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CONVERGENCE OF LOGISTIC PARAMETERS IN BAYESIAN APPROACH

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1. Introduction

In this paper, we study the statistical model of the independent, 0-1-valued observations with the following distributions:

$$P(Y_i = 1) = \frac{e^{\alpha+\beta x_i}}{1 + e^{\alpha+\beta x_i}}, \quad P(Y_i = 0) = \frac{1}{1 + e^{\alpha+\beta x_i}}$$
$$(i = 1, 2, \dots, n),$$

where x_i's are known real numbers called the *observation points*. It is sometimes more natural to consider the parameters α and β in the above logistic way than something like $e^\alpha/(1 + e^\alpha)$'s. For example, let us cosider a random variable Y on $\{0, 1\}$ with small $P[Y = 1]$, where the value 1 stands for a serious accident which we must avoid definitely. Since we are sensitive on the value $P[Y = 1]$, we take the measurement $\log P[Y = 1]$ instead of the value itself. In this case, the logistic parametrization is suitable. In the same reason, it is natural to assume that the prior distribution α and β is uniform, that is, the joint prior density for (α, β) is given by $p(\alpha, \beta) \equiv 1$ on $\mathbf{R}^2$. Then we discuss the posterior distribution on (α, β) under a set of observations $Y_i = y_i$ $(i = 1, \dots, n)$.

By the Bayes formula, the posterior probability density, is given by

$$p(\alpha, \beta | y_1, \dots, y_n) = c^{-1} \prod_{i=1}^{n} \left(\frac{e^{\alpha+\beta x_i}}{1 + e^{\alpha+\beta x_i}} \right)^{y_i} \left(\frac{1}{1 + e^{\alpha+\beta x_i}} \right)^{1-y_i}$$

if and only if the normalizing constant exists, that is

$$c := \iint \prod_{i=1}^{n} \left(\frac{e^{\alpha+\beta x_i}}{1 + e^{\alpha+\beta x_i}} \right)^{y_i} \left(\frac{1}{1 + e^{\alpha+\beta x_i}} \right)^{1-y_i} d\alpha d\beta < \infty.$$

We obtain in Theorem 1 a necessary and sufficient condition for the existence of the posterior probability distribution, or equivalently, for $c < \infty$.

Theorem 1. *A necessary and sufficient condition for $c < \infty$ is that $1 \leq m \leq n-1$ and*

$$\min\left\{\sum_{i\in S} x_i : \sharp S = m\right\} < \sum_{i=1}^{n} x_i y_i < \max\left\{\sum_{i\in S} x_i : \sharp S = m\right\},$$

where we put $m = \sum_{i=1}^{n} y_i$ and $S \subset \{1, 2, \ldots, n\}$.

Under this condition, we consider α and β to be random variables and use the notations A and B for α and β in this sense to avoid a confusion with their sample values α and β.

We are interested in the convergence of the random variables A, B under the observations $y_1, y_2, \ldots, y_{kn}$ satisfying that k number of the observation points are fixed where the same number n of observations are allocated and the ratio of 1 among them converges as $n \to \infty$ to a value in (0, 1). That is, we assume the following set of observation points:

$$x_{i,j} \quad (i = 1, \ldots, k \ ; \ j = 1, \ldots, n)$$

with

$$x_{i,1} = \cdots = x_{i,n} := x_i \quad (i = 1, \ldots, k)$$

and $x_i < x_{i+1}$ for $i = 1 \cdots k-1$ with a fixed integer k not less than 2. Let $y_{i,j}$ be the set of corresponding observations, for which we assume that

$$p_i := \lim_{n\to\infty} \frac{t_i}{n} \quad (i = 1, \ldots, k)$$

exist for

$$t_i := \sum_{j=1}^{n} y_{i,j}$$

and it holds that $0 < p_i < 1 \quad (i = 1, \ldots, k)$.

Then, the posterior density $p(\alpha, \beta | t_1, \ldots, t_k)$ for (A, B) under these observations satisfies that

$$\begin{aligned} p(\alpha, \beta | t_1, \ldots, t_k) &= {c_n}^{-1} \prod_{i=1}^{k} \prod_{j=1}^{n} \left(\frac{e^{\alpha+\beta x_{i,j}}}{1+e^{\alpha+\beta x_{i,j}}}\right)^{y_{i,j}} \left(\frac{1}{1+e^{\alpha+\beta x_{i,j}}}\right)^{1-y_{i,j}} \\ &= {c_n}^{-1} \frac{\exp\{\sum_{i=1}^{k} t_i(\alpha+\beta x_i)\}}{\prod_{i=1}^{k}\{1+\exp(\alpha+\beta x_i)\}^n} \end{aligned} \tag{1}$$

$$= {c_n}^{-1} \exp \left[n \sum_{i=1}^{k} \left\{ \frac{t_i}{n}(\alpha + \beta x_i) - \log(1 + \exp(\alpha + \beta x_i)) \right\} \right]$$

where c_n is the normalizing constant. We put

$$f(\alpha, \beta) := \sum_{i=1}^{k} \left[p_i(\alpha + \beta x_i) - \log\{1 + \exp(\alpha + \beta x_i)\} \right] \tag{2}$$

$$G_n(\alpha, \beta) := \sum_{i=1}^{k} \left[\frac{t_i}{n}(\alpha + \beta x_i) - \log\{1 + \exp(\alpha + \beta x_i)\} \right].$$

