Σ^2} of Σ^2 with respect to the metric induced by the first order isotropic holomorphic curve $\psi = \imath\circ\varphi : \Sigma^2 \to \mathbb{CP}^n$ is given by

$$\begin{aligned}(2.11)\quad K^{\Sigma^2} &= 2 - \frac{1}{2}(\|\sigma^\varphi\|^2 + \|A^\imath_{N_1}\|^2 + \|A^\imath_{N_2}\|^2)\\
&= 2 - \frac{1}{\rho^2}\Big(\|\sigma^\varphi(\partial/\partial t_1, \partial/\partial t_1)\|^2 + \|\sigma^\varphi(\partial/\partial t_1, \partial/\partial t_2)\|^2\\
&\quad + \langle A^\imath_{N_1}\varphi_*(\partial/\partial t_1), \varphi_*(\partial/\partial t_1)\rangle^2 + \langle A^\imath_{N_1}\varphi_*(\partial/\partial t_1), \varphi_*(\partial/\partial t_2)\rangle^2\\
&\quad + \langle A^\imath_{N_2}\varphi_*(\partial/\partial t_1), \varphi_*(\partial/\partial t_1)\rangle^2 + \langle A^\imath_{N_2}\varphi_*(\partial/\partial t_1), \varphi_*(\partial/\partial t_2)\rangle^2\Big)\\
&= 2 - \frac{1}{\rho^2}\Big(2(\|\mathbf{p}^{**}_{1,1}\|^2 + \|\mathbf{p}^{**}_{1,2}\|^2) + (\|\mathbf{p}_1\|^2 - \|\mathbf{p}_2\|^2)^2 + 4(\mathbf{p}_1\cdot\mathbf{p}_2)^2\Big)\\
&= 1 - \frac{2}{\rho^2}\Big(\|\mathbf{p}^{**}_{1,1}\|^2 + \|\mathbf{p}^{**}_{1,2}\|^2 - 2\big(\|\mathbf{p}_1\|^2\|\mathbf{p}_2\|^2 - (\mathbf{p}_1\cdot\mathbf{p}_2)^2\big)\Big).\end{aligned}$$

3. Immersions of some circle bundles over surfaces in complex quadric to sphere

Let M^3 be a 3-dimensional submanifolds foliated by (oriented) great circles of unit sphere $S^n(1)$ with an immersion $\Phi : M^3 \to S^n(1)$, and let $C(p)$ be the great circle of the foliation through $p \in M^3$. We note that the foliation on M^3 is regular in the sense of Palais [16, p. 13] (i.e. every point has a coordinate chart distinguished by the foliation, such that each leaf intersects the chart in at most one 2-dimensional slice). This implies that the space of leaves Σ^2 is an 2-dimensional manifold, for each $C(p)$ is complete. Since $C(p)$ is an element of Q^{n-1}, we have a map $\widetilde{\varphi} : M^3 \to Q^{n-1}$ defined by $\widetilde{\varphi}(p) = C(p)$. Then we can easily see that $\widetilde{\varphi}$ factors through an immersion $\varphi : \Sigma^2 \to Q^{n-1}$ (cf. [5, p. 142, Theorem 4.6]).

EXAMPLE 3.1. Let M^3 be a hypersurface of $S^4(1)$ on which type number (i.e., rank of shape operator A) of M is 2. Then each integral curve of 1-dimensional distribution $\ker A$ on M^3 is a part of great circle of $S^4(1)$. In particular, minimal Cartan hypersurface (i.e., the minimal isoparametric hypersurface with 3 distinct constant principal curvatures c, 0, $-c$ and $c \neq 0$) of unit 4-sphere is foliated by great circles of S^4.

EXAMPLE 3.2. As a generalization of Example 3.1, let M^3 be a 3-dimensional submanifold of $S^n(1)$ with $n \geq 5$. Suppose dimension of subspace $V(p)$ of tangent space $T_p(M^3)$, defined by

$$V(p) = \{X \in T_p(M^3) | \sigma^M(X, Y) = 0 \text{ for any } Y \in T_p(M^3)\},$$

is 1 for each $p \in M$, where σ^M denotes second fundamental form of M^3 in $S^n(1)$. Then each integral curve of 1-dimensional distribution $V(p)$ on M^3 is a part of great circle of $S^n(1)$.

EXAMPLE 3.3. Let Σ^2 be a 2-dimensional surface of $\mathbb{CP}^m$. Then $M^3 = \pi^{-1}(\Sigma^2)$ is a 3-dimensional submanifold foliated by great circles $\pi^{-1}(p)$ for $p \in \Sigma^2$ of $S^{2m+1}(1)$.

EXAMPLE 3.4. Let $S^1(c_1) \times S^2(c_2)$ be a Riemann product of the circle of radius $1/\sqrt{c_1}$ in $\mathbb{R}^2$ and the round 2−sphere of radius $1/\sqrt{c_2}$ on which $1/c_1 + 1/c_2 = 1$ holds. We parameterize the immersion $\Phi : S^1(c_1) \times S^2(c_2) \to S^4(1)$ into the unit 4-sphere as:

$$\Big(\frac{1}{\sqrt{c_1}}(\cos\theta, \sin\theta), \frac{1}{\sqrt{c_2}}(\cos u \cos v, \cos u \sin v, \sin u)\Big) \mapsto$$
$$\Big(\frac{1}{\sqrt{c_1}}(\cos\theta, \sin\theta), \frac{1}{\sqrt{c_2}}\big(\cos(\theta + u)\cos v, \cos(\theta + u)\sin v, \sin(\theta + u)\big)\Big).$$

Then integral curves of the vector field $\partial/\partial\theta$ are great circles of S^4.

