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CIRCLE PACKINGS ON COMPLEX AFFINE TORI

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1. Introduction

A set of closed disks on a plane is called a circle packing when they are arranged as follows. For each pair of distinct disks, they are disjoint or tangent. Moreover, the complement of the union of all disks is a disjoint union of triangular regions. Thus by taking the dual to the circle packing, we obtain a triangulation. We call its isotopy class a combinatorial type of the packing. A circle packing can be defined not only on a plane but also on a surface with a metric obviously by regarding a circle as a set of points with equal distance to the center. In this global case, a disk in the packing may be tangent to itself and a dual to the packing may be a triangulation in general sense. A surface will mean a surface with a reference homeomorphism (known as a marking) from a fixed surface throughout this paper.

It is not hard to show, and in fact will be shown in the third section that the set of flat tori, tori with Euclidean structures, which admit circle packings is dense in the space of all flat tori. However that set is known to be fairly poor by Andreev and Thurston. To describe this rigid property more precisely, recall that the space of flat tori up to similarity can be identified with the upper half plane $\mathbf{H}$. It was shown in [3] among other things that, to each triangulation, there is a unique flat torus up to similarity that admits a circle packing with a prescribed one as its combinatorial type. Then since the set of isotopy classes of triangulations on the torus is countable, only countably many flat tori admit circle packings in particular. This implies the fact, which we call Andreev-Thurston rigidity, that the flat structure on a torus with a packing does not admit any deformations which carry circle packings of the constant combinatorial type.

Let us consider a torus with another, more relaxed geometric structure in Thurston's sense on which the circle packing still makes sense. Namely we enlarge the transformation group to the complex affine group. The complex affine geometry is modeled on the complex plane $\mathbf{C}$ with the group of complex affine transformations, or equivalently similarity transformations. It is expressed in general as $z \mapsto az + b$ ($a, b \in \mathbf{C}$, $a \neq 0$). Remark that by restricting the group to the Euclidean isometry, we obtain the flat geometry. Although there are no canonical metrics on an affine torus, transition maps are similar transformations and preserve the shapes of figures. Therefore we still can define circles on an affine torus. A circle on an affine torus is defined as a point set which the developing map of the torus send to the union of circles on $\mathbf{C}$.

It is well known, for instance in [1], that, for each flat torus, there is a complex one-dimensional family of affine tori that are conformally equivalent to the prescribed flat one, and hence the set of affine tori can be identified with the product $\mathbf{H} \times \mathbf{C}$. A conformal map sends a circle to a circle infinitesimally but not actually. Hence the image of a packing on an affine torus by a conformal map is distorted in the flat torus. However one may still expect that the circle packing on affine tori have more flexible nature for the deformation in contrast with the rigidity for flat tori.

In fact, we will see in the fourth section that there are particular affine tori with this flexible nature.

Theorem 3.3. *There is a real two-dimensional continuous family of affine tori that admit packings of type $T(\omega_0, 0)$. Moreover, the family covers the space of flat tori up to similarity by two to one manner except one point.*

This theorem describes how much we can deform affine structures so as to carry packings with one particular combinatorial type. Hence it is obviously not powerful enough to see the global picture of the set of affine tori that admit circle packings in any sense. However since, as we will see later, an affine structure on tori can be well understood by looking at simple plane figure, quadrilaterals, it is strong enough to conclude

Theorem 3.4. *The set of affine tori that admit circle packings is dense in the set of all affine tori.*

Next section is for preparation. In the third section, known results about the flat torus and the main theorems about the affine torus are described. The proofs of the main theorems are given in the fourth section.

I would like to thank Sadayoshi Kojima for helpful advice and Kazuo Masuda for his help on calculation.

2. Preliminary

A 2-manifold M is called an affine surface if, for every pair of local charts φ_1 and φ_2, the transition map $\varphi_1 \circ \varphi_2^{-1}$ is a restriction of a complex affine map that is expressed as $z \mapsto az + b$ (a, $b \in \mathbf{C}$, $a \neq 0$). Moreover, if every transition map is a restriction of a complex affine map with $|a| = 1$, M is called a flat surface. It is known that the torus is the unique closed 2-manifold that admits an affine structure, see [2].

