Arakawa, T.
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AUTOMORPHISM GROUPS OF COMPACT RIEMANN SURFACES WITH INVARIANT SUBSETS

TATSUYA ARAKAWA

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1. Introduction

Let X denote a compact Riemann surface of genus g and G a subgroup of the automorphsim group $\mathrm{Aut}\,X$. If $g \geq 2$, we have the following theorem of Hurwitz:

Hurwitz's Theorem. $|G| \leq 84(g-1)$.

This estimate is not always best possible. In fact, Macbeath [12] showed the following:

Theorem (cf. [12] or [3]). (i) *There exist examples of X with $|\mathrm{Aut}\,X| = 84(g-1)$ with arbitrary large $g = g(X)$.*
(ii) *On the other hand, there exist arbitrary large g such that, for any X of genus g, $|\mathrm{Aut}\,X| < 84(g-1)$.*

For example, we have:

Proposition (cf. [16] or [9]). *If $g = 2$, then $|\mathrm{Aut}\,X| \leq 48$.*

We will use this inequality in §4.2 and §4.3 after giving a proof in §4.1.

Let $B \subset X$ be a finite subset of X with $|B| = k$ (≥ 1). Then Oikawa [13] showed the following:

Theorem 1 (cf. [13, Theorem 1]). *Suppose $2g+k \geq 3$. If a subgroup $G \subset \mathrm{Aut}\,X$ satisfies $GB = B$ then we have*:

$$|G| \leq 12(g-1) + 6k.$$

If the order k of B is not so large, this estimate is stronger than that of Hurwitz's. There are many results to determine the best bounds for the order of G for given g and k (as well as for $|\mathrm{Aut}\,X|$) (cf. [9], [11], [13], [16] and [17]).

Now we set:

Problem. Let $B_1, B_2, \dots, B_N$ be finite subsets of X with $B_i \cap B_j = \emptyset$ $(i \neq j)$ and $|B_j| = k_j$ $(k_1 \geq k_2 \geq \cdots \geq k_N \geq 1)$. Let $G \subset \operatorname{Aut} X$ be a subgroup which satisfies

$$GB_j = B_j \quad (1 \leq j \leq N).$$

Estimate the order of G in this situation for given integers g and $k_1, k_2, \dots, k_N$.

In the present paper, we will study these problems in the cases of $N = 2$ and 3:

Probrem'. Let B_1, B_2 and B_3 be three disjoint finite subsets of X with $|B_1| = k$, $|B_2| = l$ and $|B_3| = m$ $(k \geq l \geq m,\ l \geq 1$ and $m \geq 0)$. Let $G \subset \operatorname{Aut} X$ be a (finite) subgroup which satisfies

$$GB_j = B_j \quad (j = 1, 2, 3).$$

Estimate the order of G for given integers g, k, l and m.

In §2, we will give bounds for $|G|$ similar to that in Theorem 1 (Theorems 2 and 3) and in §3, we will show that there are infinitely many examples of automorphism groups of compact Riemann surfaces with several invariant subsets which attain the upper bounds given in §2 (Examples 1, 2, 3, 4 and Remark 3). In these aruguments, we will treat only the cases of $g = g(X) \geq 2$. The cases of $g = 0, 1$ will be stated elsewhere. In §4, three applications of the main results will be given, that is, we will apply Theorems 1, 2 and 3 to sets of Weierstrass points in §4.1, to branch loci of branched coverings in §4.2 and to singular fiber loci of pencils of curves in §4.3.

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2. Upper bounds for the order of G

The main results of this paper is the following:

Theorem 2. *Let X be a compact Riemann surface of genus $g \geq 2$ and B_1, B_2 be two disjoint finite subsets of X with orders $|B_1| = k$ and $|B_2| = l$ $(k \geq l \geq 1)$.*

Let G be a (finite) subgroup of Aut X *which satisfies $GB_j = B_j$ $(j = 1, 2)$. Then we have*:

$$|G| \leq 8(g-1) + k + 4l.$$

Theorem 3. *Let X be a compact Riemann surface of genus $g \geq 2$ and B_1, B_2 and B_3 be three disjoint finite subsets of X with orders $|B_1| = k$, $|B_2| = l$ and $|B_3| = m$ $(k \geq l \geq m \geq 1)$.*

Let G be a (finite) subgroup of $\operatorname{Aut} X$ *which satisfies $GB_j = B_j$ $(j = 1, 2, 3)$. Then we have*:

$$|G| \leq 2(g-1) + k + l + m.$$

REMARK 1. If we apply the estimate of Theorem 1 to the situation of Theorem 2 (resp. Theorem 3) above, we get the following:

$$|G| \leq 12(g-1) + 6l \quad (\text{resp. } 12(g-1) + 6m).$$

Similary, by Theorem 2, we have $|G| \leq 8(g-1)+l+4m$ in the situation of Theorem 3.

2.1. Preparations for the proof of Theorems 2 and 3 Let X be a compact Riemann surface of genus $g \geq 2$ and let G be a (finite) subgroup of $\operatorname{Aut} X$. Then we get a finite Galois covering $\varpi : X \to Y$ of degree $n = |G|$ where Y is the quotient X/G. Let $\pi = \pi(Y) \geq 0$ denote the genus of Y and $Q_1, Q_2, \ldots, Q_r \in Y$ $(r \geq 0)$ the branch points of ϖ. Suppose $|\varpi^{-1}(Q_j)| = n/e_j$ $(1 \leq j \leq r)$ with $(2 \leq)\ e_1 \leq e_2 \leq \cdots \leq e_r (\leq n)$. Then, by Riemann-Hurwitz formula, we have:

$$(*) \qquad \frac{2g-2}{n} = 2\pi - 2 + \sum_{j=1}^{r} \left(1 - \frac{1}{e_j}\right).$$

By standard arguments of branched coverings of compact Riemann surfaces (cf. For example, [3, Chap. 5] and [8, V.1]), we get the following:

CLAIM 1. (i) If $\pi \geq 2$ then we have $n \leq g - 1$.
(ii) If $\pi = 1$ then we have $r \geq 1$ and $n \leq 4(g-1)$. If moreover $r \geq 2$, then we have $n \leq 2(g-1)$.

Proof. If $\pi \geq 2$, then, by $(*)$, we have $(2g-2)/n \geq 2\pi - 2 \geq 2$, that is, $n \leq g-1$. If $\pi = 1$, then since $g \geq 2$, we have $0 < (2g-2)/n = \sum_j^r (1 - 1/e_j)$ and hence $r \geq 1$. Therefore we get

$$\frac{2g-2}{n} \geq 1 - \frac{1}{e_1} \geq \frac{1}{2},$$

that is, $n \leq 4(g-1)$.

If $\pi = 1$ and $r \geq 2$, then we have

$$\frac{2g-2}{n} \geq \left(1 - \frac{1}{e_1}\right) + \left(1 - \frac{1}{e_2}\right) \geq 1,$$

that is, $n \leq 2(g-1)$. □

CLAIM 2. $|G| \leq 8(g-1)$ unless $\pi = 0$ and:

(i)	$r = 4$	and	(e_1, e_2, e_3, e_4)	$= (2, 2, 2, 3)$		
(ii)	$r = 3$	and	(e_1, e_2, e_3)	$= (3, 4, 4)$	or	$(3, 4, 5)$
(iii)	$r = 3$	and	(e_1, e_2, e_3)	$= (3, 3, \alpha)$		$(4 \leq \alpha \leq 11)$
(iv)	$r = 3$	and	(e_1, e_2, e_3)	$= (2, 7, \beta)$		$(7 \leq \beta \leq 9)$
(v)	$r = 3$	and	(e_1, e_2, e_3)	$= (2, 6, \beta)$		$(6 \leq \beta \leq 11)$
(vi)	$r = 3$	and	(e_1, e_2, e_3)	$= (2, 5, \beta)$		$(5 \leq \beta \leq 19)$
(vii)	$r = 3$	and	(e_1, e_2, e_3)	$= (2, 4, \gamma)$		$(\gamma \geq 5)$
(viii)	$r = 3$	and	(e_1, e_2, e_3)	$= (2, 3, \gamma)$		$(\gamma \geq 7)$.