The *maximal likelihood estimator* $(\hat{\alpha_n}, \hat{\beta_n})$ is, by definition, a point (α, β) which maximize $G_n(\alpha, \beta)$. Similary, $(\hat{\alpha}, \hat{\beta})$ is defined to be (α, β) which maximize $f(\alpha, \beta)$.

Theorem 2. *The maximal likelihood estimator* $(\hat{\alpha}_n, \hat{\beta}_n)$ *exists uniquely.*

Theorem 3. *It holds that* $(\hat{\alpha}, \hat{\beta})$ *exists uniquely,* $(\hat{\alpha}_n, \hat{\beta}_n)$ *converges to* $(\hat{\alpha}, \hat{\beta})$ *as* $n \to \infty$.

Theorem 4. *The random variable* (A, B) *converges to* $(\hat{\alpha}, \hat{\beta})$ *in law.*

Corollary 1 (Lehmann [5], A. Ibragimov and R.Z. Khas'Minskii [10]). *Assume that* $t_i/n = p_i + o(n^{-1})$ $(i = 1, \dots, k)$ *as* $n \to \infty$. *Then the distribution of the random variable* $((A - \hat{\alpha})/\sqrt{n}, (B - \hat{\beta})/\sqrt{n})$ *converges to the* 2*-dimensional centered normal distribution with the covariance matrix* M^{-1}, *where*

$$M = \begin{pmatrix} u & v \\ v & w \end{pmatrix}$$

with

$$u = \sum_{i=1}^{k} \frac{\exp(\hat{\alpha} + \hat{\beta} x_i)}{(1 + \exp(\hat{\alpha} + \hat{\beta} x_i))^2}$$

$$v = \sum_{i=1}^{k} \frac{x_i \exp(\hat{\alpha} + \hat{\beta} x_i)}{(1 + \exp(\hat{\alpha} + \hat{\beta} x_i))^2}$$

$$w = \sum_{i=1}^{k} \frac{x_i^2 \exp(\hat{\alpha} + \hat{\beta} x_i)}{(1 + \exp(\hat{\alpha} + \hat{\beta} x_i))^2} .$$

The aim of this paper is to justify the Bayesian approach for the logistic parameters by proving the consistency in Theorem 4 and the approximate normality in Corollary 1. The consistency for the natural parameters $e^{\alpha+\beta x_i}/(1 + e^{\alpha+\beta x_i})$ $(i = 1, \dots, k)$

with the uniform distribution on $[0, 1]^k$ as their joint prior distribution is just the law of large number. One of the difficulties in our case is that the prior distribution is not a finite measure, so that we have to start with a condition for the posterior distribution to be a probability measure. As we already remarked, the logistic parameters are sometimes more natural than the natural parameters. This fact is also discussed in [1]. We refer to [2], [3], [4] for the meanings of Bayesian approach. Heberman [6] discussed the logit model with continuum observations, but did not discuss the binary data case which we discuss in this paper. Johan W. Pratt [7] discussed log likelihood for his model, but the model did not contain our case. Cox [9] gave a way to get maximum likelihood estimates, but he did not discuss the existence and uniqueness. Our results contains some of V.T. Farewell [11].

2. Proof of Theorem 1

For a given set of observation points x_i $(i = 1, \dots, n)$ and a set of corresponding observations $y_i \in \{0, 1\}$ with $m := \sum_{i=1}^n y_i$ and $M := \sum_{i=1}^n x_i y_i$, we define a subset Ω of $\mathbf{R}^2$ as the closed convex set generated by the set

$$\left\{ \left(\sharp S,\ \sum_{i \in S} x_i \right);\ S \subset \{1, \dots, n\} \right\}.$$

Let $\partial\Omega$ be the boundary of Ω. Then, the claimed condition in Theorem 1 is equivalent to $P := (m, M) \in \Omega \setminus \partial\Omega$, so that it is sufficient to prove that $c < \infty$ if and only if $P \in \Omega \setminus \partial\Omega$.

We put

$$Q_j = (\alpha_j,\ \beta_j) := \left(j,\ \min\left\{ \sum_{i \in S} x_i\ ;\ \sharp S = j \right\} \right)$$

for $j = 0, 1, \dots, n$, and

$$Q_j = (\alpha_j,\ \beta_j) := \left(2n - j,\ \max\left\{ \sum_{i \in S} x_i\ ;\ \sharp S = 2n - j \right\} \right)$$

for $j = n, n+1, \dots, 2n$. Then, it is easy to see that $\partial\Omega$ is the polygon $Q_0 Q_1 \cdots Q_{2n-1} Q_{2n}$ with $Q_{2n} = Q_0$. Let

$$\overrightarrow{PQ_j} = (r_j \cos\theta_j,\ r_j \sin\theta_j) \quad (j = 0, 1, \dots, 2n-1)$$

with $r_i \geq 0$ and $\theta_0 < \theta_1 < \cdots < \theta_{2n-1} < \theta_0 + 2\pi =: \theta_{2n}$.