Let $\varphi : \Sigma^2 \to Q^{n-1}$ be a conformal immersion from a Riemann surface Σ^2 to the complex quadric Q^{n-1} as in §2, and let $P(Q^{n-1}, S^1)$ be the circle bundle over Q^{n-1} (cf. §1), which is the pullback bundle of the Hopf fibration $S^{2n+1}(\mathbb{CP}^n, S^1)$, where P is the space of ordered two orthonormal vectors in $\mathbb{R}^{n+1}$. We denote the pullback bundle over Σ^2 with respect to φ as $\pi_\varphi : \varphi^*P \to \Sigma^2$, and let $M^3 = \varphi^*P$. By the definition, there is a bundle chart $\{(U_\alpha, \varphi_\alpha)\}$ $(\alpha \in \Lambda)$ of φ^*P such that

$$u \in \pi_\varphi^{-1}(U_\alpha) \to (\pi_\varphi(u), \varphi_\alpha(u)) \in U_\alpha \times S^1$$

gives a homeomorphism. If $U_\alpha \cap U_\beta \neq \emptyset$, then $\varphi_\beta(u)(\varphi_\alpha(u))^{-1}$ gives rise to the transition function

$$x \in U_\alpha \cap U_\beta \to \Theta_{\beta\alpha}(x) \in S^1 = \mathbb{R}/2\pi\mathbb{Z}.$$

Note that if $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, then we can see that

$$\Theta_{\gamma\alpha}(p) = \Theta_{\gamma\beta}(p) + \Theta_{\beta\alpha}(p), \quad p \in U_\alpha \cap U_\beta \cap U_\gamma.$$

For each $\alpha \in \Lambda$, we can take the (differentiable) section $\rho_\alpha : U_\alpha \to \pi_\varphi^{-1}(U_\alpha)$ as

$$\pi_\varphi(\rho_\alpha(p)) = p, \quad \varphi_\alpha(\rho_\alpha(p)) = e \quad \text{(identity)}.$$

If $U_\alpha \cap U_\beta \neq \emptyset$, then we have

$$\rho_\alpha(p) = \rho_\beta(p)\Theta_{\beta\alpha}(p), \quad p \in U_\alpha \cap U_\beta. \tag{3.1}$$

Since $\rho_\alpha(p)$ and $\rho_\beta(p)$ are viewed as elements in P, we may write them as

$$\rho_\alpha(p) = (\mathbf{e}_\alpha(p), \mathbf{f}_\alpha(p)), \quad \rho_\beta(p) = (\mathbf{e}_\beta(p), \mathbf{f}_\beta(p)),$$

where $\mathbf{e}_\alpha(p)$, $\mathbf{f}_\alpha(p)$ and $\mathbf{e}_\beta(p)$, $\mathbf{f}_\beta(p)$ are oriented orthonormal basis of the two-plane $\varphi(p) \in Q^{n-1}$. Then (3.1) is written as

$$\begin{aligned}(\mathbf{e}_\alpha(p), \mathbf{f}_\alpha(p)) = (&\cos\Theta_{\beta\alpha}(p)\mathbf{e}_\beta(p) + \sin\Theta_{\beta\alpha}(p)\mathbf{f}_\beta(p), \\ &-\sin\Theta_{\beta\alpha}(p)\mathbf{e}_\beta(p) + \cos\Theta_{\beta\alpha}(p)\mathbf{f}_\beta(p)).\end{aligned} \tag{3.2}$$

Note that the pullback bundle $M^3 = \varphi^* P$ is also realized as the quotient space $\Lambda \times \Sigma^2 \times S^1/\sim$, where

$$(\alpha, p, \theta) \sim (\beta, q, \zeta) \Longleftrightarrow p = q \in U_\alpha \cap U_\beta, \text{ and } \zeta = \theta + \Theta_{\beta\alpha}. \tag{3.3}$$

For each $p \in \Sigma^2$, the fiber $\pi_\varphi^{-1}(p)$ with respect to $\pi_\varphi : \varphi^* P \to \Sigma^2$ is identified with the great circle $\varphi(p) \in Q^{n-1}$ (cf. Remark 1.1).

We define the map $\Phi : M^3 = \varphi^* P \to S^n(1)$ as

$$\Phi([\alpha, p, \theta]) = \cos\theta\mathbf{e}_\alpha(p) + \sin\theta\mathbf{f}_\alpha(p), \tag{3.4}$$

where $[\alpha, p, \theta]$ is the equivalence class of $(\alpha, p, \theta) \in \Lambda \times \Sigma^2 \times S^1$. By (3.2) and (3.3), Φ is well-defined. We can see that Φ maps each fiber $\pi_\varphi^{-1}(p)$ for $p \in \Sigma^2$ to the corresponding great circle $\varphi(p) \in Q^{n-1}$

If $p \in U_\alpha \subset \Sigma^2$, then $p \mapsto (\mathbf{e}_\alpha(p), \mathbf{f}_\alpha(p))$ gives a lift of $\varphi|_{U_\alpha} : U_\alpha \to Q^{n-1}$ to P. For simplicity we denote $(\mathbf{e}(p), \mathbf{f}(p))$ instead of $(\mathbf{e}_\alpha(p), \mathbf{f}_\alpha(p))$, and we use the same notations as §1. We may view Φ as a $\mathbb{R}^{n+1}$-valued function on M^3. Using (2.2), we get that the first order differential of Φ is

$$\Phi_\theta = \frac{\partial\Phi}{\partial\theta} = -\sin\theta\mathbf{e} + \cos\theta\mathbf{f}, \tag{3.5}$$

$$\Phi_j = \frac{\partial\Phi}{\partial t_j} = \cos\theta(\lambda_j\mathbf{f} + \mathbf{p}_j) + \sin\theta(-\lambda_j\mathbf{e} + \mathbf{q}_j) \quad (j = 1, 2), \tag{3.6}$$