Proof. By Claim 1 above, we may assume $\pi = 0$. Then, since $2g - 2 > 0$, we have $r \geq 3$.

(a) (The cases of $r \geq 4$.) If $r \geq 5$, then we have $(2g-2)/n \geq -2 + 5/2 = 1/2$ and hence $n \leq 4(g-1)$.

If $r = 4$, then we have $e_4 \geq 3$.

Since $n \leq 6(g-1)$ for $e_3 \geq 3$ and $n \leq 8(g-1)$ for $e_4 \geq 4$, the only possible exception is $(e_1, e_2, e_3, e_4) = (2, 2, 2, 3)$.

Hereafter we suppose $r = 3$. If $e_1 \geq 4$, then we have $n \leq 8(g-1)$. So we may assume $e_1 = 2$ or 3.

(b) (The cases of $r = 3$ and $e_1 = 3$.) Suppose $e_1 = 3$. If $e_2 \geq 5$, then we have

$$\frac{2g-2}{n} \geq 1 - \frac{1}{3} - \frac{2}{5} = \frac{4}{15},$$

that is,

$$n \leq \frac{15}{2}(g-1) < 8(g-1).$$

If $e_2 = 4$ and $e_3 \geq 6$, then we have

$$\frac{2g-2}{n} \geq 1 - \frac{1}{3} - \frac{1}{4} - \frac{1}{6} = \frac{1}{4},$$

that is $n \leq 8(g-1)$.

If $e_2 = 3$, then we have $e_3 \geq 4$. If moreover, $e_3 \geq 12$, then

$$\frac{2g-2}{n} \geq 1 - \frac{2}{3} - \frac{1}{12} = \frac{1}{4},$$

that is $n \leq 8(g-1)$.

Consequently we have $n \leq 8(g-1)$ unless

$$(e_1, e_2, e_3) = (3, 4, 4),\ (3, 4, 5) \quad \text{or} \quad (3, 3, \alpha)\ (4 \leq \alpha \leq 11).$$

(c) (The cases of $r = 3$ and $e_1 = 2$.) Suppose $e_1 = 2$. Then $e_2 \geq 3$. If $e_2 \geq 8$, then we have

$$\frac{2g-2}{n} \geq 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4},$$

and hence $n \leq 8(g-1)$. So we may assume $3 \leq e_2 \leq 7$.

(c1) Suppose $e_2 = 7$. Then we have

$$\frac{2g-2}{n} = 1 - \frac{1}{2} - \frac{1}{7} - \frac{1}{e_3} \quad (e_3 \geq 7),$$

that is, $n = \{28e_3/(5e_3 - 14)\}(g-1)$ $(e_3 \geq 7)$. Hence we have $n \leq 8(g-1)$ for $e_3 \geq 10$.

(c2) By the same arguments as in (c1) above, we have

$$n \leq 8(g-1) \quad \text{for} \quad \begin{cases} e_3 \geq 12 & \text{if} \quad e_2 = 6. \\ e_3 \geq 20 & \text{if} \quad e_2 = 5. \end{cases}$$

(c3) We have $e_3 \geq 5$ for $e_2 = 4$ and $e_3 \geq 7$ for $e_2 = 3$. In these cases, n is always larger than $8(g-1)$. □

2.2. Proof of Theorem 2 To verify the inequality in Theorem 2, it suffices to see the eight cases (i), ..., (viii) in Claim 2. Note that since $GB_j = B_j$ $(j = 1, 2)$, B_j is nothing but a (finite and disjoint) union of set theoretic fibers of the covering $\varpi : X \to Y = X/G$.

(i) Suppose $r = 4$ and $(e_1, e_2, e_3, e_4) = (2, 2, 2, 3)$. Then by $(*)$, we have n $(= |G|) = 12(g-1)$. On the other hand, for a point $Q \in Y$, we have:

$$|\varpi^{-1}(Q)| = \begin{cases} 6(g-1) & \text{if} \quad Q = Q_1, Q_2, Q_3 \\ 4(g-1) & \text{if} \quad Q = Q_4 \\ 12(g-1) & \text{otherwise,} \end{cases}$$

and hence we have $k \geq 6(g-1)$ and $l \geq 4(g-1)$, that is, $8(g-1) + k + 4l \geq 8(g-1) + 6(g-1) + 16(g-1) = 30(g-1) > 12(g-1) = n$.

(ii) to (viii) are parallel. We only treat the case of (viii):

Suppose $r = 3$ and $(e_1, e_2, e_3) = (2, 3, \gamma)$ $(\gamma \geq 7)$.

By $(*)$, we have $n = \{12\gamma/(\gamma-6)\}(g-1)$. Moreover we have:

$$|\varpi^{-1}(Q)| = \begin{cases} \dfrac{6\gamma}{\gamma-6}(g-1) & \text{if } \quad Q = Q_1, \\ \dfrac{4\gamma}{\gamma-6}(g-1) & \text{if } \quad Q = Q_2, \\ \dfrac{12}{\gamma-6}(g-1) & \text{if } \quad Q = Q_3, \\ \dfrac{12\gamma}{\gamma-6}(g-1) & \text{otherwise.} \end{cases}$$

Therefore we have $k+4l \geq \{4\gamma/(\gamma-6)\}(g-1)+4\cdot\{12/(\gamma-6)\}(g-1) = \{(4\gamma+48)/(\gamma-6)\}(g-1)$, which implies

$$n = \frac{12\gamma}{\gamma-6}(g-1) \leq 8(g-1)+k+4l.$$

2.3. Proof of Theorem 3 Let $\varpi : X \to Y$ be the Galois covering with $Y = X/G$ and $Q_1, Q_2, \ldots, Q_r \in Y$ the branch points of ϖ with $(n >)\ |\varpi^{-1}(Q_1)| \geq |\varpi^{-1}(Q_2)| \geq \cdots \geq |\varpi^{-1}(Q_r)|\ (\geq 1)$. By Claim 1, we may assume $\pi = \pi(Y) \leq 1$.

Suppose $\pi = 1$. If $r \leq 2$, then we have $k \geq n = |G|$, while if $r \geq 2$, we have $n \leq 2(g-1)$ by Claim 1 again.

Now suppose $\pi = 0$ and hence $r \geq 3$. Then, by the formula $(*)$, we have

$$\begin{aligned} \frac{2g-2}{n} &= -2+\sum_{j=1}^{r-3}\left(1-\frac{1}{e_j}\right)+\left(1-\frac{1}{e_{r-2}}\right)+\left(1-\frac{1}{e_{r-1}}\right)+\left(1-\frac{1}{e_r}\right) \\ &\geq 1-\left(\frac{1}{e_{r-2}}+\frac{1}{e_{r-1}}+\frac{1}{e_r}\right). \end{aligned}$$

Since $k \geq n/e_{r-2}$, $l \geq n/e_{r-1}$ and $m \geq n/e_r$, the above inequality implies the following one:

$$|G| = n \leq 2(g-1)+\frac{n}{e_{r-2}}+\frac{n}{e_{r-1}}+\frac{n}{e_r} \leq 2(g-1)+k+l+m. \qquad \square$$

3. Examples

In this section, we will show that there are infintely many examples of (X, B, G) (resp. (X, B_1, B_2, G) or (X, B_1, B_2, B_3, G)) such that the equality sign of the inequality in Theorem 1 (resp. Theorem 2 or 3) holds.