Now we prove the "if" part. Assume that $P \in \Omega \setminus \partial\Omega$. Since P is in the interior of the convex set Ω, we have

$$\tau := \max_{0\leq j\leq 2n-1} \frac{\theta_{j+1}-\theta_j}{2} < \frac{\pi}{2}$$
$$c_0 := \min_{0\leq j\leq 2n-1} r_j > 0.$$

Define

$$\Omega_j := \left\{ (r\cos\phi,\ r\sin\phi);\ \frac{\theta_{j-1}+\theta_j}{2} \leq \phi < \frac{\theta_j+\theta_{j+1}}{2}\ ,\ r>0 \right\}$$

for $j = 0, 1, \ldots, 2n-1$, where $\theta_{-1} := \theta_{2k-1} - 2\pi$. Then it holds that

$$\bigcup_{j=0}^{2n-1} \Omega_j = \mathbf{R}^2 \setminus \{(0,0)\}$$

and that

$$\begin{aligned}
c &= \iint \frac{\exp(\alpha m + \beta M)}{\prod_{i=1}^{n}(1+\exp(\alpha+\beta x_i))} d\alpha d\beta \\
&= \iint \frac{\exp(\alpha m + \beta M)}{\sum_S \exp\left(\alpha \sharp S + \beta \sum_{i\in S} x_i\right)} d\alpha d\beta \\
&= \sum_{j=0}^{2n-1} \iint_{\Omega_j} \frac{1}{\sum_S \exp\left((\sharp S - m)\alpha + \left(\sum_{i\in S} x_i - M\right)\beta\right)} d\alpha d\beta \\
&\leq \sum_{j=0}^{2n-1} \iint_{\Omega_j} \exp((m-\alpha_j)\alpha + (M-\beta_j)\beta) d\alpha d\beta.
\end{aligned}$$

Since $|\theta_j - \phi| \leq \tau < \pi/2$ for any $(r\cos\phi,\ r\sin\phi) \in \Omega_j$, we have

$$(\alpha_j - m)\alpha + (\beta_j - M)\beta \geq c_0 r \cos\tau$$

for any $(\alpha, \beta) = (r\cos\phi,\ r\sin\phi) \in \Omega_j$. Thus,

$$c \leq \sum_{j=0}^{2k-1} \iint_{\Omega_j} \exp(-c_0 r \cos\tau)\, r dr d\phi < \infty.$$

Now we prove the "only if" part. Assume that $P \in \partial\Omega$. That is, P is one of the vertices of the polygon $Q_0 Q_1 \cdots Q_{2n-1} Q_0$. Let $P = Q_j$ and γ be the angle $Q_{j-1} P Q_{j+1}$ in the region of Ω. Then $\gamma \leq \pi$. Therefore, it is possible to take a half line $l = \{(m + r\cos\theta, M + r\sin\theta); r \geq 0\}$ satisfying that $\angle Q_{j-1} P l \leq \pi/2$ and

$\angle Q_{j+1}Pl \leq \pi/2$. This implies that $\angle Q(S)Pl \leq \pi/2$ for any $S \subset \{1, \ldots, n\}$ with $Q(S) := (\sharp S, \ \sum_{i \in S} x_i) \neq P$.

Let

$$\Gamma := \{(\alpha, \beta) \in \mathbf{R}^2; |\alpha \sin\theta - \beta\cos\theta| \leq 1, \ \ \alpha\cos\theta + \beta\sin\theta < 0\}.$$

Then, for any $(\alpha, \beta) \in \Gamma$ and $S \subset \{1, \ldots, n\}$, it hollds that

$$\alpha(u - m) + \beta(v - M) \leq \rho,$$

where $(u, v) := Q(S)$ and ρ is the diameter of Ω. Thus, we have

$$\begin{aligned} c &= \iint \frac{\exp(\alpha m + \beta M)}{\prod_{i=1}^n (1 + \exp(\alpha + \beta x_i))} d\alpha d\beta \\ &= \iint \frac{\exp(\alpha m + \beta M)}{\sum_S \exp\left(\alpha\sharp S + \beta \sum_{i \in S} x_i\right)} d\alpha d\beta \\ &= \iint \frac{1}{\sum_S \exp(\alpha(u - m) - \beta(v - M))} d\alpha d\beta \\ &\geq \iint_\Gamma \frac{1}{\sum_S \exp(\alpha(u - m) - \beta(v - M))} d\alpha d\beta \\ &\geq \iint_\Gamma \frac{1}{2^n e^\rho} d\alpha d\beta \\ &= 2^{-n} e^{-\rho} \iint_\Gamma d\alpha d\beta = \infty \ . \end{aligned}$$