and

$$\Phi \wedge \Phi_\theta = \mathbf{e} \wedge \mathbf{f}. \tag{3.7}$$

Denote

$$\Psi_j := \Phi_j - \lambda_j\Phi_\theta \tag{3.8}$$

$$= \cos\theta \mathbf{p}_j + \sin\theta \mathbf{q}_j, \quad (j = 1, 2).$$

So

$$\begin{aligned}\Psi_1 \wedge \Psi_2 &= \cos^2\theta(\mathbf{p}_1 \wedge \mathbf{p}_2) + \cos\theta\sin\theta(\mathbf{p}_1 \wedge \mathbf{q}_2 - \mathbf{p}_2 \wedge \mathbf{q}_1)\\ &\quad + \sin^2\theta(\mathbf{q}_1 \wedge \mathbf{q}_2).\end{aligned}$$

Hence we have

Proposition 3.5. *Let $\varphi : \Sigma^2 \to Q^{n-1}$ be a conformal immersion from a Riemann surface Σ^2 to the complex quadric Q^{n-1}, and let $P(Q^{n-1}, S^1)$ be the circle bundle over Q^{n-1}, where $P = SO(n+1)/SO(n-1)$ is the space of ordered two orthonormal vectors in $\mathbb{R}^{n+1}$. Then*

(1) *The map Φ from the pullback bundle $\varphi^* P$ to $S^n(1)$ defined by* (3.4) *maps each fiber $\pi_\varphi^{-1}(p)$ for $p \in \Sigma^2$ of the circle bundle $\pi_\varphi : P \to \Sigma^2$ to the corresponding great circle $\varphi(p) \in Q^{n-1}$ of $S^n(1)$.*

(2) *Φ is regular at $[\alpha, p, \theta] \in \varphi^* P$ if and only if at $p \in U_\alpha \subset \Sigma^2$, φ satisfies*

$$\cos^2\theta(\mathbf{p}_1 \wedge \mathbf{p}_2) + \cos\theta\sin\theta(\mathbf{p}_1 \wedge \mathbf{q}_2 - \mathbf{p}_2 \wedge \mathbf{q}_1) + \sin^2\theta(\mathbf{q}_1 \wedge \mathbf{q}_2) \neq 0.$$

(3) *If $\varphi(\Sigma^2)$ is not contained in a totally geodesic Q^{n-2} in Q^{n-1}, then $\Phi(\varphi^* P)$ is not contained in a totally geodesic $S^{n-1}(1)$ in $S^n(1)$.*

We suppose that Φ is an immersion, i.e., with respect to a basis $\{\Phi_\theta, \Psi_1, \Psi_2\}$ of the tangent space $T_{(\mathbf{p},\theta)}M$, the metric of M induced by Φ is given as follows:

$$\begin{aligned}&\|\Phi_\theta\|^2 = 1, \quad \Phi_\theta \cdot \Psi_1 = \Phi_\theta \cdot \Psi_2 = 0,\\ &\|\Psi_1\|^2 = \|\mathbf{p}_1\|^2\cos^2\theta + 2\mathbf{p}_1 \cdot \mathbf{q}_1 \cos\theta\sin\theta + \|\mathbf{q}_1\|^2\sin^2\theta,\\ &\Psi_1 \cdot \Psi_2 = \mathbf{p}_1 \cdot \mathbf{p}_2 \cos^2\theta + (\mathbf{p}_1 \cdot \mathbf{q}_2 + \mathbf{p}_2 \cdot \mathbf{q}_1)\cos\theta\sin\theta + \mathbf{q}_1 \cdot \mathbf{q}_2 \sin^2\theta,\\ &\|\Psi_2\|^2 = \|\mathbf{p}_2\|^2\cos^2\theta + 2\mathbf{p}_2 \cdot \mathbf{q}_2 \cos\theta\sin\theta + \|\mathbf{q}_2\|^2\sin^2\theta.\end{aligned}$$

We find the condition whether the tangent vectors Φ_θ of each great circle corresponding to a two-plane $\varphi(p) \subset \mathbb{R}^n$ is a null direction of the second fundamental form σ^Φ of $\Phi : M^3 \to S^n(1)$ or not. Since $D_{\Phi_\theta}\Phi_\theta = -\Phi$, clearly

$$\sigma^\Phi(\Phi_\theta, \Phi_\theta) = 0, \tag{3.9}$$

where D is a flat connection of $\mathbb{R}^{n+1}$. By the fact that

$$D_{\Phi_\theta}\Psi_j = -\sin\theta \mathbf{p}_j + \cos\theta \mathbf{q}_j \quad (j = 1, 2) \tag{3.10}$$

is orthogonal to Φ and Φ_θ, we have

$$\sigma^\Phi(\Phi_\theta, \Psi_j) = 0 \Leftrightarrow \Psi_1 \wedge \Psi_2 \wedge D_{\Phi_\theta}\Psi_j = 0.$$

Hence

$$\sigma^{\Phi}(\Phi_\theta, \Psi_1) = 0 \iff -\mathbf{p}_1 \wedge \mathbf{q}_1 \wedge (\cos\theta\mathbf{p}_2 + \sin\theta\mathbf{q}_2) = 0 \iff \begin{cases} \mathbf{p}_1 \wedge \mathbf{q}_1 \wedge \mathbf{p}_2 = 0 \\ \mathbf{p}_1 \wedge \mathbf{q}_1 \wedge \mathbf{q}_2 = 0, \end{cases} \tag{3.11}$$

and

$$\sigma^{\Phi}(\Phi_\theta, \Psi_2) = 0 \iff \mathbf{p}_2 \wedge \mathbf{q}_2 \wedge (\cos\theta\mathbf{p}_1 + \sin\theta\mathbf{q}_1) = 0 \iff \begin{cases} \mathbf{p}_2 \wedge \mathbf{q}_2 \wedge \mathbf{p}_1 = 0 \\ \mathbf{p}_2 \wedge \mathbf{q}_2 \wedge \mathbf{q}_1 = 0. \end{cases} \tag{3.12}$$