3.1. Examples of small genus Let us begin with examples of genus two and three:

EXAMPLE 1. Let X be a compact Riemann surface of genus two given by the equation

$$y^2 = (x - a_1)(x - a_2)(x - a_3)(x - a_4)(x - a_5)(x - a_6)$$

where $a_1, \dots, a_6 \in \mathbf{P}^1$ are the vertices of the regular octahedron ($\cong \mathbf{P}^1$). Then we get an automorphism group G of X of order 48, generated by the octahedral group $O_{24} \subset \operatorname{Aut}\mathbf{P}^1$ and the hyperelliptic involution $\tau \in \operatorname{Aut} X$. Note that, as we have mentioned in §1, it is known that the order of an automorphism groups of a compact Riemann surface of genus two is at most 48. Hence, in fact, we have $G = \operatorname{Aut} X$.

Now let R_1, R_2 and $R_3 \subset X$ denote the set theoretic pull backs of the centers of the edges, the centers of the faces and the vertices of the octahedron, respectively. Then we have:

(i) $GR_i = R_i$ $(i = 1, 2, 3)$.

(ii) $|R_1| = 24$, $|R_2| = 16$ and $|R_1| = 6$.

We therefore get the following equalities:

(I) Let $B = R_3 \subset X$. Then we have $|G| = 12(g - 1) + 6|B|$.

(II) Let $B_1 = R_2$ and $B_2 = R_3$. Then we have $|G| = 8(g - 1) + |B_1| + 4|B_2|$.

(III) Let $B_1 = R_1$, $B_2 = R_2$ and $B_3 = R_3$. Then we have $|G| = 2(g - 1) + |B_1| + |B_2| + |B_3|$.

EXAMPLE 2. Let X be the compact Riemann surface of genus three which has a group of automorphism $G = \operatorname{Aut} X$ of order 168 ($= 84(g(X) - 1)$). Then, it is well known that we have a Galois covering $\varphi : X \to \mathbf{P}^1$ of degree 168 with three branch points $Q_1, Q_2, Q_3 \in \mathbf{P}^1$ such that $|R_1| = 84$, $|R_2| = 56$ and $|R_3| = 24$ where $R_i = \varphi^{-1}(Q_i)$ $(i = 1, 2, 3)$ (cf. [3, Chap. 5] or [6, VII.3.10]).

Here we have $GR_i = R_i$ $(i = 1, 2, 3)$ and get the following:

(I) Let $B = R_3$. Then we have $|G| = 12(g - 1) + 6|B|$.

(II) Let $B_1 = R_2$ and $B_2 = R_2$. Then we have $|G| = 8(g - 1) + |B_1| + 4|B_2|$.

(III) Let $B_1 = R_1$, $B_2 = R_2$ and $B_3 = R_3$. Then we have $|G| = 2(g - 1) + |B_1| + |B_2| + |B_3|$.

3.2. Examples of higher genus (I) As we have mentioned in §1, Macbeath [12] showed that there exist examples of compact Riemann surfaces of arbitrary large genus where the equality sign of Hurwitz's Theorem holds. In this subsection we will apply the same arguments to the situations of Theorems 1, 2 and 3 and show some results similar to Macbeath's.

Lemma 1 (cf. [12]. See also [3, Chap. 4]). *Let $\varpi : W \to X$ denote an unramified Galois covering of compact Riemann surfaces and let $D = \varpi_*\pi_1(W, P) \subset \pi_1(X, Q)$ be the image of the fundamental group of W ($P \in W$ and $Q = \varpi(P) \in X$ are base points).*

Let $\varphi : X \to X$ be an automorphism of X. If the following condition $(**)$ holds, then there exists a uniquely determined automorphism $\tilde{\varphi}$ of W such that $\varpi\tilde{\varphi} = \varphi\varpi$:

$(**)$ *There is a path α on X connecting Q and $\varphi(Q)$ such that $\alpha^{-1}(\varphi_* D)\alpha = D$.*

REMARK 2. Since ϖ is Galois, D is a normal subgroup of $\pi_1(X, Q)$. Hence the condition $(**)$ above does not depend on the choice of the path α.

Since the mapping $\pi_1(X, P) \to \pi_1(X, P)$ given by $\gamma \mapsto \alpha^{-1}\varphi(\gamma)\alpha$ is a group automorphism, we get:

Corollary 1. *For a compact Riemann surface X, let $D \subset \pi_1(X, Q)$ be a characteristic subgroup, that is, for any group automorphism $\sigma : \pi_1(X, Q) \to \pi_1(X, Q)$, $\sigma(D) = D$. Let $\varpi : W \to X$ denote the (unramified) Galois covering which corresponds to D. Then there exists a (injective) homomorphism*

$$\operatorname{Aut} X \to \operatorname{Aut} W.$$

EXAMPLE 3. Let X, R_1, R_2, R_3 and G be as in Example 1.

Since X is of genus two, there exist generators α_1, β_1, α_2 and β_2 of $\pi_1(X, P)$ with the only relation

$$\alpha_1^{-1}\beta_1^{-1}\alpha_1\beta_1 \cdot \alpha_2^{-1}\beta_2^{-1}\alpha_2\beta_2 = 1.$$

Let $n \geq 2$ be an integer and let $\psi : \pi_1(X, P) \to (\mathbf{Z}/n\mathbf{Z})^4$ be the (surjective) homomorphism given by

$$\begin{cases} \psi(\alpha_1) = (1, 0, 0, 0), \\ \psi(\beta_1) = (0, 1, 0, 0), \\ \psi(\alpha_2) = (0, 0, 1, 0), \\ \psi(\beta_2) = (0, 0, 0, 1). \end{cases}$$

Then $D := \operatorname{Ker}\psi \subset \pi_1(X, P)$ is a characteristic subgroup of index n^4.

Now let $W \to X$ be the umramified Galois covering corresponding to D. Then, by Corollary 1, we get a subgroup $\tilde{G} \subset \operatorname{Aut} W$ generated by G and the Galois group $G(W/X) \subset \operatorname{Aut} W$.

Let $\tilde{R}_1$, $\tilde{R}_2$ and $\tilde{R}_3 \subset W$ be the pull back of R_1, R_2 and $R_3 \subset X$, respectively. Then we have $\tilde{G}\tilde{R}_i = \tilde{R}_i$ $(i = 1, 2, 3)$. Moreover, since the covering $W \to X$ is unramified, we have:

(i) $g(W) - 1 = n^4(g(X) - 1) = n^4$.

(ii) $|\tilde{R}_1| = 24n^4$, $|\tilde{R}_2| = 16n^4$, and $|\tilde{R}_3| = 6n^4$.

(iii) $|\tilde{G}| = |G|n^4 = 48n^4$.

Therefore we get the following equalities:

(I) $|\tilde{G}| = 12(g(W) - 1) + 6|\tilde{R}_3|$ and
(II) $|\tilde{G}| = 8(g(W) - 1) + |\tilde{R}_2| + 4|\tilde{R}_3| \quad (|\tilde{R}_2| > |\tilde{R}_3|)$.
(III) $|\tilde{G}| = 2(g(W) - 1) + |\tilde{R}_1| + |\tilde{R}_2| + |\tilde{R}_3| \quad (|\tilde{R}_1| > |\tilde{R}_2| > |\tilde{R}_3|)$.

REMARK 3. We can also apply the same arguments as above to Example 2 and get examples of $g(W) = 2n^6$, $|\tilde{G}| = 168n^6$, $|\tilde{R}_1| = 84n^4$, $|\tilde{R}_2| = 56n^4$ and $|\tilde{R}_3| = 24n^4$, that is, the equalities (I), (II) and (III) above hold.

In particular, the pairs $(W, \tilde{G})$ for $n = 2, 3, \ldots$ in this case are nothing but the examples where the upper bound of Hurwitz's theorem is attained, which are given by Macbeath [12].

3.3. Examples of higher genus (II) In this subsection, we will give another example where the equality sign in Theorem 3 holds:

EXAMPLE 4. For $g \geq 2$, let $\zeta = \exp\left(2\pi\sqrt{-1}/(2g+2)\right)$. Let X be the hyperelliptic Riemann surface of genus g given by the equation

$$y^2 = (x-1)(x-\zeta)(x-\zeta^2)\cdots(x-\zeta^{2g+1}).$$

Let B_1, B_2 and $B_3 \subset X$ be the (finite) subset given as $\{y = 0\}$, $\{x = 0\}$ and $\{x = \infty\}$, respectively. Then we have $|B_1| = 2g + 2$ and $|B_2| = |B_3| = 2$.