Let S be an affine torus. Let $\varphi_1 : U_1 \to \mathbf{C}$ and $\varphi_2 : U_2 \to \mathbf{C}$ be local charts. If $U_1 \cap U_2 \neq \emptyset$, $\varphi_1 \circ \varphi_2^{-1}$ is a restriction of a complex affine map τ. Thus, if $U_1 \cap U_2$ is connected, we can extend φ_1 to a map $\varphi : U_1 \cup U_2 \to \mathbf{C}$ by defining

$$\varphi(z) = \begin{cases} \varphi_1(z) & \text{if } z \in U_1, \\ \tau \circ \varphi_2(z) & \text{if } z \in U_2. \end{cases}$$

By gluing local charts over and over again along paths in S, we get a developing map $d : \widetilde{S} \to \mathbf{C}$, where $\widetilde{S}$ is the universal cover of S.

We fix two oriented simple closed curves a and b that are representatives of generators of the fundamental group of S. By the uniformization theorem, we can identify $\widetilde{S}$ with $\mathbf{C}$. Covering transformations of $\widetilde{S}$ act as translations on $\mathbf{C}$. With an appropriate normalization, the covering transformations corresponding to a and b can be represented by translations $z \mapsto z+1$ and $z \mapsto z+\omega$ where $\operatorname{Im}\omega > 0$. We call $\omega \in \mathbf{H}$ the Teichmüller parameter of an affine torus S.

Since $\widetilde{S}$ can be identified with $\mathbf{C}$, a developing map d can be regarded as a self map of $\mathbf{C}$. We explicitly represent the developing map as a function defined on $\mathbf{C}$. At first, since covering transformations descend to affine maps on $\mathbf{C}$, d satisfies the identities, $d(z+1) = a_A d(z) + b_A$, $d(z+\omega) = a_B d(z) + b_B$, where a_A, b_A, a_B and b_B should be some constant. Since d is a conformal map, $d'(z) \neq 0$ $(z \in \mathbf{C})$ where $d'(z)$ is the derivative of d at z. Then d''/d' is well-defined and a constant function c on $\mathbf{C}$. We call $c \in \mathbf{C}$ the affine parameter of the affine torus S. Affine tori with the same Teichmüller parameter are classified by the affine parameter c. If an affine torus is flat, then the affine parameter c is 0.

If the affine parameter c is 0, that is to say, the affine torus is flat, we can choose d so that $d(z) = z$. Therefore the developed image of the simple closed curves a and b forms a lattice composed of parallelograms. The coordinates of vertices of one parallelogram are 0, 1, ω and $\omega + 1 \in \mathbf{C}$, and the other parallelograms can be obtained from the parallelogram by translations.

If the affine parameter c is not 0, that is to say, the affine torus is not flat, we can choose d so that $d(z) = e^{cz}$. The developed image of the simple closed curves a and b winds around the origin as in Fig. 2. The coordinates of vertices of one building block of a lattice are 1, e^c, $e^{c\omega}$ and $e^{c(\omega+1)}$, and other regions can be obtained from the region by the transformation $z \mapsto e^{kc+lc\omega}z$ on $\mathbf{C}$ where k, l are integers. In this case, the developing map is not a homeomorphism between $\widetilde{S}$ and $\mathbf{C}$, but can be considered as a homeomorphism between $\widetilde{S}$ and $\overline{\mathbf{C}}$, where $\overline{\mathbf{C}}$ is the universal cover of $\mathbf{C} - \{O\}$. We use this interpretation if necessary.

The space of affine tori can be identified with $\mathbf{H} \times \mathbf{C}$. The space of all flat tori lies on $\mathbf{H} \times \{0\}$. We denote by $T(\omega, c)$ an affine torus with a Teichmüller parameter ω and an affine parameter c.

Recall that an open disk on an affine torus is a point set on the torus whose developed image is a disjoint union of Euclidean disks.