Consequently we have

Proposition 3.6. *Let $\varphi : \Sigma^2 \to Q^{n-1}$ be an immersion from a surface to a complex quadric, and let $\Phi : M^3 = \varphi^* P \to S^n(1)$ be the corresponding immersion defined by* (3.4)*. Then the tangent vectors Φ_θ of each great circle corresponding to a two-plane $\varphi(p) \subset \mathbb{R}^n$ is a null direction of the second fundamental form σ^Φ of Φ if and only if*

$$\dim \operatorname{span}\{\mathbf{p}_1, \mathbf{q}_1, \mathbf{p}_2, \mathbf{q}_2\} \leq 2.$$

Because of Proposition 2.1, we get

Corollary 3.7. *Under the same assumption as in* Proposition 3.6, *if φ is either holomorphic or anti-holomorphic, then the tangent vectors Φ_θ of each great circle corresponding to a two-plane $\varphi(p) \subset \mathbb{R}^n$ is a null direction of the second fundamental form σ^Φ of Φ.*

REMARK 3.8. In Example 3.4, generalized Clifford torus $\Phi : S^1(c_1) \times S^2(c_2) \to S^4(1) \subset S^n(1)$ is given by a *totally real* surface Σ^2 in Q^{n-1}. But there is no null direction of the second fundamental form σ^Φ of Φ.

Next we try to find the condition such that the immersion $\Phi : M^3 = \varphi^* P \to S^n(1)$ is minimal. Since $\Phi \wedge \Phi_\theta = \mathbf{e} \wedge \mathbf{f}$, Φ is minimal if and only if

$$\begin{aligned} \Psi_1 \wedge \Psi_2 \wedge \{\|\Psi_2\|^2 D_{\Psi_1}\Psi_1 - (\Psi_1 \cdot \Psi_2)(D_{\Psi_1}\Psi_2 + D_{\Psi_2}\Psi_1) \\ + \|\Psi_1\|^2 D_{\Psi_2}\Psi_2\} \equiv 0 \mod (\mathbf{e}, \mathbf{f}). \end{aligned} \tag{3.13}$$

Differentiating Ψ_j by $\Psi_k = \Phi_k - \lambda_k \Phi_\theta$ (see (3.5)), we get

$$D_{\Psi_k}\Psi_j = \cos\theta(\mathbf{p}_{j,k} - \lambda_k \mathbf{q}_j) + \sin\theta(\mathbf{q}_{j,k} + \lambda_k \mathbf{p}_j),$$

where $\mathbf{p}_{j,k} = \partial \mathbf{p}_j / \partial t_k$ and $\mathbf{q}_{j,k} = \partial \mathbf{q}_j / \partial t_k$ in $\mathbb{R}^{n+1}$, respectively. Put

$$\begin{aligned} A_{j,k} &= \mathbf{p}^*_{j,k} - \lambda_k \mathbf{q}_j, \quad \mathbf{p}^*_{j,k} = \mathbf{p}_{j,k} + (\mathbf{p}_j \cdot \mathbf{p}_k)\mathbf{e} + (\mathbf{p}_j \cdot \mathbf{q}_k)\mathbf{f}, \\ B_{j,k} &= \mathbf{q}^*_{j,k} + \lambda_k \mathbf{p}_j, \quad \mathbf{q}^*_{j,k} = \mathbf{q}_{j,k} + (\mathbf{q}_j \cdot \mathbf{p}_k)\mathbf{e} + (\mathbf{q}_j \cdot \mathbf{q}_k)\mathbf{f}, \end{aligned}$$

i.e., $\mathbf{p}^*_{j,k}$ and $\mathbf{q}^*_{j,k}$ are orthogonal components of $\mathbf{p}_{j,k}$ and $\mathbf{q}_{j,k}$ to span$\{\mathbf{e}, \mathbf{f}\}$ in $\mathbb{R}^{n+1}$, respectively. Then we obtain

$$D_{\Psi_k} \Psi_j \equiv \cos\theta A_{j,k} + \sin\theta B_{j,k} \mod (\mathbf{e}, \mathbf{f}).$$

Note that (2.2), (3.2) and $\partial^2 \Phi / \partial t_j \partial t_k = \partial^2 \Phi / \partial t_k \partial t_j$ imply

$$A_{j,k} = A_{k,j} \text{ and } B_{j,k} = B_{k,j}.$$

By direct calculations, we have

$$\begin{aligned} &\|\Psi_2\|^2 D_{\Psi_1}\Psi_1 - (\Psi_1 \cdot \Psi_2)(D_{\Psi_1}\Psi_2 + D_{\Psi_2}\Psi_1) + \|\Psi_1\|^2 D_{\Psi_2}\Psi_2 \\ \equiv\ & \cos^3\theta C_0 + \cos^2\theta \sin\theta C_1 + \cos\theta \sin^2\theta C_2 + \sin^3\theta C_3, \quad \mod (\mathbf{e}, \mathbf{f}) \end{aligned}$$