On the other hand, let $G \subset \operatorname{Aut} X$ be the subgroup generated by the mapping $(x, y) \mapsto (\zeta x, y)$ and the hyperellptic involution on X.

Then we have $GB_j = B_j$ $(j = 1, 2, 3)$. Furthermore, since $|G| = 4g + 4$, we conclude:

$$|G| = 2(g-1) + |B_1| + |B_2| + |B_3| \quad (|B_1| > |B_2| = |B_3|).$$

4. Applications

In this section, we will give three applications of Theorems 1, 2 and 3. In §4.1, we take the sets of Weierstrass points of compact Riemann surface of small genus as the invariant subsets. In §4.2, we consider automorphsim groups of branched coverings of compact Riemann surfaces, where branch loci are invariant subsets. In §4.3, pencils of algebraic curves on algebraic surfaces are treated, and in this case, the singular fiber loci in the base curves will be the invariant subsets of automorphism groups.

4.1. Weierstrass points and automorphism groups of compact Riemann surfaces of genus three

4.1.1. In this subsection, we will summarize some basic facts on *Weierstrass points* of compact Riemann surfaces. See for example, [8, III. 5] for details.

Let X denote a compact Riemann surface of genus g (≥ 2). A point $P \in X$ is said to be a *Weierstarass point* of X if $h^0(X, gP) \geq 2$. Let $W \subset X$ denote the set of all Weierstarass points of X. Then we have:

Proposition 1 (cf. [8, III.5.9 and 5.11]). *There exists a mapping* $\tau : X \to \mathbf{Z}_{\geq 0}$, *called the weight of points, which satisfies the following conditions*:

(i) *For a point* $P \in X$, $P \in W$ *if and only if* $\tau(P) > 0$.
(ii) $\sum_{P\in W} \tau(P) = g^3 - g$.
(iii) $\tau(Q) \leq (g^2 - g)/2$ *for all* $Q \in X$.
(iv) X *is hyperelliptic if and only if there is at least one* $P \in W$ *such that the equality sign in* (iii) *holds if and only if so for all* $P \in W$.

Corollary 2. *For a compact Riemann surface* X, *we have* $2g+2 \leq |W| \leq g^3 - g$. *Moreover* $|W| = 2g+2$ *if and only if* X *is hyperelliptic.*

For example, if $g = 2$, we have $|W| = 6$ and hence, we can deduce a well known fact that a compact Riemann surface of genus two is always hyperelliptic.

Now we define subsets W_r $(r > 0)$ of W by $W_r := \{P \in W;\ \tau(P) = r\}$. Then we have:

$$W = \bigcup_{r=1}^{\infty} W_r \quad \text{(disjoint union).}$$

Note that W_r is possibly an empty set. In particular by Proposition 1 (iii) above, we always have $W_r = \emptyset$ for $r > (g^2 - g)/2$.

4.1.2. Let X, $W \subset X$, $\tau(Q)$ and $W_r \subset W$ be as in §4.1.1. The following is a direct consequence of the definition of τ:

Lemma 2. *For any automorphism* $\sigma \in \operatorname{Aut} X$, *we have* $\tau(\sigma(Q)) = \tau(Q)$ *for all* $Q \in X$.

Corollary 3. *For any subgroup* $G \subset \operatorname{Aut} X$, *we have* $GW = W$ *and* $GW_r = W_r$ $(r \geq 1)$.

By this corollary, we can apply Theorems 1 and 2 to invariant subsets $W_r \subset X$ to estimate the order of automorphism groups or the number of Weierstrass points (when $g = 2$ or 3).

(a) Suppose $g(X) = 2$. Then we have $|W| = |W_1| = 6$. Hence, by Theorem 1 and Cororally 2, we get the following inequality, which we have mentioned in §1 and §3 (See also Remark 5 in §4.2.5, below):

$$|\operatorname{Aut} X| \leq 48.$$

(b) Suppose $g(X) = 3$ and set $w = |W|$. Then we have:
 (1) $0 \leq \tau(Q) \leq 3$,

(2) $\sum_{P \in W} \tau(P) = 24$ and hence

(3) $8 \leq w \leq 24$.

Suppoose X is hyperelliptic. Then, since $GW = W$ with $w = 8$, we get the following inequality just as in (a) above:

$$|\operatorname{Aut} X| \leq 72.$$

REMARK 4. In fact, we have the best bound for the order of automorphism groups of hyperelliptic curves of genus three (cf. [16], [9] or Remark 5 in §4.2.5): $|\operatorname{Aut} X| \leq 48$.

In the general case, by Hurwitz's theorem we have seen in §1, we have $|\operatorname{Aut} X| \leq 168$.

Suppose $|\operatorname{Aut} X| = 168$. Then it is known that all Weierstrass points of X are of weight one, and hence $w = 24$ (cf. [3, 6.2]).

Applying Theorem 1 to $W \subset X$, we get a "new proof" of this fact:

Since $168 = |\operatorname{Aut} X| \leq 24 + 6w$, *we have* $w \geq 24$. *On the other hand,* $w \leq 24$ *by* (3) *above. Therefore we get* $w = 24$.

Now suppose X is nonhyperelliptic. Then, since $W_3 = \emptyset$, we have $W = W_1 \cup W_2$ and hence we can apply Theorem 2 to $W_1, W_2 \subset X$ instead of "Theorem 1 to $W \subset X$" in the above arguments (unless either $W_1 = \emptyset$ or $W_2 = \emptyset$).

As for the order of W_j $(j = 1, 2)$, we have $|W_1| + |W_2| = w$ and $|W_1| + 2|W_2| = 24$, that is, $(|W_1|, |W_2|) = (2w - 24, 24 - w)$. Therefore we also have:

$$\begin{cases} 0 < |W_1| \leq |W_2| & \text{if} \quad 13 \leq w \leq 16 \\ |W_1| > |W_2| > 0 & \text{if} \quad 17 \leq w \leq 23. \end{cases}$$

Note that since $W_3 = \emptyset$, we have $w \geq 12$. Moreover $W_1 = \emptyset$ if $w = 12$ and $W_2 = \emptyset$ if $w = 24$.

Consequently we get the following by Theorems 1 and 2 (and Remark 4 above):

Proposition 2. *Let X be a compact Riemann surface of genus three and let w be the number of Weierstrass points of X. Then we have*:

$$|\operatorname{Aut} X| \leq \begin{cases} 48 & w = 8 \\ 96 & w = 12 \\ 7w - 56 & 13 \leq w \leq 16 \\ -2w + 88 & 17 \leq w \leq 23 \\ 168 & w = 24. \end{cases}$$

Note that if $w = 12$ (or 8, 24), we cannot use Theorem 2.

Corollary 4. *For a compact Riemann surface X of genus three with $W_j \neq \emptyset$ ($j = 1, 2$) we have*:

$$|\operatorname{Aut} X| \leq 56.$$

Proof. Since $W_j \neq \emptyset$ ($j = 1, 2$), we have $13 \leq w \leq 23$ and hence

$$|\operatorname{Aut} X| \leq \begin{cases} 7w - 56 \ (\leq 56) & 13 \leq w \leq 16 \\ -2w + 88 \ (\leq 54) & 17 \leq w \leq 23. \end{cases} \qquad \square$$

4.1.3. The arguments in §4.1.2 are applicable to compact Riemann surfaces of higher genus. Here we will see two cases, that is:
(a) X is nonhyperelliptic with $g = 4$ and
(b) X is hyperelliptic with $g = 5$.
The notations are same as those is §4.1.1 and §4.1.2.