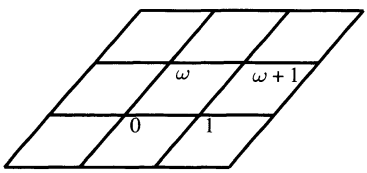

Fig. 1.
Developed image in the case $c = 0$

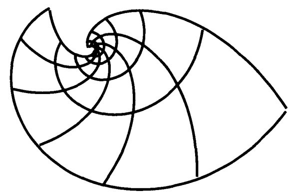

Fig. 2.
Developed image in the case $c \neq 0$

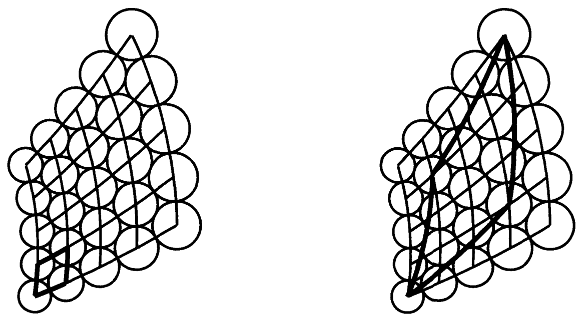

Fig. 3. Fundamental regions of $T(\omega, c)$ and $T(\omega', c')$ for $\omega' = (1 + 3\omega)/(3 + 1\omega)$

Lemma 2.1. *If there exists a circle packing on an affine torus $T(\omega, c)$, then for any $k, l, m, n \in \mathbf{Z}$ such that $k + l\omega \neq 0$ and $(m + n\omega)/(k + l\omega) \in \mathbf{H}$, there also exists a circle packing on $T(\omega', c')$ where $\omega' = (m + n\omega)/(k + l\omega)$ and $c' = (k + l\omega)c$.*

Proof. If there is a circle packing on $T(\omega, c)$, there exists a circle packing on $\overline{\mathbf{C}}$ which is invariant by the action of the holonomy group of $T(\omega, c)$. It is easy to see that the holonomy group of $T(\omega', c')$ is included in the holonomy group of $T(\omega, c)$. Hence this circle packing is invariant by the action of the holonomy group of $T(\omega', c')$, and $T(\omega', c')$ admits a circle packing. □

3. Theorem

Several deep results about circle packings on flat tori have been known. For instance, a special case of Theorem 13.7.1 in [3] can be simplified to the following statement for the application to our case.

Theorem 3.1 (Thurston). *Let S be a torus and τ a triangulation of S. Then there is a unique flat tori up to similarity such that the 1-skeleton of τ is isotopic to the dual of some circle packing.*

This result implies the following countable denseness easily.

Corollary 3.2. *The set of flat tori that admit circle packings is countable and dense in the space of all flat tori.*

Proof. Since, for each $n \in \mathbf{N}$, the set of combinatorially equivalent classes of triangulations with n triangles is finite, the set of combinatorially equivalent classes of triangulations is countable. Since the mapping class group is countable as a set, the set of isotopy classes of triangulations is countable. By Theorem 3.1, for each triangulation, a unique flat torus admits a circle packing whose dual is isotopic to 1-skeleton of the triangulation. Hence the set of flat tori that admit circle packings is countable in the Teichmüller space.

The flat torus corresponding to $\omega_0 = (1 + \sqrt{3})/2$ admits a circle packing with one circle, whose dual is developed to the regular hexagonal tiling on $\mathbf{C}$. Then, by Lemma 2.1, for each $m \in \mathbf{Z}$ and $n, k \in \mathbf{N}$, the flat torus corresponding to $(m + n\omega_0)/k$ admits a circle packing. So the set of flat tori that admit some circle packings is dense in the Teichmüller space. □

Our main concern in this paper is to discuss the corresponding results for circle packings on affine tori. We are primarily interested in a packing combinatorially equivalent to a circle packing with one circle which the flat torus $T(\omega_0, 0)$ admits. Let us call such a packing of type $T(\omega_0, 0)$. Finding affine structures on the torus which support a circle packing of this type, the following result is obtained.

Theorem 3.3. *There is a real two-dimensional continuous family of affine tori that admit packings of type $T(\omega_0, 0)$. Moreover, the family covers the space of flat tori up to similarity by two to one manner except one point.*

With detailed analysis using Lemma 2.1, the following fact derives from the above result.

Theorem 3.4. *The set of affine tori that admit circle packings is dense in the set of all affine tori.*

These theorems are the main results in this paper and proved in the next section.