where

$$\begin{aligned} C_0 &= \|\mathbf{p}_2\|^2 A_{1,1} - 2(\mathbf{p}_1 \cdot \mathbf{p}_2) A_{1,2} + \|\mathbf{p}_1\|^2 A_{2,2}, \\ C_1 &= \|\mathbf{p}_2\|^2 B_{1,1} - 2(\mathbf{p}_1 \cdot \mathbf{p}_2) B_{1,2} + \|\mathbf{p}_1\|^2 B_{2,2}, \\ &\quad + 2(\mathbf{p}_2 \cdot \mathbf{q}_2) A_{1,1} - 2(\mathbf{p}_1 \cdot \mathbf{q}_2 + \mathbf{p}_2 \cdot \mathbf{q}_1) A_{1,2} + 2(\mathbf{p}_1 \cdot \mathbf{q}_1) A_{2,2} \\ C_2 &= \|\mathbf{q}_2\|^2 A_{1,1} - 2(\mathbf{q}_1 \cdot \mathbf{q}_2) A_{1,2} + \|\mathbf{q}_1\|^2 A_{2,2}, \\ &\quad + 2(\mathbf{p}_2 \cdot \mathbf{q}_2) B_{1,1} - 2(\mathbf{p}_1 \cdot \mathbf{q}_2 + \mathbf{p}_2 \cdot \mathbf{q}_1) B_{1,2} + 2(\mathbf{p}_1 \cdot \mathbf{q}_1) B_{2,2} \\ C_3 &= \|\mathbf{q}_2\|^2 B_{1,1} - 2(\mathbf{q}_1 \cdot \mathbf{q}_2) B_{1,2} + \|\mathbf{q}_1\|^2 B_{2,2}. \end{aligned}$$

Since $\cos^5\theta, \cos^4\theta \sin\theta, \ldots, \sin^5\theta$ are independent functions in (3.13), we get

Proposition 3.9. *Let $\varphi : \Sigma^2 \to Q^{n-1}$ be an immersion from a surface to a complex quadric, and let $\Phi : M^3 = \varphi^* P \to S^n(1)$ be the corresponding immersion defined by* (3.4). *Then Φ is minimal if and only if the following equations hold*:

$$\begin{aligned} &\mathbf{p}_1 \wedge \mathbf{p}_2 \wedge C_0 = 0, \\ &(\mathbf{p}_1 \wedge \mathbf{q}_2 - \mathbf{p}_2 \wedge \mathbf{q}_1) \wedge C_0 + \mathbf{p}_1 \wedge \mathbf{p}_2 \wedge C_1 = 0, \\ &\mathbf{q}_1 \wedge \mathbf{q}_2 \wedge C_0 + (\mathbf{p}_1 \wedge \mathbf{q}_2 - \mathbf{p}_2 \wedge \mathbf{q}_1) \wedge C_1 + \mathbf{p}_1 \wedge \mathbf{p}_2 \wedge C_2 = 0, \\ &\mathbf{q}_1 \wedge \mathbf{q}_2 \wedge C_1 + (\mathbf{p}_1 \wedge \mathbf{q}_2 - \mathbf{p}_2 \wedge \mathbf{q}_1) \wedge C_2 + \mathbf{p}_1 \wedge \mathbf{p}_2 \wedge C_3 = 0, \\ &\mathbf{q}_1 \wedge \mathbf{q}_2 \wedge C_2 + (\mathbf{p}_1 \wedge \mathbf{q}_2 - \mathbf{p}_2 \wedge \mathbf{q}_1) \wedge C_3 = 0, \\ &\mathbf{q}_1 \wedge \mathbf{q}_2 \wedge C_3 = 0. \end{aligned}$$

Corollary 3.10. *Under the same assumption as* Proposition 3.9, *suppose* $n \geq 5$ *and* $\dim \operatorname{span}\{\mathbf{p}_1, \mathbf{q}_1, \mathbf{p}_2, \mathbf{q}_2\} = 4$. *Then* Φ *is minimal if and only if the following equations hold:*

$$\begin{aligned} C_0 &= \mu_0 \mathbf{p}_1 + \nu_0 \mathbf{p}_2, \\ C_1 &= \mu_1 \mathbf{p}_1 + \mu_0 \mathbf{q}_1 + \nu_1 \mathbf{p}_2 + \nu_0 \mathbf{q}_2, \\ C_2 &= \mu_2 \mathbf{p}_1 + \mu_1 \mathbf{q}_1 + \nu_2 \mathbf{p}_2 + \nu_1 \mathbf{q}_2, \\ C_3 &= \mu_2 \mathbf{q}_1 + \nu_2 \mathbf{q}_2, \end{aligned}$$

where $\mu_0, \mu_1, \mu_2, \nu_0, \nu_1, \nu_2$ *are some functions on* Σ^2.

4. Three dimensional submanifolds of the sphere given by holomorphic curves of the complex quadric

In this section, as a special case of §2, we investigate 3-dimensional submanifold M^3 of $S^n(1)$ given by *holomorphic curve* Σ^2 of Q^{n-1}. We use the same notation as §2 and §3.

Let $\varphi : \Sigma^2 \to Q^{n-1}$ be a holomorphic immersion from a Riemann surface Σ^2 to the complex quadric Q^{n-1} and let $\pi_\varphi : \varphi^* P \to \Sigma^2$ be the pullback bundle of the circle bundle $P(Q^{n-1}, S^1)$ ($P = SO(n+1)/SO(n-1)$ is the set of ordered orthonormal 2-vectors in $\mathbb{R}^{n+1}$) with respect to φ. We consider the map $\Phi : \varphi^* P \to S^n(1)$ defined by (3.4). Using (2.2), we get that the first order differential of Φ is

$$\begin{aligned} \Phi_\theta &= \frac{\partial \Phi}{\partial \theta} = -\sin\theta \mathbf{e} + \cos\theta \mathbf{f}, \\ \Phi_1 &= \frac{\partial \Phi}{\partial t_1} = \cos\theta(\lambda_1 \mathbf{f} + \mathbf{p}_1) + \sin\theta(-\lambda_1 \mathbf{e} - \mathbf{p}_2), \\ \Phi_2 &= \frac{\partial \Phi}{\partial t_2} = \cos\theta(\lambda_2 \mathbf{f} + \mathbf{p}_2) + \sin\theta(-\lambda_2 \mathbf{e} + \mathbf{p}_1). \end{aligned}$$