(a) Suppose $g(X) = 4$. Then, by Proposition 1, we have:
(1) $0 \leq \tau(Q) \leq 6$,
(2) $\sum_{P \in W} \tau(P) = 60$ and hence
(3) $10 \leq w \leq 60$.

Now suppose moreover, X is nonhyperelliptic. Then we have the following disjoint union:

$$W = W_1 \cup W_2 \cup W_3 \cup W_4 \cup W_5.$$

Let us apply main theorems to these subsets:

(a1) Suppose $W_5 \neq \emptyset$. Then we can apply Theorem 1 to the invariant subset $W_5 \subset X$ and get

$$|\operatorname{Aut} X| \leq 6|W_5| + 36.$$

Since $|W_5| \leq 12$, we also have $|\operatorname{Aut} X| \leq 108$.

(a2) Suppose $W_4, W_5 \neq \emptyset$. Then we can apply Theorem 2 to the invariant subsets $W_4, W_5 \subset X$ and get:

$$|\operatorname{Aut} X| \leq \begin{cases} |W_4| + 4|W_5| + 24 & \text{if} \quad |W_4| \geq |W_5| \\ 4|W_4| + |W_5| + 24 & \text{if} \quad |W_4| < |W_5|. \end{cases}$$

Since $4|W_4| + 5|W_5| \leq 60$, we also have $|\operatorname{Aut} X| \leq 55$.

(a3) Suppose $W_3, W_4, W_5 \neq \emptyset$. Then we can apply Theorem 3 to the invariant subsets $W_3, W_4, W_5 \subset X$ and get

$$|\operatorname{Aut} X| \leq |W_3| + |W_4| + |W_5| + 6.$$

Since $3|W_3|+4|W_4|+5|W_5| \le 60$, we also have $|\operatorname{Aut} X| \le 25$.

(b) Suppose $g(X)=5$ and X is hyperelliptic. Then, by Corollary 2, we have $|W| = 12$. Hence applying Theorem 1 to $W \subset X$, we have:

$$|\operatorname{Aut} X| \le 120.$$

Note that this inequality coinsides with the best bound in [9] and [16] (cf. Remark 5 in §4.2.5).

4.2. Automorphism groups of branched coverings of compact Riemann surfaces

4.2.1. Let X, Y denote compact Riemann surfaces and $\varphi : Y \to X$ a branched covering of Riemann surfaces.

An automorphism of φ is, by definition, a pair of an automorphism τ of Y and an automorphism σ of X such that $\varphi\tau = \sigma\varphi$.

Let $\operatorname{Aut}(\varphi)$ denote the group of all automorphisms of φ. Then by the natural map $(\tau, \sigma) \mapsto \sigma$, we get the following exact sequence of groups

$$1 \to G(Y/X) \to \operatorname{Aut}(\varphi) \to H_\varphi \to 1$$

where $G(Y/X)$ is the covering transformation group of φ, which is given by

$$G(Y/X) := \{\tau \in \operatorname{Aut} Y;\ \varphi\tau = \varphi\}$$

and $H_\varphi \subset \operatorname{Aut} X$ is the image of the above natural map.

In this subsection, we will attempt to give bounds for the order of $\operatorname{Aut}(\varphi)$ when φ is *not* unramified and the degree of φ is small (and $g(X) \ge 2$).

In §4.2.2, we will apply Theorems 1, 2 and 3 to the branch loci of the covering φ on X to estimate the order of H_φ and in §4.2.3 we will see the relation between the order of $G(Y/X)$ and branch locus of φ. The bounds of $\operatorname{Aut}(\varphi)$ will be given in §4.2.4 and then in §4.2.5, we will see some special cases where $\operatorname{Aut}(\varphi)$ coinsides with $\operatorname{Aut} Y$.

4.2.2. Let X, Y, φ and H_φ be as above. In the following, we always assume $d := \deg\varphi \ge 2$, $q := g(X) \ge 2$ and hence $g := g(Y) > 3$.

We set $B_j := \{Q \in X;\ |\varphi^{-1}(Q)| = j\}$ $(j \ge 1)$. Then B_j's are mutually disjoint finite subsets of X for $0 < j < d$ and $B = B_1 \cup \cdots \cup B_{d-1} \subset X$ is the branch locus of the covering $\varphi : Y \to X$. Note that since we assume φ is not unramified, B is not empty.

Let $b_j \in \mathbf{Z}_{\ge 0}$ denote the order of B_j. Then, by Riemann-Hurwitz formula, we have:

$$(***) \qquad (d-1)b_1 + (d-2)b_2 + \cdots + 2b_{d-2} + b_{d-1} = 2g - 2 - d(2q-2).$$

Now since we have $H_\varphi B_j = B_j$ for all j, we can apply Theorems 1, 2, and 3 to these invariant subsets B_j's to get bounds of $|H_\varphi|$.

(a) Suppose $d = \deg(\varphi) = 2$. Then, by the formula $(***)$, we have $|B| = b_1 = 2g - 4q + 2$ (> 0). Hence applying Theorem 1 to $B_1 \subset X$, we get the following:

CLAIM 3. $|H_\varphi| \leq 12(g - q)$.

(b) Suppose $d = \deg(\varphi) = 3$. Then $B = B_1 \cup B_2$ and hence by $(***)$, we have $2b_1 + b_2 = 2g - 6q + 4$. Let us apply Theorem 1 or 2 to $B_1, B_2 \subset X$.

(b1) If $b_1 = 0$, then $b_2 = 2g - 6q + 4$. Hence, by Theorem 1, we have:

$$|H_\varphi| \leq 12(q-1) + 6b_2 = 12g - 24q + 12.$$

(b2) If $b_1, b_2 \neq 0$ then since $b_1 + 4b_2 = 8g - 24q + 16 - 7b_1$ and $4b_1 + b_2 = 2g - 6q + 4 + 2b_1$, Hence, by Theorem 2, we have:

$$|H_\varphi| \leq \begin{cases} 2g + 2q - 4 + 2b_1 & \text{if} \quad 1 \leq b_1 \leq b_2. \\ 8g - 16q + 8 - 7b_1 & \text{if} \quad 1 \leq b_2 \leq b_1. \end{cases}$$

(b3) If $b_2 = 0$, then $b_1 = g - 3q + 2$. Hence, by Theorem 1, we have

$$|H_\varphi| \leq 12(q-1) + 6b_1 = 6g - 6q.$$

Consequently we get:

CLAIM 4.

$$|H_\varphi| \leq \begin{cases} 12g - 24q + 12 & \text{if} \quad b_1 = 0, \\ 2g + 2q - 4 + 2b_1 & \text{if} \quad 1 \leq b_1 < \dfrac{2g - 6q + 4}{3}, \\ 8g - 16q + 8 - 7b_1 & \text{if} \quad \dfrac{2g - 6q + 4}{3} \leq b_1 < g - 3q + 2, \\ 6g - 6q & \text{if} \quad b_1 = g - 3q + 2. \end{cases}$$

(c) Suppose $d = \deg(\varphi) = 4$. Then $B = B_1 \cup B_2 \cup B_3$ and by $(***)$, we have $3b_1 + 2b_2 + b_3 = 2g - 8q + 6$. Applying Theorems 1, 2 or 3, we get:

CLAIM 5. (i) Suppose exactly one of B_j's is nonempty. Then we have:

$$|H_\varphi| \leq \begin{cases} 4g - 4q & \text{if} \quad b_1 \neq 0 \quad \text{and} \quad b_2 = b_3 = 0. \\ 6g - 12q + 6 & \text{if} \quad b_2 \neq 0 \quad \text{and} \quad b_1 = b_3 = 0. \\ 12g - 36q + 24 & \text{if} \quad b_3 \neq 0 \quad \text{and} \quad b_1 = b_2 = 0. \end{cases}$$

(ii) Suppose exactly one of B_j's is empty.