EXAMPLE 1. We consider the case where $x_1 = x_2 = \cdots = x_{n_1} = u \neq v = x_{n_1+1} = x_{n_1+2} = \cdots = x_{n_1+n_2}$ and

$$\sum_{i=1}^{n_1} y_i = m_1 \quad , \quad \sum_{i=n_1+1}^{n_1+n_2} y_i = m_2$$

with $0 < m_1 < n_1$ and $0 < m_2 < n_2$. Then we have

$$\begin{aligned} c &= \iint \frac{\exp(m_1(\alpha + u\beta))}{(1 + \exp(\alpha + u\beta))^{n_1}} \frac{\exp(m_2(\alpha + v\beta))}{(1 + \exp(\alpha + v\beta))^{n_2}} d\alpha d\beta \\ &= \frac{1}{|u - v|} B(n_1 - m_1, m_1) B(n_2 - m_2, m_2). \end{aligned}$$

3. Proof of Theorem 2

Note that

$$\frac{\partial G_n}{\partial \alpha} = \sum_{i=1}^{k}\left(\frac{t_i}{n} - 1 + \frac{1}{1+\exp(\alpha+\beta x_i)}\right) =: g_1(\alpha, \beta)$$
$$\frac{\partial G_n}{\partial \beta} = \sum_{i=1}^{k}\left(x_i\left(\frac{t_i}{n} - 1\right) + \frac{x_i}{1+\exp(\alpha+\beta x_i)}\right) =: g_2(\alpha, \beta).$$

Since

$$\frac{\partial g_1(\alpha,\ \beta)}{\partial \alpha} = -\sum_{i=1}^{k}\frac{\exp(\alpha+\beta x_i)}{(1+\exp(\alpha+\beta x_i))^2} < 0$$
$$g_1(-\infty,\ \beta) = \sum_{i=1}^{k}\frac{t_i}{n_i} > 0$$
$$g_1(\infty,\ \beta) = \sum_{i=1}^{k}\left(\frac{t_i}{n_i} - 1\right) < 0$$

for any α, β, there exists a unique $\overline{\alpha} = \overline{\alpha}(\beta)$ for any β such that $g_1(\overline{\alpha}, \beta) \equiv 0$.

Then since

$$\frac{d\overline{\alpha}}{d\beta} = -\frac{\partial g_1/\partial\beta}{\partial g_1/\partial\alpha} = -\frac{\sum_{i=1}^{k}\left\{x_i\exp(\overline{\alpha}+\beta x_i)/[1+\exp(\overline{\alpha}+\beta x_i)]^2\right\}}{\sum_{i=1}^{k}\left\{\exp(\overline{\alpha}+\beta x_i)/[1+\exp(\overline{\alpha}+\beta x_i)]^2\right\}},$$

we have

$$\frac{dg_2(\overline{\alpha}, \beta)}{d\beta} = \frac{\partial g_2}{\partial\alpha}\frac{d\overline{\alpha}}{d\beta} + \frac{\partial g_2}{\partial\beta}$$
$$= \left(\sum_{i=1}^{k}\frac{\exp(\overline{\alpha}+\beta x_i)}{[1+\exp(\overline{\alpha}+\beta x_i)]^2}\right)^{-2} \times\left\{\left(\sum_{i=1}^{k}\frac{\exp(\overline{\alpha}+\beta x_i)}{[1+\exp(\overline{\alpha}+\beta x_i)]^2}\right)\left(\sum_{i=1}^{k}\frac{x_i^2\exp(\overline{\alpha}+\beta x_i)}{[1+\exp(\overline{\alpha}+\beta x_i)]^2}\right) - \left(\sum_{i=1}^{k}\frac{x_i\exp(\overline{\alpha}+\beta x_i)}{[1+\exp(\overline{\alpha}+\beta x_i)]^2}\right)^2\right\}$$
$$< 0$$

by the Cauchy-Schwarz inequality.

We consider $\overline{\alpha}/\beta$ as $\beta \to \infty$. Let $p \in [-\infty, +\infty]$ be any one of limit points of $\overline{\alpha}/\beta$ as $\beta \to \infty$. We denote by $\lim_{\beta * \to \infty}$ the limit as $\beta \to \infty$ along a subset such that $\overline{\alpha}/\beta \to p$.

Case 1: If $-p < x_1$, then

$$\begin{aligned} 0 &= \lim_{\beta * \to \infty} g_1(\overline{\alpha}, \beta) \\ &= \sum_{i=1}^{k} \left(\frac{t_i}{n} - 1 \right) < 0, \end{aligned}$$

which is absurd.

Case 2: If $-p > x_k$, then

$$\begin{aligned} 0 &= \lim_{\beta * \to \infty} g_1(\overline{\alpha}, \beta) \\ &= \sum_{i=1}^{k} \frac{t_i}{n} > 0, \end{aligned}$$

which is absurd.