As in §3, we denote

$$\begin{aligned} \Psi_1 &:= \Phi_1 - \lambda_1 \Phi_\theta = \cos\theta \mathbf{p}_1 - \sin\theta \mathbf{p}_2, \\ \Psi_2 &:= \Phi_2 - \lambda_2 \Phi_\theta = \cos\theta \mathbf{p}_2 + \sin\theta \mathbf{p}_1. \end{aligned}$$

Using Proposition 3.5, we get

Proposition 4.1. *Let* $\varphi : \Sigma^2 \to Q^{n-1}$ *be a holomorphic immersion from a Riemann surface* Σ^2 *to the complex quadric* Q^{n-1}, *and let* $\varphi^* P(\Sigma^2, S^1)$ *be the pullback bundle of the circle bundle* $P(Q^{n-1}, S^1)$ ($P = SO(n+1)/SO(n-1)$) *with respect to* φ. *Then the map* $\Phi : \varphi^* P \to S^n(1)$ *defined by* (3.4) *is regular at each point in* $\pi_\varphi^{-1}(x)$ *for* $x \in \Sigma^2$ *if and only if* x *is not a real point for* φ (Definition 2.7). *Consequently if* *the holomorphic curve $\varphi : \Sigma^2 \to Q^{n-1}$ is first order isotropic, then the corresponding map $\Phi : \varphi^* P \to S^n(1)$ is always an immersion.*

With respect to a basis $\{\Phi_\theta, \Psi_1, \Psi_2\}$ of the tangent space $T_{[\alpha,\mathbf{p},\theta]}M$, the metric of $\varphi^* P$ induced by Φ is given as follows:

$$\begin{aligned}
\|\Phi_\theta\|^2 &= 1, \quad \Phi_\theta \cdot \Psi_1 = \Phi_\theta \cdot \Psi_2 = 0,\\
\|\Psi_1\|^2 &= \|\mathbf{p}_1\|^2 \cos^2\theta - 2\mathbf{p}_1 \cdot \mathbf{p}_2 \cos\theta \sin\theta + \|\mathbf{p}_2\|^2 \sin^2\theta,\\
\Psi_1 \cdot \Psi_2 &= \mathbf{p}_1 \cdot \mathbf{p}_2(\cos^2\theta - \sin^2\theta) + (\|\mathbf{p}_1\|^2 - \|\mathbf{p}_2\|^2)\cos\theta \sin\theta,\\
\|\Psi_2\|^2 &= \|\mathbf{p}_2\|^2 \cos^2\theta + 2\mathbf{p}_1 \cdot \mathbf{p}_2 \cos\theta \sin\theta + \|\mathbf{p}_1\|^2 \sin^2\theta.
\end{aligned}$$

Put

$$\begin{aligned}
\rho &= \|\mathbf{p}_1\|^2 + \|\mathbf{p}_2\|^2,\\
\rho_1 &= \|\mathbf{p}_1\|^2 - \|\mathbf{p}_2\|^2, \qquad (4.1)\\
\rho_2 &= 2\mathbf{p}_1 \cdot \mathbf{p}_2.
\end{aligned}$$

Note that the holomorphic immersion $\varphi : \Sigma^2 \to Q^{n-1}$ is first order isotropic (cf. Definition 2.7) if and only if $\rho_1 = \rho_2 = 0$. Then we have

$$\begin{aligned}
\|\Psi_1\|^2 &= \frac{1}{2}(\rho + \rho_1 \cos 2\theta - \rho_2 \sin 2\theta),\\
\Psi_1 \cdot \Psi_2 &= \frac{1}{2}(\rho_1 \sin 2\theta + \rho_2 \cos 2\theta),\\
\|\Psi_2\|^2 &= \frac{1}{2}(\rho - \rho_1 \cos 2\theta + \rho_2 \sin 2\theta),
\end{aligned}$$

and

$$\begin{aligned}
\Delta := \|\Psi_1\|^2\|\Psi_2\|^2 - (\Psi_1 \cdot \Psi_2)^2 &= \frac{\rho^2 - \rho_1^2 - \rho_2^2}{4}\\
&= \|\mathbf{p}_1\|^2\|\mathbf{p}_2\|^2 - (\mathbf{p}_1 \cdot \mathbf{p}_2)^2 > 0.
\end{aligned}$$

Next, we calculate the second fundamental form of $\Phi : M^3 \to S^n(1)$. By (3.9), (3.11) and (3.12), we have

$$\sigma^\Phi(\Phi_\theta, \Phi_\theta) = \sigma^\Phi(\Phi_\theta, \Psi_j) = 0 \quad (j = 1, 2).$$

(3.10) yields that

$$\begin{aligned}
D_{\Psi_j}\Psi_1 &= \cos\theta(\mathbf{p}_{1,j} + \lambda_j \mathbf{p}_2) + \sin\theta(-\mathbf{p}_{2,j} + \lambda_j \mathbf{p}_1),\\
D_{\Psi_j}\Psi_2 &= \cos\theta(\mathbf{p}_{2,j} - \lambda_j \mathbf{p}_1) + \sin\theta(\mathbf{p}_{1,j} + \lambda_j \mathbf{p}_2).
\end{aligned}$$