(α) If $B_1 = \emptyset$, then we have:

$$|H_\varphi| \leq \begin{cases} 2g-2+2b_2 & \text{if } 1 \leq b_2 < \dfrac{2g-8q+6}{3}. \\ 8g-24q+16-7b_2 & \text{if } \dfrac{2g-8q+6}{3} \leq b_2 < g-4q+3. \end{cases}$$

(β) If $B_2 = \emptyset$, then we have:

$$|H_\varphi| \leq \begin{cases} 2g-2+b_1 & \text{if } 1 \leq b_1 < \dfrac{g-4q+3}{2}. \\ 8g-24q+16-11b_1 & \text{if } \dfrac{g-4q+3}{2} \leq b_1 < \dfrac{2g-8q+6}{3}. \end{cases}$$

(γ) If $B_3 = \emptyset$, then we have:

$$|H_\varphi| \leq \begin{cases} g+4q-5+\dfrac{5}{2}b_1 & \text{if } 1 \leq b_1 < \dfrac{2g-8q+6}{5}. \\ 4g-8q+4-5b_1 & \text{if } \dfrac{2g-8q+6}{5} \leq b_1 < \dfrac{2g-8q+6}{3}. \end{cases}$$

(iii) Suppose each of B_j's is nonempty. Then we have:

$$|H_\varphi| \leq 2g-6q+4-2b_1-b_2.$$

4.2.3. Let the notations be as above. Let us observe the relation between the order of the covering transformation group $G(Y/X)$ and those of B_j's.

Let us recall the following fundamental facts on covering transformations:

Proposition (cf. [3, Chap. 4]). (i) *If $\tau \in G(Y/X)$ is not identity, then all fixed points of τ are lying over the branch locus of $\varphi : Y \to X$ on X.*
(ii) *$|G(Y/X)|$ is a divisor of d = $\deg(\varphi)$ and hence $|G(Y/X)| \leq d$ = $\deg(\varphi)$. Moreover the equality sign holds if and only if φ is a Galois covering if and only if $G(Y/X)$ transitively acts on each fiber of φ.*

By these properties, we can deduce the following claims:

CLAIM 6. Suppose $d = 2$. Then we have $|G(Y/X)| = 2$.

CLAIM 7. Suppose $d = 3$. Then we have:
(i) $|G(Y/X)| = 1$ or 3.
(ii) If $|G(Y/X)| = 3$, then $b_2 = 0$ and hence $b_1 = g-3q+2$.

CLAIM 8. Suppose $d = 4$. Then we have:
(i) $|G(Y/X)| = 1$, 2 or 4.
(ii) If $\varphi : Y \to X$ is Galois, then $b_3 = 0$ and hence $3b_1 + 2b_2 = 2g - 8q + 6$.

4.2.4. We continue the previous notations. Now we can give bounds for the order of $\mathrm{Aut}(\varphi)$:

Proposition 3. *Suppose $d = 2$. Then we have*:

$$|\mathrm{Aut}(\varphi)| \leq 24(g - q).$$

Proposition 4. *Suppose $d = 3$.*
(i) *If φ is Galois, then we have*:

$$|\mathrm{Aut}(\varphi)| \leq 18(g - q).$$

(ii) *If φ is not Galois, then we have*:

$$|\mathrm{Aut}(\varphi)| \leq \begin{cases} 12g - 24q + 12 & \text{if } b_1 = 0, \\ 2g + 2q - 4 + 2b_1 & \text{if } 1 \leq b_1 < \dfrac{2g - 6q + 4}{3}, \\ 8g - 16q + 8 - 7b_1 & \text{if } \dfrac{2g - 6q + 4}{3} \leq b_1 < g - 3q + 2, \\ 6g - 6q & \text{if } b_1 = g - 3q + 2. \end{cases}$$

Proposition 5. *Suppose $d = 4$.* (i) *If φ is Galois, then we have $3b_1 + 2b_2 = 2g - 8q + 6$ and*:

$$|\mathrm{Aut}(\varphi)| \leq \begin{cases} 24g - 48q + 24 & \text{if } b_1 = 0, \\ 4g + 16q - 20 + 10b_1 & \text{if } 1 \leq b_1 < \dfrac{2g - 8q + 6}{5}, \\ 16g - 32q + 16 - 20b_1 & \text{if } \dfrac{2g - 8q + 6}{5} \leq b_1 < \dfrac{2g - 8q + 6}{3}, \\ 16g - 16q & \text{if } b_1 = \dfrac{2g - 8q + 6}{3}. \end{cases}$$

(ii) *If φ is not Galois, then we have*:

$$
|\operatorname{Aut}(\varphi)| \le \begin{cases}
8g-8q & \text{if } b_1 \neq 0 \text{ and } b_2 = b_3 = 0. \\
12g-24q+12 & \text{if } b_2 \neq 0 \text{ and } b_1 = b_3 = 0. \\
24g-72q+48 & \text{if } b_3 \neq 0 \text{ and } b_1 = b_2 = 0. \\
4g-4+4b_2 & \text{if } b_1 = 0 \\
 & \text{and } 1 \le b_2 < \dfrac{2g-8q+6}{3}. \\
16g-48q+32-14b_2 & \text{if } b_1 = 0 \\
 & \text{and } \dfrac{2g-8q+6}{3} \le b_2 < g-4q+3. \\
4g-4+2b_1 & \text{if } b_2 = 0 \\
 & \text{and } 1 \le b_1 < \dfrac{g-4q+3}{2}. \\
16g-48q+32-22b_1 & \text{if } b_2 = 0 \\
 & \text{and } \dfrac{g-4q+3}{2} \le b_1 < \dfrac{2g-8q+6}{3}. \\
2g+8q-10+5b_1 & \text{if } b_3 = 0 \\
 & \text{and } 1 \le b_1 < \dfrac{2g-8q+6}{5}. \\
8g-16q+8-10b_1 & \text{if } b_3 = 0 \\
 & \text{and } \dfrac{2g-8q+6}{5} \le b_1 < \dfrac{2g-8q+6}{3}. \\
4g-12q+8-4b_1-2b_2 & \text{if } b_1, b_2, b_3 \neq 0.
\end{cases}
$$

Proof. These are direct consequences of the equality

$$|\operatorname{Aut}(\varphi)| = |G(Y/X)| \cdot |H_\varphi|$$

and the Claims in §4.2.2 and §4.2.3. □

4.2.5. Let X denote a compact Riemann surface of genus g (≥ 2). For a non-negative integer q, X is said to be *q-hyperelliptic* if there exists a double covering $\varphi : X \to Y$ where Y is a compact Riemann surface of genus q (cf. [8, V.1]).

In the present paper, we say X is *q-trigonal* if there exists a triple covering $\varphi : X \to Y$ where Y is a compact Riemann surface of genus q.

Note that 0-hyperelliptic (resp. 0-trigonal) is the usual hyperelliptic (resp. trigonal).

Now let us recall Castelnuovo-Severi's theorem with corollaries (cf. [6] or [3, Chap. 3]):

Castelnuovo-Severi's theorem. *Let X, Y and Z denote compact Riemann surfaces of genus g, q and r, respectively. Let $\varpi : X \to Y$ and $\rho : X \to Z$ be surjective holomorphic maps of degree m and n, respectively.*

Suppose there does not exist a compact Riemann surface W of genus $g' < g$ with a surjective holomorphic map $\mu; X \to W$, $\varpi' : W \to Y$ and $\rho' : W \to Z$ such that $\varpi = \varpi' \mu$ and $\rho = \rho' \mu$. Then we have:

$$g \leq mq + nr + (m - 1)(n - 1)$$

Corollary 5 (See also [6, V.1, Corollary 1] for (i)). (i) *Suppose X is q-hyperelliptic with $g > 4q + 1$. Then the double covering map from X to a genus q Riemann surface is unique up to isomorphism, that is, if $\varphi : X \to Y$ and $\varphi' : X \to Y'$ are two double coverings where Y and Y' are compact Riemann surfaces of genus q, then there exists an isomporphism $\tau : Y \to Y'$ which satisfies $\varphi' = \tau\varphi$.*
(ii) *Suppose X is q-trigonal with $g > 6q + 4$. Then the tirple covering map from X to a genus q Riemann surface is unique up to isomorphisms* (*in the same sense as in* (i).)

