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EXPONENTIAL DECAY OF POSITIVITY PRESERVING SEMIGROUPS ON L^p

MASANORI HINO

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1. Introduction

In recent studies on invariant measures of diffusions on infinite-dimensional spaces, several approaches have been considered. In [6], existence, uniqueness and regularity of invariant measures and asymptotic properties of solutions of stochastic evolution equations were investigated. In purely analytic approaches, a study of invariant measures for Markovian semigroups on L^2 associated with quadratic forms was made in [5] as a generalization of [19, 20, 21, 10], while the regularity of the measures μ solving some elliptic equations $L^*\mu = 0$ was proved in [4, 3], where L is an operator of type $Lu = \mathrm{tr}(Au'') + B \cdot \nabla u$.

In this paper, for given Markovian (or more generally, positivity preserving) semigroups $\{P_t\}$, we discuss conditions for the existence of invariant measures, and for the exponential decay of $\{P_t\}$ to a projection operator. We treat this problem in an analytic way, so we always impose the condition that $\{P_t\}$ is also a strongly continuous semigroup on L^p for some $p \in (1, \infty)$.

As for the existence of invariant measures, the situation becomes quite simple if $\{P_t\}$ is eventually compact. But we do not assume this since such compactness seems hard to be expected in the case that the underlying space is infinite dimensional. In Gross' paper [9] concerning physical ground states for Hermitian operators, similar situations were dealt with and the so-called hyperboundedness of semigroups was used as a replacement of compactness. We apply his idea to our problem; under the condition (I) regarding integrability (see Definition 2.1) for semigroups or resolvents, we prove the existence of invariant measures by approximating the underlying space by a sequence of finite number of sets. The result improves the corresponding ones in [5, 10].

In order to discuss the exponential decay of $\{P_t\}$ in the L^p sense, we introduce a kind of ergodicity condition, (E) (see Definition 3.1). We may say that it is a substitution for strict positivity of transition densities; we do not expect the existence of such densities in our concerning infinite-dimensional cases. This type of condition appeared in Kusuoka's article [13] and was further researched by Aida [1] to discuss the spectral gap of the generators of symmetric Markovian semigroups. It turns out that this is useful also in our framework. Indeed, we prove that P_t decays exponentially to a projection operator in the L^p sense if and only if the conditions (I) and (E) hold for the semigroup. This can be regarded as a generalization of some results by Aida [1] and Mathieu [17] to the L^p category.

Typical examples that we can conclude exponential decay from our results are conservative Markovian semigroups on L^2, associated with bilinear forms obtained by adding drift terms to strongly local symmetric Dirichlet forms which satisfy logarithmic Sobolev inequalities. They seem to lie outside the range of examples to which usual perturbation methods apply.

The organization of this paper is as follows: In the section 2, we introduce and study the condition (I) and prove the existence of invariant measures. In the section 3, we define the condition (E) and give the characterization of the exponential decay for semigroups. In the section 4, some criteria for (E) are given. In the section 5, we give a few examples.

2. Existence of invariant measures

Let $(X, \mathcal{F}, m)$ be a measure space with total measure 1. We assume that $L^p = L^p(X, \mathcal{F}, m)$ is separable for some (hence for all) $p \in (1, \infty)$. We consider only real-valued functions. Let $\|\cdot\|_p$ denote the L^p-norm with respect to m. Let L^p_+ (resp. $L^p_{>0}$) be the set of all nonnegative (resp. strictly positive) functions in L^p. For a measurable function f, we set $f_+ = f \vee 0$ and $f_- = (-f) \vee 0$. For a bounded linear operator S, $\|S\|$ denotes the operator norm and S^* the dual operator. A bounded operator S on L^p is called positivity preserving if $Sf \in L^p_+$ for all $f \in L^p_+$. We fix $p \in (1, \infty)$ and denote by q the conjugate exponent of p; $1/p + 1/q = 1$. The L^q space is considered as the dual space of L^p.

Throughout this paper, S is assumed to be a positivity preserving bounded operator on L^p, and $\{P_t\}$ a positivity preserving, strongly continuous semigroup on L^p. Let $\{R_\alpha\}$ and A denote the associated resolvent operators and the generator, respectively.

We discuss the existence of a non-zero function $\rho \in L^q$ such that

$$\int_X P_t f \cdot \rho \, dm = \int_X f \rho \, dm \qquad \text{for every } t > 0 \text{ and } f \in L^p,$$

in other words, $\rho \in \bigcap_{t>0} \operatorname{Ker}(1 - P_t^*)$. By noting that the relation $\bigcap_{t>0} \operatorname{Ker}(1 - P_t^*) = \operatorname{Ker} A^* = \operatorname{Ker}(1 - \alpha R_\alpha^*)$ for any α, it suffices to study $\operatorname{Ker}(1 - S^*)$ for a positivity preserving operator S. For this purpose, we introduce the following condition.