Case 3: If there exists x_{i_0} such that $x_{i_0} < -p < x_{i_0+1}$, then we have

$$\begin{aligned} 0 &= \lim_{\beta * \to \infty} g_1(\overline{\alpha}, \beta) \\ &= \sum_{i=i_0+1}^{k} \left(\frac{t_i}{n} - 1 \right) + \sum_{i=1}^{i_0} \frac{t_i}{n} . \end{aligned}$$

Hence,

$$\begin{aligned} \lim_{\beta * \to \infty} g_2(\overline{\alpha}, \beta) &= \sum_{i=i_0+1}^{k} x_i \left(\frac{t_i}{n} - 1 \right) + \sum_{i=1}^{i_0} x_i \frac{t_i}{n} \\ &< x_{i_0} \left[\sum_{i=i_0+1}^{k} \left(\frac{t_i}{n} - 1 \right) + \sum_{i=1}^{i_0} \frac{t_i}{n} \right] = 0. \end{aligned}$$

Case 4: If $p = x_{i_0}$ for some $i_0 = 1, 2, \ldots, k$, then

$$\begin{aligned} 0 &= \lim_{\beta * \to \infty} g_1(\overline{\alpha}, \beta) \\ &= \sum_{i=i_0+1}^{k} \left(\frac{t_i}{n} - 1 \right) + \sum_{i=1}^{i_0-1} \frac{t_i}{n} + \lim_{\beta * \to \infty} \frac{1}{1 + \exp(\overline{\alpha} + p\beta)} . \end{aligned}$$

Hence,

$$\lim_{\beta*\to\infty} g_2(\overline{\alpha},\ \beta) =$$

$$\sum_{i=i_0+1}^{k} x_i\left(\frac{t_i}{n}-1\right)+\sum_{i=1}^{i_0-1} x_i\frac{t_i}{n}+x_{i_0}\lim_{\beta*\to\infty}\frac{1}{1+\exp(\overline{\alpha}+p\beta)}$$

$$< x_{i_0}\left[\sum_{i=i_{i_0}+1}^{k}\left(\frac{t_i}{n}-1\right)+\sum_{i=1}^{i_0-1}\frac{t_i}{n}+\lim_{\beta*\to\infty}\frac{1}{1+\exp(\overline{\alpha}+\lambda\beta)}\right]=0.$$

Thus, $\lim_{\beta*\to\infty} g_2(\overline{\alpha},\ \beta) < 0$.

In the same way, we can prove that $\lim_{\beta**\to-\infty} g_2(\overline{\alpha},\ \beta) > 0$. Therefore, there exists a unique $\hat{\beta}_n$ such that $g_2(\overline{\alpha},\ \hat{\beta}_n) = 0$. Putting $\hat{\alpha}_n = \overline{\alpha}(\hat{\beta}_n)$, we have proved that $(\hat{\alpha}_n, \hat{\beta}_n)$ is the unique point which maximizes the function $G_n(\alpha, \beta)$.

4. Proof of Theorem 3

The unique existence of $(\hat{\alpha}, \hat{\beta})$ can be proved exactly in the same way as for that of $(\hat{\alpha}_n, \hat{\beta}_n)$.

Let us take $\delta > 0$ and n_0 such that for any $n \geq n_0$,

$$\delta \leq \frac{t_i}{n} \leq 1-\delta \quad (i = 1, \ldots, k).$$

Lemma 1. *Let*

$$\varphi(x, p) := px - \log(1+e^x)$$

be a function on $x \in \mathbf{R}$ and $p \in \mathbf{R}$ with $0 < \delta \leq p \leq 1-\delta < 1$ for some $\delta > 0$. Then, we have

(i) $$\max_{x\in\mathbf{R}} \varphi(x, p) = p\log p + (1-p)\log(1-p) \leq \delta\log\delta + (1-\delta)\log(1-\delta) < 0,$$

(ii) $$\max_{\delta\leq p\leq 1-\delta} \varphi(x, p) \leq -\delta|x|$$

and

(iii) $$\left|\frac{\varphi(x, p')}{\varphi(x, p)} - 1\right| \leq C|p'-p| \text{ for some constant } C > 0.$$

Proof. (i) Since

$$\frac{\partial\varphi}{\partial x} = p - 1 + \frac{1}{1+e^x}$$

is a monotone decreasing function in x and takes value 0 at $x = \log p - \log(1-p)$, we have

$$\begin{aligned}\max_{x\in\mathbf{R}}\varphi(x,p) &= \varphi(\log p-\log(1-p),p)\\ &= p\log p+(1-p)\log(1-p)\\ &\le \delta\log\delta+(1-\delta)\log(1-\delta)<0.\end{aligned}$$

(ii) For any $x\ge 0$, we have

$$\varphi(x,p)\le px-\log e^x\le -\delta x.$$

On the other hand, for any $x<0$, we have

$$\varphi(x,p)\le px\le \delta x.$$

Thus we have (ii).