For an integer $q \geq 2$, suppose Y is q-hyperelliptic (resp. q-trigonal) with $g > 4q + 1$ (resp. $g > 6q + 4$) and $\varpi : Y \to X$ the unique double covering (resp. triple covering) with $g(X) = q$. Then by the uniquness of ϖ, we have:

$$\operatorname{Aut} Y = \operatorname{Aut}(\varpi),$$

and hence, to conclude that the inequalities in Propositions 3 and 4 in §4.2.4 also give the bounds for $\operatorname{Aut} Y$, we only need to mention that by the assumption $g > 4q + 1$ (resp. $g > 6q + 4$), the double (resp. triple) covering $\varphi : X \to Y$ is not unramified.

In particular we have

Proposition 6. *Let q denote an integer with $q \geq 2$. Then*:
(i) *For a q-hyperelliptic Riemann surface X of genus g with $g > 4q + 1$, we have*:

$$|\operatorname{Aut} X| \leq 24(g - q).$$

(ii) *For a q-trigonal Riemann surface X of genus g with $g > 6q + 4$ and the unique triple covering $\varphi : X \to Y$ with $g(Y) = q$, we have*:
(a) *If φ is Galois, then*:

$$|\operatorname{Aut} X| \leq 18(g - q).$$

(b) *If φ is not Galois, then*:

$$|\operatorname{Aut} X| \leq \begin{cases} 12g - 24q + 12 & \text{if } b_1 = 0, \\ 2g + 2q - 4 + 2b_1 \left(< \dfrac{10}{3}g - 2q - \dfrac{3}{4}\right) & \text{if } 1 \leq b_1 < \dfrac{2g - 6q + 4}{3}, \\ 8g - 16q + 8 - 7b_1 \left(\leq \dfrac{10}{3}g - 2q - \dfrac{3}{4}\right) & \\ \qquad \qquad \text{if } \dfrac{2g - 6q + 4}{3} \leq b_1 < g - 3q + 2, & \\ 6g - 6q & \text{if } b_1 = g - 3q + 2 \end{cases}$$

where $b_1 = |B_1|$ *with* $B_1 = \{Q \in Y; |\varphi^{-1}(Q)| = 1\}$.

REMARK 5. In the case of $q = 0$, that is, the case where X is a usual hyperelliptic (resp. trigonal) compact Riemann surface, there are better estimates than that in Proposition 1 (resp. Corollary 3) above (cf. [9] and [16] (resp. [2, Theorem 1])):

Proposition. (i) *Let X be a hyperelliptic compact Riemann surface of genus $g \geq 2$. Then we have*:

$$|\operatorname{Aut} X| \leq \begin{cases} 4(g+1) & \text{if } g \neq 2, 3, 5, 9. \\ 48 & \text{if } g = 2, 3. \\ 120 & \text{if } g = 5, 9. \end{cases}$$

(ii) *Let X be a trigonal compact Riemann surface of genus $g \geq 5$. Then we have*:

$$|\operatorname{Aut} X| \leq \begin{cases} 6(g+1) & \text{if } g \neq 6, 10, 18. \\ 72 & \text{if } g = 6. \\ 180 & \text{if } g = 10, 18. \end{cases}$$

4.3. Automorphism groups of fibrations of curves

4.3.1. Let $f : S \to C$ denote a relatively minimal fibrations of algebraic curves of genus $g \geq 2$ over a nonsingular algebraic curve C of genus $\pi \geq 0$. Then we have:

Lemma 3 (cf. [5, Corollary (i)]).

$$K_S^2 \geq 8(g-1)(\pi - 1)$$

where K_S is the canonical bundle of the algebraic surface S.

We will assume that $\pi = \pi(C) \geq 2$ in the following aruguments. Note that then, the algebraic surface S is of general type, and hence we have $|\operatorname{Aut} S| < \infty$.

An automorphism of the fibration f is, by definition, a pair of an automorphism $\tilde{\sigma}$ of S and an automorphism σ of C such that $f\tilde{\sigma} = \sigma f$. Let $\mathbf{G} = \mathrm{Aut}(f)$ denote the group of all automorphisms of f. Then we have an estimate for the order of $\mathbf{G}$ by G. Xiao [19] and Z. Chen [7]:

Proposition 7 (cf. [19, Proposition 1]). $|\mathbf{G}| \leq 882K_S^2$ *(for* $\pi \geq 2$*).*

Proof. By the definition of the automorphism of the fibration f, we get the following exact sequence of groups:

$$1 \to K \to \mathbf{G} \to H \to 1$$

where H is a subgroup of $\mathrm{Aut}\, C$ and $K = \{(\tilde{\sigma}, \mathrm{id}_C) \in G\}$.

Let $F \subset S$ denote a general fiber of f. Then there exists an inclusion $K \to \mathrm{Aut}\, F$ and hence we have $|K| \leq 84(g-1)$ by Hurwitz's theorem. On the other hand, by the assumption that $\pi(C) \geq 2$, we have $|H| \leq 84(\pi - 1)$ by Hurwitz's theorem, again. Consequently we have $|\mathbf{G}| = |K||H| \leq 7056(g-1)(\pi-1)$, which implies the assertion by Lemma 3. □

Since $|K| \leq 48$ in the proof above when $g = 2$ (cf. Proposition in §1), we also have:

Proposition 8 (cf. [7, Proposition 1]). *If* $g = 2$, *then* $|\mathbf{G}| \leq 504K_S^2$.

Chen [7] also shows the following:

Proposition 9 (cf. [7, Proposition 1]). *If* $g = 2$ *and* f *is not locally trivial, then* $|\mathbf{G}| \leq 126K_S^2$.

Now let us assume the fibration f has some singular fibers $F_1, F_2, \ldots \subset S$. In the following, we will attempt to give upper bounds for $|\mathbf{G}|$ which depend on the data of F_j's as an application of Theorems 1, 2 and 3.

Let e denote the topological Euler number and let $\epsilon(F_j)$ be the *Euler contribution* of F_j, that is, $\epsilon(F_j) := e(F_j) + 2g - 2$. Then we have:

Proposition 10 (cf. [4, Proposition III.11.4]).
(i) $\epsilon(F_j) \geq 1$.
(ii) $e(S) = (2-2g)(2-2\pi) + \sum_j \epsilon(F_j)$.

On the other hand, we have the following *slope inequality* of the fibration f:

Proposition 11 (cf. [10], [18]. See also [14], [15]).

$$K_{S/C}^2 \geq \frac{4(g-1)}{g}\chi_f$$

where $K_{S/C}^2 = K_S^2 - 8(g-1)(\pi-1)$ *and* $\chi_f = \chi(\mathcal{O}_S) - (g-1)(\pi-1)$.

By Propositions 10, 11 above and Noether's formula, we conclude the following inequality:

Proposition 12 (cf. [1]).

$$K_{S/C}^2 \geq \frac{g-1}{2g+1}\sum_j \epsilon(F_j).$$

Now let us estimate the order of G when f has some singular fibers.

4.3.2. Set $B := \{P \in C;\ f^{-1}(P) \subset S$ is a singular fiber$\} \subset C$. Then, for the subgroup $H \subset \operatorname{Aut} C$ given in the proof of Proposition 7 above, we have $HB = B$. Hence, by Theorem 1, we have $|H| \leq 12(\pi-1) + 6k$ where $k = |B|$ $(< \infty)$.

On the other hand, we have

$$K_{S/C}^2 \geq \frac{g-1}{2g+1}\sum_j \epsilon(F_j) \geq \frac{g-1}{2g+1}\cdot k$$

by Proposition 12 and Proposition 10 (i).