4. Proof

We explicitly represent the necessary condition for an affine torus $T(\omega, c)$ to admit a circle packing of type $T(\omega_0, 0)$. The developed image of an affine torus is divided into quadrilateral fundamental regions by the simple closed curves a and b. Vertices of a fundamental region are 1, e^c, $e^{c\omega}$ and $e^{c(\omega+1)}$. The radii of circles centered at these

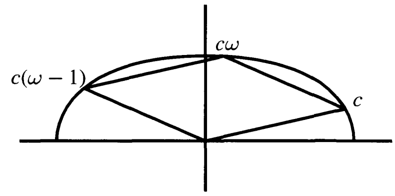

Fig. 4.

vertices on $\mathbf{C}$ must be r, $r|e^c|$, $r|e^{c\omega}|$ and $r|e^{c(\omega+1)}|$ respectively so that the union of the circles on $\mathbf{C}$ is invariant by the holonomy. Thus the condition can be described by

$$\begin{cases} |e^c - 1| = r + r|e^c|, \\ |e^{c\omega} - 1| = r + r|e^{c\omega}|, \\ |e^{c\omega} - e^c| = r|e^c| + r|e^{c\omega}|, \\ |\operatorname{Im} c| < \pi, \\ |\operatorname{Im} c\omega| < \pi, \\ |\operatorname{Im} c(\omega-1)| < \pi. \end{cases}$$

Note that inequality conditions are necessary so that the circles actually make a packing on $\overline{\mathbf{C}}$.

Three equations above can be transformed to

$$\frac{|e^c - 1|}{|e^c| + 1} = \frac{|e^{cw} - 1|}{|e^{cw}| + 1} = \frac{|e^{cw} - e^c|}{|e^{cw}| + |e^c|} (= r).$$

Using the identity $1 - (|e^z - 1|/(|e^z| + 1))^2 = (\cos \operatorname{Im}(z/2)/\cosh \operatorname{Re}(z/2))^2$, the equation can be expressed as

$$\text{(1)} \qquad \frac{\cos \operatorname{Im}(c/2)}{\cosh \operatorname{Re}(c/2)} = \frac{\cos \operatorname{Im}\{(c\omega)/2\}}{\cosh \operatorname{Re}\{(c\omega)/2\}} = \frac{\cos \operatorname{Im}\{c(\omega-1)/2\}}{\cosh \operatorname{Re}\{c(\omega-1)/2\}}.$$

Let f be a smooth function such that $f : \mathbf{C}' \to \mathbf{R}$, $f(z) = \cos \operatorname{Im}(z/2)/\cosh \operatorname{Re}(z/2)$, where $\mathbf{C}' = \{z \in \mathbf{C} \mid |\operatorname{Im} z| < \pi\}$. The range of f is $0 < f(z) \leq 1$. For $0 < t \leq 1$, let L_t be the level set $L_t = \{z \in \mathbf{C}' \mid f(z) = t\}$. Then the equation (1) is translated into the following condition.

$$\text{(2)} \qquad c, c\omega, c(\omega-1) \in L_{f(c)}.$$

Note that 0, c, $c\omega$, and $c(\omega-1)$ form a parallelogram. Conversely, for $c \in \mathbf{C}'$ such that $c \neq 0$, assume that there exist the points z_1 and $z_2 \in L_{f(c)}$ such that 0, c, z_1 and z_2 form a parallelogram. By setting $\omega = z_1/c$, the pair (ω, c) satisfies the condition (2).

By definition of f, L_t is homeomorphic to S^1 and surrounds the origin, for each

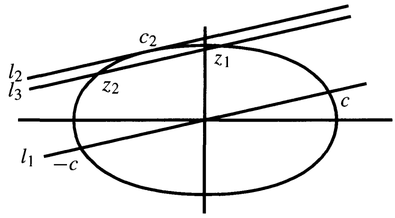

Fig. 5.

$0 < t < 1$. By the symmetry of f, L_t is symmetric about the real axis, the imaginary axis and the origin. The region bounded by L_t is a convex set.

Using these natures, we have

Lemma 4.1. *For any nonzero $c \in \mathbf{C}'$, there is the unique $\omega \in \mathbf{H}$ such that $T(\omega, c)$ admits a packing of type $T(\omega_0, 0)$.*

Proof. In Fig. 5, there are points z_1 and z_2 such that 0, c, z_1 and z_2 form a parallelogram. Moreover, since the region bounded by $L_{f(c)}$ is convex, there exist only two such pairs (z_1, z_2) and (z_1', z_2'). $\omega \in \mathbf{H}$ is derived from one of the pairs, and $1 - \omega \notin \mathbf{H}$ from the other. □

With this lemma, we can define the following function.