(iii) Since

$$\left|\frac{\partial\log\varphi}{\partial p}\right|=\left|\frac{x}{\varphi}\right|\le\frac{1}{\delta}$$

by (ii), we have

$$|\log\varphi(x,p')-\log\varphi(x,p)|\le\frac{1}{\delta}|p'-p|,$$

which implies (iii). □

Lemma 2. *For any $x_i\ne x_j$, there exists a constant $C>0$ such that*

$$(\alpha+\beta x_i)^2+(\alpha+\beta x_j)^2\ge C(\alpha^2+\beta^2)$$

holds for any α and β.

Proof. We have

$$\begin{aligned}&(\alpha+\beta x_i)^2+(\alpha+\beta x_j)^2\\ &\quad=2\left(\alpha+\beta\frac{x_i+x_j}{2}\right)^2+2\left(\beta\frac{x_i-x_j}{2}\right)^2\\ &\quad\ge C_1\beta^2\end{aligned}$$

and

$$(\alpha+\beta x_i)^2+(\alpha+\beta x_j)^2$$

$$
\begin{aligned}
&\geq \frac{x_j^2}{x_i^2+x_j^2}(\alpha+\beta x_i)^2 + \frac{x_i^2}{x_i^2+x_j^2}(\alpha+\beta x_j)^2 \\
&= \frac{(\alpha x_j + \beta x_i x_j)^2 + (\alpha x_i + \beta x_i x_j)^2}{x_i^2+x_j^2} \\
&= \frac{2\{\alpha(x_i - x_j)/2\}^2 + 2\{\alpha(x_i+x_j)/2 + \beta x_i x_j\}^2}{x_i^2+x_j^2} \\
&\geq C_2\alpha^2
\end{aligned}
$$

with some positive constants C_1 and C_2. Thus we have

$$(\alpha+\beta x_i)^2 + (\alpha+\beta x_j)^2 \geq C(\alpha^2+\beta^2)$$

with $C := (1/2)\min\{C_1, C_2\} > 0$. □

Lemma 3. *There exists a constant* $D > 0$ *such that*

$$G_n(\alpha,\beta) \leq -D(\alpha^2+\beta^2)^{1/2}$$

for any $n \geq n_0$ *and* $(\alpha,\beta) \in \mathbf{R}^2$.

Proof. Since

$$G_n(\alpha,\beta) = \sum_{i=1}^{k} \varphi\left(\alpha+\beta x_i, \frac{t_i}{n}\right),$$

where φ is defined in Lemma 1, we have

$$
\begin{aligned}
G_n(\alpha,\beta) &\leq -\delta(|\alpha+\beta x_i| + |\alpha+\beta x_j|) \\
&\leq -\delta\{(\alpha+\beta x_i)^2 + (\alpha+\beta x_j)^2\}^{1/2} \\
&\leq -\delta C(\alpha^2+\beta^2)^{1/2} \\
&= -D(\alpha^2+\beta^2)^{1/2}
\end{aligned}
$$

with $D = \delta C$ by Lemmas 1 and 2. □

Now we shall complete the proof of Theorem 3, since

$$G_n(0,0) = -k\log 2$$

and by Lemma 3, for any (α,β) with $\alpha^2+\beta^2 > (k\log 2/C)^2$

$$G_n(\alpha,\beta) < -k\log 2,$$

it holds that

$$\hat{\alpha}_n^2 + \hat{\beta}_n^2 \leq \left(\frac{k \log 2}{C}\right)^2.$$

Since G_n converges to f uniformly in any bounded region as $n \to \infty$, for any subsequence $\{n'\}$ of $\{n\}$ such that

$$\alpha^* := \lim_{n' \to \infty} \hat{\alpha}_{n'} , \quad \beta^* := \lim_{n' \to \infty} \hat{\beta}_{n'}$$

exist, it holds that

$$\begin{aligned} \lim_{n' \to \infty} G_n(\hat{\alpha}_{n'}, \hat{\beta}_{n'}) &= \lim_{n' \to \infty} f(\hat{\alpha}_{n'}, \hat{\beta}_{n'}) \\ &= f(\alpha^*, \beta^*) \ \leq \ f(\hat{\alpha}, \hat{\beta}). \end{aligned}$$

On the other hand, since

$$\begin{aligned} & |f(\hat{\alpha}, \hat{\beta}) - G_n(\hat{\alpha}_n, \hat{\beta}_n)| \\ = & \left| \max_{\alpha^2+\beta^2 \leq (k \log 2/D)^2} f(\alpha, \beta) - \max_{\alpha^2+\beta^2 \leq (k \log 2/D)^2} G_n(\alpha, \beta) \right| \\ \leq & \sup_{\alpha^2+\beta^2 \leq (k \log 2/D)^2} |f(\alpha, \beta) - G_n(\alpha, \beta)| \to 0 \end{aligned}$$

as $n \to \infty$, $f(\alpha^*, \beta^*) = f(\hat{\alpha}, \hat{\beta})$. The uniqueness of the (α, β) which maximizes $f(\alpha, \beta)$ implies that $(\alpha^*, \beta^*) = (\hat{\alpha}, \hat{\beta})$. This also implies that $\hat{\alpha}_n \to \hat{\alpha}$ and $\hat{\beta}_n \to \hat{\beta}$ as $n \to \infty$, which completes the proof.