Therefore we have:

Proposition 13. (i) $|\mathbf{G}| \leq [\{1008(2g+1)(\pi-1) + 504(2g+1)k\}/\{8(2g+1)(\pi-1) + k\}]K_S^2$

(ii) *If* $g = 2$, *then* $|\mathbf{G}| \leq [\{2880(\pi-1) + 1440k\}/\{40(\pi-1) + k\}]K_S^2$.

Since we assume $\pi \geq 2$, we also have:

Corollary 6. (i) $|\mathbf{G}| \leq [\{1008(2g+1) + 504(2g+1)k\}/\{8(2g+1) + k\}]K_S^2$

(ii) *If* $g = 2$, *then* $|\mathbf{G}| \leq [\{2880 + 1440k\}/\{40 + k\}]K_S^2$.

REMARK 6. Let $F = f^{-1}(P) \subset S$ be a singular fiber with $\epsilon = \epsilon(F)$ (≥ 1). Then we can take the finite subset $B_F = \{Q \in C; f^{-1}(Q) \cong F\}$ of order k_F $(\leq k)$ instead of B in Propostion 13, and morover, we have

$$K_{S/C}^2 \geq \frac{g-1}{2g+1}\cdot \epsilon k_F.$$

We therefore get a (possibly) better estimate:

(i) $|\mathbf{G}| \leq [\{1008(2g+1)(\pi-1)+504(2g+1)k_F\}/\{8(2g+1)(\pi-1)+\epsilon k_F\}]K_S^2$,
(ii) *If* $g=2$, $|\mathbf{G}| \leq [\{2880(\pi-1)+1440k_F\}/\{40(\pi-1)+\epsilon k_F\}]K_S^2$.

For example, let $f : S \to C$ be a relatively minimal fibration of genus two curves and suppose f has k singular fibers of the following form:

$$(****) \qquad F = 2E_0 + E_1 + E_2 + E_3 + E_4 + E_5 + E_6$$

where $E_j \cong \mathbf{P}^1$ $(0 \leq j \leq 6)$ with $E_0E_j = 1$ $(1 \leq j \leq 6)$ and $E_iE_j = 0$ $(1 \leq i < j \leq 6)$.
Then we have $\epsilon = 10$ and $k_F \leq k$ in the inequality (ii) above and hence:

$$|\mathbf{G}| \leq \frac{2880(\pi-1)+1440k}{40(\pi-1)+10k}K_S^2.$$

EXAMPLE 5. If moreover, we set $\pi = 2$ and $k = 6$ in the above inequality, then

$$|\mathbf{G}| \leq 115.2K_S^2.$$

Let us construct an example where the equality sign in this inequality holds. (See [14], [15] or [7] for details of the following construction, especially the computations of the invariants of S.)

Let X, $R_j \subset X$ $(j = 1, 2, 3)$ and $G \subset \mathrm{Aut}(X)$ be as in Example 1 and let Z be the product $X \times \mathbf{P}^1$ with projections $p_1 : Z \to X$ and $p_2 : Z \to \mathbf{P}^1$. Let R be the divisor on Z given as $R := p_1^{-1}(R_3) + p_2^{-1}(a_1) + \cdots + p_2^{-1}(a_6)$. Let $S' \to Z$ denote the double covering branched along R and let $S \to S'$ be the minimal resolution of singularities.

Then the natural projection $f : S \to X$ is a relatively minimal genus two fibration over a genus two curve X which has six singular fibers of the form $(****)$ (over the points of $R_3 \subset X$). In this case, the equality sign in Proposition 12 holds and hence we have $K_S^2 = 20$.

Now let $\mathbf{G} \subset \mathrm{Aut}\, S$ denote the subgroup generated by $G \subset \mathrm{Aut}\, X$, $O_{24} \subset \mathrm{Aut}\, \mathbf{P}^1$ and the involution of the double covering $S' \to Z$. Then we have:
(i) $\mathbf{G} \subset \mathrm{Aut}(f)$ and
(ii) $|\mathbf{G}| = 48 \times 24 \times 2 = 2304$, that is $|\mathbf{G}| = 115.2K_S^2$.

EXAMPLE 5'. If we take Example 3 instead of Example 1 in the above arguments, we get examples of $\pi = n^4 + 1$, $k = 6n^4$ and

$$|\mathbf{G}| = \frac{2880(\pi-1)+1440k}{40(\pi-1)+10k}K_S^2$$

for any $n \geq 1$.

4.3.3. Suppose the fibration $f : S \to C$ has two singular fibers $F_1 = f^{-1}(P_1)$ and $F_2 = f^{-1}(P_2)$ with $\epsilon = \epsilon(F_1)$ and $\delta = \epsilon(F_2)$ $(\epsilon, \delta \geq 1)$ such that $F_1 \not\cong F_2$. Let B_{F_1},

B_{F_2} be the subsets of C given in the Remark 6 above for the singular fibers F_1 and F_2, respectively. Then we have $B_{F_1} \cap B_{F_2} = \emptyset$.

Suppose $|B_{F_1}| = k \geq l = |B_{F_2}|$ (≥ 1). Then, since $HB_{F_j} = B_{F_j}$ ($j = 1, 2$), we have $|H| \leq 8(\pi - 1) + k + 4l$ by Theorem 2. On the other hand, by Proposition 12, we have

$$K^2_{S/C} \geq \frac{g-1}{2g+1}(k\epsilon + l\delta).$$

We therefore get:

Proposition 14. (i) $|\mathbf{G}| \leq [\{672(2g+1)(\pi-1) + 84(2g+1)(k+4l)\}/\{8(2g+1)(\pi-1) + k\epsilon + l\delta\}]K^2_S$
(ii) *If* $g = 2$, *then* $|\mathbf{G}| \leq [\{1920(\pi-1) + 240(k+4l)\}/\{40(\pi-1) + k\epsilon + l\delta\}]K^2_S$.

Corollary 7. (i) $|\mathbf{G}| \leq [\{672(2g+1) + 84(2g+1)(k+4l)\}/\{8(2g+1) + k\epsilon + l\delta\}]K^2_S$
(ii) *If* $g = 2$, *then* $|\mathbf{G}| \leq [\{1920 + 240(k+4l)\}/\{40 + k\epsilon + l\delta\}]K^2_S$.

4.3.4. If the fibration $f : S \to C$ has three singular fibers F_1, F_2 and F_3 with $F_i \not\cong F_j$ $(i \neq j)$, $\epsilon(F_1) = \epsilon$, $\epsilon(F_2) = \delta$, $\epsilon(F_3) = \gamma$ and $k = |B_{F_1}| \geq l = |B_{F_2}| \geq m = |B_{F_3}|$ (≥ 1), then, by Theorem 3, we have:

Proposition 15. (i) $|\mathbf{G}| \leq [\{168(2g+1)(\pi-1) + 84(2g+1)(k+l+m)\}/\{8(2g+1)(\pi-1) + k\epsilon + l\delta + m\gamma\}]K^2_S$
(ii) *If* $g = 2$, *then* $|\mathbf{G}| \leq [\{480(\pi-1) + 240(k+l+m)\}/\{40(\pi-1) + k\epsilon + l\delta + m\gamma\}]K^2_S$.

Corollary 8. (i) $|\mathbf{G}| \leq [\{168(2g+1) + 84(2g+1)(k+l+m)\}/\{8(2g+1) + k\epsilon + l\delta + m\gamma\}]K^2_S$
(ii) *If* $g = 2$, *then* $|\mathbf{G}| \leq [\{480 + 240(k+l+m)\}/\{40 + k\epsilon + l\delta + m\gamma\}]K^2_S$.

REMARK 7. In Propositons 13, 14, 15 and Corollary 6, 7, 8 (and Remark 6) above, we do not have to assume that the fibers F_i (or F) are singular. We only need the assumption that the subsets B_{F_i} (or B_F) of C are finite.

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Department of Information
and Computer Engineering
Gunma National College of Technology
580 Toriba, Maebashi
Gunma 371-8530, Japan
e-mail: arakawa@ice.gunma-ct.ac.jp