Since span$\{\Phi\} + T_\Phi(M^3)$ is spanned by $\mathbf{e}, \mathbf{f}, \mathbf{p}_1, \mathbf{p}_2$, second fundamental form of Φ is (cf. (2.9) and (2.10))

$$\begin{aligned}\sigma_{11} &:= \sigma^\Phi(\Psi_1, \Psi_1) = \cos\theta \mathbf{p}^{**}_{1,1} - \sin\theta \mathbf{p}^{**}_{1,2},\\ \sigma_{12} &:= \sigma^\Phi(\Psi_1, \Psi_2) = \cos\theta \mathbf{p}^{**}_{1,2} + \sin\theta \mathbf{p}^{**}_{1,1},\\ \sigma_{22} &:= \sigma^\Phi(\Psi_2, \Psi_2) = -\cos\theta \mathbf{p}^{**}_{1,1} + \sin\theta \mathbf{p}^{**}_{1,2}.\end{aligned}$$

Hence the mean curvature vector H^Φ of $\Phi : M^3 \to S^n(1)$ is

$$\begin{aligned}H^\Phi &= \frac{1}{\Delta}(\|\Psi_2\|^2\sigma_{11} - 2\Psi_1 \cdot \Psi_2\sigma_{12} + \|\Psi_1\|^2\sigma_{22})\\ &= \frac{2}{\Delta}\Big(-\cos\theta(\rho_1\mathbf{p}^{**}_{1,1} + \rho_2\mathbf{p}^{**}_{1,2}) + \sin\theta(-\rho_1\mathbf{p}^{**}_{1,2} + \rho_2\mathbf{p}^{**}_{1,1})\Big).\end{aligned}$$

Consequently we obtain

Theorem 1. *Let $\varphi : \Sigma^2 \to Q^{n-1}$ be a holomorphic immersion from a Riemann surface Σ^2 to complex quadric Q^{n-1}, and let $\pi_\varphi : \varphi^*P \to \Sigma^2$ be the pullback bundle of the circle bundle $P(Q^{n-1}, S^1)$ with respect to φ, where $P = SO(n+1)/SO(n-1)$ is the set of ordered two orthonormal vectors. Suppose the map $\Phi : \varphi^*P \to S^n(1)$ defined by* (3.4), *i.e. each fiber $\pi_\varphi^{-1}(p)$ for $p \in \Sigma^2$ is mapped to the corresponding great circle $\varphi(p) \in Q^{n-1}$, is an immersion. Then Φ is minimal* (*i.e.* $H^\Phi = 0$) *if and only if either Φ is totally geodesic* ($\mathbf{p}^{**}_{1,1} = \mathbf{p}^{**}_{1,2} = 0$) *or the holomorphic curve φ is first order isotropic* ($\rho_1 = \rho_2 = 0$).

Theorem 1 and Proposition 4.1 imply

Theorem 2. *Let χ be a full minimal immersion from 2-sphere S^2* (*resp. a pseudo holomorphic map* [4] *from a Riemann surface* Σ^2) *to $S^{2m}(1)$ and let $\pi_\Xi : \Xi^*P \to S^2$* (*resp.* Σ^2) *be the pullback bundle of the circle bundle $P(Q^{2m-1}, S^1)$* ($P = SO(2m+1)/SO(2m-1)$) *with respect to the directrix curve $\Xi : S^2$* (*resp.* Σ^2) *$\to Q^{2m-1}$. Then the immersion $\Phi : \varphi^*P \to S^{2m}(1)$ defined by* (3.4), *i.e. each fiber $\pi_\Xi^{-1}(p)$ for $p \in S^2$* (*resp.* Σ^2) *is mapped to the corresponding great circle $\Xi(p) \in Q^{2m-1}$, is full and minimal.*

REMARK 4.2. In Theorem 2, the minimal immersion $\Phi : \Xi^*P \to S^{2m}(1)$ is realized as a *tube* of radius $\pi/2$ over the minimal 2-sphere S^2 or the pseudo-holomorphic map Σ^2 with respect to the $(m-1)$-th normal space. More precisely, let e_{2m-1}, e_{2m} be a *local* orthonormal frame field of the $(m-1)$-th normal space on some open neighborhood U of either a minimal S^2 or a pseudo holomorphic Σ^2. Then on $\pi_\Xi^{-1}(U) = U \times S^1$, Φ is given by

$$\Phi(x, \theta) = \cos\theta e_{2m-1} + \sin\theta e_{2m}.$$

EXAMPLE 4.3. Let $\psi_4 : \mathbb{CP}^1 \to Q^3 \subset \mathbb{CP}^4$ be the Veronese curve of order 4 in Example 2.2. Then the minimal immersion Φ from the pullback bundle over $\mathbb{CP}^1$ with respect to ψ_4 to $S^4(1)$ given by (3.4) is nothing but the Cartan minimal hypersurface (cf. Example 3.1).

Put

$$g^{11} = \|\Psi_2\|^2/\Delta, \quad g^{12} = -\Psi_1 \cdot \Psi_2/\Delta, \quad g^{22} = \|\Psi_1\|^2/\Delta.$$