EXAMPLE 2. For Example 1, we have

$$\begin{aligned} \hat{\alpha}_n &= \frac{v}{v-u} \log \frac{m_1}{n_1 - m_1} + \frac{u}{u-v} \log \frac{m_2}{n_2 - m_2} \\ \hat{\beta}_n &= \frac{1}{u-v} \log \frac{m_1(n_2 - m_2)}{m_2(n_1 - m_1)} . \end{aligned}$$

5. Proof of Theorem 4

Lemma 4. *It holds that*

$$\sum_{i=1}^{k} \varphi\left(\alpha + \beta x_i, \frac{t_i}{n}\right) = f(\alpha, \beta)(1 + O(\delta_n))$$

where $\delta_n := \max_i |(t_i/n) - p_i|$ *and* $O(\delta_n)$ *is uniform in* α *and* β *as* $n \to \infty$.

Proof. Take $\delta > 0$ such that $2\delta < \min_i p_i$ and $\max_i p_i + 2\delta < 1$. Then by (1), there exists n_0 such that for any $n \geq n_0$, it holds that

$$\left|\frac{t_i}{n} - p_i\right| < \delta \quad (i = 1, \ldots, k).$$

Then by (iii) of Lemma 1, there exists a constant C such that

$$\varphi\left(\alpha + \beta x_i, \frac{t_i}{n}\right) = \varphi(\alpha + \beta x_i \ , p_i)(1 + \xi_{i,n})$$

with $|\xi_{i,n}| \leq C|(t_i/n) - p_i|$ for any $i = 1, \ldots, k$. Therefore, we have

$$\sum_{i=1}^{k} \varphi\left(\alpha + \beta x_i, \frac{t_i}{n}\right) = f(\alpha, \beta)(1 + \xi_n)$$

with

$$|\xi_n| \leq C \max_i \left|\frac{t_i}{n} - p_i\right| = O(\delta_n). \qquad \square$$

To prove Theorem 4, it is sufficient to prove that for any given $\varepsilon > 0$,

$$\lim_{n\to\infty} \iint_{(\hat{\alpha}-\varepsilon,\hat{\alpha}+\varepsilon)\times(\hat{\beta}-\varepsilon,\hat{\beta}+\varepsilon)} p(\alpha, \beta | t_1, \ldots, t_k) d\alpha d\beta = 1.$$

Note that

$$p(\alpha, \beta | t_1, \ldots, t_k) = c_n{}^{-1} \exp\left[n \sum_{i=1}^{k} \varphi\left(\alpha + \beta x_i, \frac{t_i}{n}\right)\right]$$

with

$$c_n := \iint \exp\left[n \sum_{i=1}^{k} \varphi\left(\alpha + \beta x_i, \frac{t_i}{n}\right)\right] d\alpha d\beta.$$

By Theorem 3, Lemmas 1 and 3,

$$\max_{(\alpha,\beta)\in\mathbf{R}^2} f(\alpha, \beta) = f(\hat{\alpha}, \hat{\beta}) < 0$$
$$\lim_{\alpha^2+\beta^2\to\infty} f(\alpha, \beta) = -\infty. \tag{1}$$

For any $\Delta > 0$, let

$$\Omega(\Delta) := \{(\alpha, \beta) \in \mathbf{R}^2; f(\alpha, \beta) > \Lambda - \Delta\},$$

where we put $\Lambda := f(\hat{\alpha}, \hat{\beta})$. Since by Theorem 3, $(\hat{\alpha}, \hat{\beta})$ is the unique point which maximizes f together with (3) and the fact that f is continuous, we can take Δ such that

$$\Omega(5\Delta) \subset (\hat{\alpha} - \varepsilon, \hat{\alpha} + \varepsilon) \times (\hat{\beta} - \varepsilon, \hat{\beta} + \varepsilon). \tag{2}$$

Since $\Omega(\Delta)$ is a nonempty bounded open set, it has a positive area, say $S > 0$. Moreover, by (1) and Lemma 4, there exists n_1 such that for any $n \geq n_1$ and $(\alpha, \beta) \in \Omega(\Delta)$,

$$\sum_{i=1}^{k} \varphi\left(\alpha + \beta x_i, \frac{t_i}{n}\right) > \Lambda - 2\Delta.$$

Hence for any $n \geq n_1$, we have

$$\iint_{\Omega(\Delta)} \exp\left[n \sum_{i=1}^{k} \varphi\left(\alpha + \beta x_i, \frac{t_i}{n}\right)\right] d\alpha d\beta \geq e^{(\Lambda - 2\Delta)n} S. \tag{3}$$