DEFINITION 4.2. Let $\Omega(c)$ denote the value ω in the lemma, given for each nonzero $c \in \mathbf{C}'$. Further, put $\Omega(0) = \omega_0$. Thus, we define a function $\Omega : \mathbf{C}' \to \mathbf{H}$.

By definition, the torus $T(\Omega(c), c)$ has a packing of type $T(\omega_0, 0)$.

We can tell the rough behavior of Ω.

For each $t \in (0, 1)$, there are twelve particular points on L_t (Fig. 6). These points lie on the axes or the bisectors. Note that $qp_1p_3p_5$, $qp_2p_4p_6$, ... are parallelograms. As t decreases, the ellipse-like closed curve $L_{f(c)}$ becomes larger. Then, these points make loci l_1, l_2, ..., l_{12}. $\mathbf{C}'$ can be divided by these loci and the origin. We denote each region by C_1, C_2, ..., C_{12} (Fig. 7). Similarly, $\mathbf{H}$ is divided into six regions by circles $|\omega| = 1$, $|\omega - 1| = 1$ and a line $\operatorname{Re}\omega = 1/2$. We denote each region by H_1, H_2, ..., H_6 (Fig. 8).

Assume that $c \in C_1$. Let ω_c denote $\Omega(c)$. Then $c\omega_c$ and $c(\omega_c - 1)$ belong to C_3 and C_5 respectively. Since $|c\omega_c| < |c(\omega_c - 1)| < |c|$, ω_c belongs to H_1. Therefore $\Omega(C_1) \subset H_1$. Similarly, $\Omega(C_i)$ and $\Omega(C_{i+6})$ are included in H_i, for $i = 1, 2, 3, 4, 5, 6$.

Ω has the following properties.

(Continuity) Let ω_c denote $\Omega(c)$. We introduce a new variable

$$k = \cos \operatorname{Im} \frac{c}{2} \cosh \operatorname{Re} \frac{c\omega}{2} = \cos \operatorname{Im} \frac{c\omega}{2} \cosh \operatorname{Re} \frac{c}{2}.$$

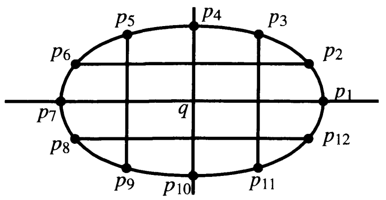

Fig. 6. 12 points on L_t

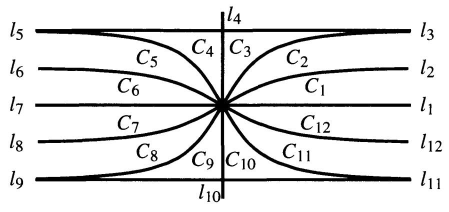

Fig. 7. 12 regions on $\mathbf{C}'$

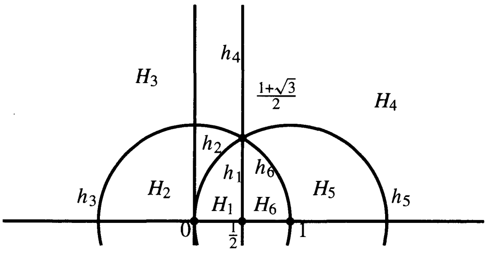

Fig. 8. Six regions on $\mathbf{H}$

Then we can solve the equation (1) and have an expression of k and $c\omega_c/2$ in term of c;

$$\begin{aligned}
k(c) &= \sqrt{k_1(c) - k_2(c)\sqrt{k_3(c)}},\\
k_1(c) &= 1 + \cos \operatorname{Im}\frac{c}{2}\cosh \operatorname{Re}\frac{c}{2} + \sin^2 \operatorname{Im}\frac{c}{2}\sinh^2 \operatorname{Re}\frac{c}{2},\\
k_2(c) &= \sin \operatorname{Im}\frac{c}{2}\sinh \operatorname{Re}\frac{c}{2},\\
k_3(c) &= 1 + 2\cos \operatorname{Im}\frac{c}{2}\cosh \operatorname{Re}\frac{c}{2} + \sin^2 \operatorname{Im}\frac{c}{2}\sinh^2 \operatorname{Re}\frac{c}{2},\\
\operatorname{Re}\frac{c\omega_c}{2} &= \varepsilon_u(c)\cosh^{-1}\left[\frac{k(c)}{\cos \operatorname{Im}(c/2)}\right],\\
\operatorname{Im}\frac{c\omega_c}{2} &= \varepsilon_v(c)\cos^{-1}\left[\frac{k(c)}{\cosh \operatorname{Re}(c/2)}\right].
\end{aligned}$$