Then the square of the length of H^Φ is

$$\begin{aligned}\|H^\Phi\|^2 &= (g^{22})^2\|\sigma_{11}\|^2 + 4(g^{12})^2\|\sigma_{12}\|^2 + (g^{11})^2\|\sigma_{22}\|^2 \\ &\quad + 4g^{22}g^{12}\sigma_{11}\cdot\sigma_{12} + 4g^{11}g^{12}\sigma_{12}\cdot\sigma_{22} + 2g^{11}g^{22}\sigma_{11}\cdot\sigma_{22} \\ &= \frac{4}{\Delta^2}\Big(\cos^2\theta\big(\rho_1^2\|\mathbf{p}_{1,1}^{**}\|^2 + 2\rho_1\rho_2(\mathbf{p}_{1,1}^{**}\cdot\mathbf{p}_{1,2}^{**}) + \rho_2^2\|\mathbf{p}_{1,2}^{**}\|^2\big) \\ &\quad + 2\sin\theta\cos\theta\big((\rho_1^2-\rho_2^2)(\mathbf{p}_{1,1}^{**}\cdot\mathbf{p}_{1,2}^{**}) + \rho_1\rho_2(\|\mathbf{p}_{1,2}^{**}\|^2 - \|\mathbf{p}_{1,1}^{**}\|^2)\big) \\ &\quad + \sin^2\theta\big(\rho_1^2\|\mathbf{p}_{1,2}^{**}\|^2 - 2\rho_1\rho_2(\mathbf{p}_{1,1}^{**}\cdot\mathbf{p}_{1,2}^{**}) + \rho_2^2\|\mathbf{p}_{1,1}^{**}\|^2\big)\Big).\end{aligned}$$

The square of the length of the second fundamental form $\|\sigma^\Phi\|^2$ is given by

$$\begin{aligned}\|\sigma^\Phi\|^2 &= (g^{11})^2\|\sigma_{11}\|^2 + (g^{22})^2\|\sigma_{22}\|^2 \\ &\quad + 2(g^{11}g^{22} + (g^{12})^2)\|\sigma_{12}\|^2 + 2(g^{12})^2\sigma_{11}\cdot\sigma_{22} \\ &\quad + 4g^{11}g^{12}\sigma_{11}\cdot\sigma_{12} + 4g^{12}g^{22}\sigma_{12}\cdot\sigma_{22}.\end{aligned}$$

Hence, using $\sigma_{11} = -\sigma_{22}$, we get

$$\begin{aligned}\|H^\Phi\|^2 - \|\sigma^\Phi\|^2 &= 2\big((g^{12})^2 - g^{11}g^{22}\big)(\|\sigma_{11}\|^2 + \|\sigma_{12}\|^2) \\ &\quad + 8(g^{22} - g^{11})g^{12}\sigma_{11}\cdot\sigma_{12} \\ &= -\frac{1}{\Delta}(\|\mathbf{p}_{1,1}^{**}\|^2 + \|\mathbf{p}_{1,2}^{**}\|^2) - \frac{1}{\Delta^2}\big(2\rho_1\rho_2\cos 4\theta + (\rho_1^2-\rho_2^2)\sin 4\theta\big) \\ &\quad \cdot\big(2\mathbf{p}_{1,1}^{**}\cdot\mathbf{p}_{1,2}^{**}\cos 2\theta + (\|\mathbf{p}_{1,1}^{**}\|^2 - \|\mathbf{p}_{1,2}^{**}\|^2)\sin 2\theta\big).\end{aligned}$$

Since $\cos 4\theta\cos 2\theta$, $\cos 4\theta\sin 2\theta$, $\sin 4\theta\cos 2\theta$ and $\sin 4\theta\sin 2\theta$ are independent functions, we finally obtain

Theorem 3. *Let $\varphi : \Sigma^2 \to Q^{n-1}$ be a holomorphic immersion from a Riemann surface Σ^2 to the complex quadric Q^{n-1}, and let Φ be the immersion from of the pullback bundle $\pi_\varphi : \varphi^*P \to \Sigma^2$ of the circle bundle $P(Q^{n-1}, S^1)$ ($P = SO(n+1)/SO(n-1)$) with respect to φ to sphere defined by which each fiber $\pi_\varphi^{-1}(p)$ for $p \in \Sigma^2$ is mapped to the corresponding great circle $\varphi(p) \in Q^{n-1}$ (cf. (3.4)).*

(1) *If the length of the mean curvature vector $\|H^\Phi\|$ with respect to Φ is constant along each great circles $\varphi(p)$ for $p \in \Sigma^2$, then M is minimal.*

(2) *The scalar curvature* $R^M = 6 + \|H^\Phi\|^2 - \|\sigma^\Phi\|^2$ *of* M^3 *is constant along each great circles* $\varphi(p)$ *for* $p \in \Sigma^2$ *if and only if the corresponding holomorphic curve* φ *satisfies either*
 (i) $\rho_1 = \rho_2 = 0$, *i.e., first order isotropic, or*
 (ii) $\|\mathbf{p}_{1,1}^{**}\|^2 = \|\mathbf{p}_{1,2}^{**}\|^2$ *and* $\mathbf{p}_{1,1}^{**} \cdot \mathbf{p}_{1,2}^{**} = 0$.

(3) *The scalar curvature* R^M *is constant on* M^3, *if and only if the corresponding holomorphic curve* φ *satisfies either*
 (i) *First order isotropic and the Gauss curvature* K^{Σ^2} *is constant, or*
 (ii) *Not first order isotropic,* $\|\mathbf{p}_{1,1}^{**}\|^2 = \|\mathbf{p}_{1,2}^{**}\|^2$, $\mathbf{p}_{1,1}^{**} \cdot \mathbf{p}_{1,2}^{**} = 0$ *and* $\|\mathbf{p}_{1,1}^{**}\|^2 + \|\mathbf{p}_{1,2}^{**}\|^2 = C\left(\|\mathbf{p}_1\|^2\|\mathbf{p}_2\|^2 - (\mathbf{p}_1 \cdot \mathbf{p}_2)^2\right)$ *for some constant* C.

(4) *Suppose the holomorphic immersion* $\varphi : \Sigma^2 \to Q^{n-1}$ *is of first order isotropic, and so the immersion from* $M^3 = \varphi^* P$ *to* $S^n(1)$ *defined by* (3.4) *is minimal. Then the scalar curvature* R^M *of* M^3 *is constant if and only if the Gauss curvature* K^{Σ^2} *of the corresponding holomorphic curve* φ *is constant.*

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