On the other hand, by (1), (2), (3) and Lemma 1, there exists n_2 such that for any $n \geq n_2$ and $(\alpha, \beta) \notin \Omega(5\Delta)$,

$$\sum_{i=1}^{k} \varphi\left(\alpha + \beta x_i, \frac{t_i}{n}\right) < \Lambda - 4\Delta.$$

Also by (1), Lemmas 3 and 4, there exists n_3 such that for any $n \geq n_3$ and $(\alpha, \beta) \in \mathbf{R}^2$,

$$\sum_{i=1}^{k} \varphi\left(\alpha + \beta x_i, \frac{t_i}{n}\right) < \frac{1}{2} f(\alpha, \beta) \leq -\frac{1}{2} C(\alpha^2 + \beta^2)^{1/2}.$$

Hence, for any η with $0 < \eta < 1$, $(\alpha, \beta) \notin \Omega(5\Delta)$, and $n \geq n_4 := n_2 \vee n_3$ we have

$$\sum_{i=1}^{k} \varphi\left(\alpha + \beta x_i, \frac{t_i}{n}\right) \leq -\frac{1}{2} C\eta(\alpha^2 + \beta^2)^{1/2} + (1 - \eta)(\Lambda - 4\Delta).$$

Therefore, taking a small $\eta > 0$ such that

$$(1 - \eta)(\Lambda - 4\Delta) < \Lambda - 3\Delta,$$

we have

$$\sum_{i=1}^{k} \varphi\left(\alpha + \beta x_i, \frac{t_i}{n}\right) \leq -C'(\alpha^2 + \beta^2)^{1/2} + \Lambda - 3\Delta$$

for any $(\alpha, \beta) \notin \Omega(5\Delta)$ and $n \geq n_4$ with some constant $C' > 0$. Hence, we have

$$\begin{aligned}
&\iint_{\mathbf{R}^2\setminus\Omega(5\Delta)} \exp\left[n\sum_{i=1}^{k}\varphi\left(\alpha+\beta x_i, \frac{t_i}{n}\right)\right] d\alpha d\beta \\
&\quad \leq \iint \exp\left[-C'n(\alpha^2+\beta^2)^{1/2} + (\Lambda - 3\Delta)n\right] d\alpha d\beta \\
&\quad \leq e^{(\Lambda-3\Delta)n} \iint \exp\left[-C'(\alpha^2+\beta^2)^{1/2}\right] d\alpha d\beta \\
&\quad \leq C'' e^{(\Lambda-3\Delta)n} \qquad (4)
\end{aligned}$$

for any $n \geq n_4$ with some constant $C'' > 0$.

Let

$$I_n := \iint_{(\hat{\alpha}-\varepsilon, \hat{\alpha}+\varepsilon)\times(\hat{\beta}-\varepsilon, \hat{\beta}+\varepsilon)} p(\alpha, \beta | t_1, \ldots, t_k) d\alpha d\beta.$$

Then by (4), we have

$$\begin{aligned}
I_n &\geq \iint_{\Omega(5\Delta)} p(\alpha, \beta | t_1, \ldots, t_k) d\alpha d\beta \\
&= c_n^{-1} \iint_{\Omega(5\Delta)} \exp\left[n\sum_{i=1}^{k}\varphi\left(\alpha+\beta x_i, \frac{t_i}{n}\right)\right] d\alpha d\beta.
\end{aligned}$$

Putting

$$J(j) := \iint_{\Omega(j\Delta)} \exp\left[n\sum_{i=1}^{k}\varphi\left(\alpha+\beta x_i, \frac{t_i}{n}\right)\right] d\alpha d\beta \qquad (5)$$

and

$$L(j) := \iint_{\mathbf{R}^2\setminus\Omega(j\Delta)} \exp\left[n\sum_{i=1}^{k}\varphi\left(\alpha+\beta x_i, \frac{t_i}{n}\right)\right] d\alpha d\beta, \qquad (6)$$

we have

$$\begin{aligned}
I_n &\geq c_n^{-1} J(5) = \frac{J(5)}{J(5)+L(5)} \\
&\geq \frac{J(1)}{J(1)+L(5)} = \frac{1}{1+\{L(5)/J(1)\}}.
\end{aligned}$$

Let $n_0 := n_1 \vee n_4$. Then, for any $n \geq n_0$, we have by (5) and (6) that

$$J(1) \geq e^{(\Lambda-2\Delta)n} S \text{ and } L(5) \leq C'' e^{(\Lambda-3\Delta)n}.$$

Thus,

$$I_n \geq \frac{1}{1 + C'' S^{-1} e^{-\Delta n}}$$

from which $\lim_{n\to\infty} I_n = 1$ follows. □

Lehmann gave conditions B(1)–B(4) for the asymptotic normality in [5]. The condition B(1) follows Theorem 4, the other conditions B(2)–B(4) are verified easily. Thus we have Corolloary 1.

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