Now, $\varepsilon_u(c)$ and $\varepsilon_v(c)$ mean the sign of $\operatorname{Re} c\omega_c$ and $\operatorname{Im} c\omega_c$, and are constant on each region C_i. Ω is continuous on C_i. $\{\Omega|_{C_i}\}$ are glued neatly. Then Ω is continuous on $\mathbf{C}'$.

(Nearly Two-to-One Property) For $t \in (0, 1]$ and $\theta \in [0, \pi]$, let $p(t, \theta)$ be the intersection point of L_t and the ray $R_\theta = \{se^{i\theta} \mid s > 0\}$. Let $r(t, \theta)$ be the distance between $p(t, \theta)$ and the origin. For any fixed t_1 and $t_2 \in (0, 1)$ such that $t_1 > t_2$, $r(t_1, \theta)$ and $r(t_2, \theta)/r(t_1, \theta)$ are monotonously decreasing as θ is increasing from 0 to π.

Suppose that there exist c and c' such that $c \neq \pm c'$ and $\Omega(c) = \Omega(c')$. Let ω_c

denote $\Omega(c)$. Now,

$$\frac{|c'|}{|c|} = \frac{|c'\omega_c|}{|c\omega_c|} = \frac{|c'(\omega_c - 1)|}{|c(\omega_c - 1)|}. \tag{3}$$

Let t and t' denote $f(c)$ and $f(c')$. There are three cases;
(i) $t = t'$ and $\arg c \neq \arg c'$,
(ii) $t \neq t'$ and $\arg c = \arg c'$,
(iii) $t \neq t'$ and $\arg c \neq \arg c'$.
In every case, we get a contradiction to the equation (3) by monotoneities of $r(t, \theta)$ and $r(t_2, \theta)/r(t_1, \theta)$.

It was proved that, if $\Omega(c)$ equals $\Omega(c')$, then $c = c'$ or $c = -c'$. Conversely, $\Omega(c)$ is equal to $\Omega(-c)$ for any c. Therefore Ω is two-to-one except $c = 0$.

(Surjectivity) Take $\omega_1 \in H_1$. There is a sufficiently small number t_1 such that $\operatorname{Im}\Omega(c) < \operatorname{Im}\omega_1$ for all $c \in L_{t_1} \cap C_1$. Now let A be the intersection of C_1 and the region inside of L_{t_1}. The boundary of A consists of part of l_1, part of l_2 and $L_{t_1} \cap C_1$. The image of the boundary ∂A by Ω is the union of part of h_1, part of h_2 and $\Omega(L_{t_1} \cap C_1)$, and then surrounds ω_1. By the continuity of Ω, ω_1 belongs to $\Omega(A)$. H_i is proved to be included in the image of Ω as well as H_1. Hence Ω is surjective.

From the properties of Ω above, Theorem 3.3 is now obvious. Finally, we prove Theorem 3.4.

At first, we fix $c_1 \in \mathbf{C}$. Let c_k and ω_k denote $c_1/k \in \mathbf{C}'$ and $\Omega(c_1/k)$ for a natural number $k > K$ where K is a sufficiently large number. By Lemma 2.1, $T((m + n\omega_k)/k, c_1)$ admits a circle packing for $m \in \mathbf{Z}$ and $n \in \mathbf{N}$. Since ω_k converges to $(1 + \sqrt{3})/2$ as k goes to infinity, $\{((m + n\omega_k)/k, c_1) \mid k > K, m \in \mathbf{Z}, n \in \mathbf{N}\}$ is dense in a horizontal section $\mathbf{H} \times \{c_1\}$. That is to say that the set of affine tori which admit some circle packings is dense in the section. Then the set is dense in $\mathbf{H} \times \mathbf{C}$.

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