DEFINITION 2.1. Set

$$\psi_S(K) = \sup_{f \in L^p_+,\ \|f\|_p \le 1} \|(Sf - K)_+\|_p, \quad K > 0.$$

We say that S satisfies the condition (I) if $\psi_S(K) < 1$ for some $K > 0$.

We also write $\psi_{S,L^p(m)}(K)$ and $(\mathrm{I})_{L^p(m)}$ instead if there is a possibility of confusion. It is easy to see that

$$\psi_S(K) = \sup_{f\in L^p,\, \|f\|_p\le 1} \|(|Sf| - K)_+\|_p$$

and

$$\sup_{f\in L^p_+,\, \|f\|_p\le c} \|(Sf - K)_+\|_p = c\psi_S(K/c), \quad c > 0. \tag{2.1}$$

The following facts are often useful to verify (I) in applications.

Lemma 2.2. (i) *If $\{|Sf_k|^p\}_{k\in\mathbb{N}}$ is uniformly integrable for any bounded sequence $\{f_k\}_{k\in\mathbb{N}}$ in L^p, then S satisfies* (I). *In particular, if S is bounded from L^p to $L^{p'}$ with $p' > p$ or to $L^p \log L$, then S satisfies* (I).
(ii) *Suppose that there are some $\alpha > 0$ and $n \in \mathbb{N}$ such that $\{|R^n_\alpha f_k|^p\}$ is uniformly integrable for any bounded sequence $\{f_k\}_{k\in\mathbb{N}}$ in L^p. If $\{P_t\}$ is differentiable at $t = t_0$, then P_{nt_0} satisfies* (I).

Proof. (i): Since the assumption is equivalent to saying that

$$\lim_{K\to\infty} \sup_{\|f\|_p\le 1} \int_{\{|Sf|>K\}} |Sf|^p\, dm = 0,$$

and the left-hand side exceeds $\lim_{K\to\infty} \psi_S(K)^p$, the assertion holds.
(ii): From the assumption, $\lim_{K\to\infty} \psi_{R^n_\alpha}(K) = 0$ and the operator $(\alpha - A)P_{t_0}$ is bounded. Let M be its operator norm. Then for $f \in L^p_+$ with $\|f\|_p \le 1$,

$$\|(P_{nt_0} f - K)_+\|_p = \|[R^n_\alpha\{(\alpha - A)P_{t_0}\}^n f - K]_+\|_p \le M^n \psi_{R^n_\alpha}\left(\frac{K}{M^n}\right),$$

by (2.1). When K is taken to be sufficiently large, the right-hand side is less than 1. This finishes the proof. □

Before studying several properties on (I), we give one more definition.

DEFINITION 2.3. For $\varphi \in L^p_{>0}$, a probability measure m_φ on X and a positivity preserving operator S_φ on $L^p(m_\varphi)$ are defined by $dm_\varphi = \varphi^p\, dm/\|\varphi\|_p^p$ and $S_\varphi f = \varphi^{-1} S(\varphi f)$ for $f \in L^p(m_\varphi)$.

Proposition 2.4. *Let S satisfy* (I). *Then the following assertions hold.*
(i) *S^* satisfies $(\mathrm{I})_{L^q(m)}$.*
(ii) *For any $\varphi \in L^p_{>0}$, S_φ satisfies $(\mathrm{I})_{L^p(m_\varphi)}$.*

Proof. (i) We may assume that $S \neq 0$. For $g \in L_+^q$ with $\|g\|_q \leq 1$ and $K > 0$, $K_i > 0$ $(i = 1, 2, 3)$,

$$\begin{aligned}
\|(S^*g - K)_+\|_q^q &\leq \int_X S^*g \cdot (S^*g - K)_+^{q-1} dm = \int_X g \cdot S\{(S^*g - K)_+^{q-1}\} dm \\
&\leq \int_X (g - K_1)_+ \cdot S\{(S^*g - K)_+^{q-1}\} dm + K_1 \int_X S\{(S^*g - K)_+^{q-1}\} dm \\
&=: I_1 + K_1 I_2.
\end{aligned}$$

From the relation $\|(S^*g - K)_+^{q-1}\|_p = \|(S^*g - K)_+\|_q^{q/p} \leq \|S^*\|^{q/p} = \|S\|^{q/p}$ and (2.1), we have

$$\begin{aligned}
I_1 &\leq K_2 \int_X (g - K_1)_+ dm + \int_X g[S\{(S^*g - K)_+^{q-1}\} - K_2]_+ dm \\
&\leq K_2 K_1^{1-q} \|g\|_q^q + \|g\|_q \|[S\{(S^*g - K)_+^{q-1}\} - K_2]_+\|_p \\
&\leq K_2 K_1^{1-q} + \|(S^*g - K)_+^{q-1}\|_p \psi_S\left(\frac{K_2}{\|(S^*g - K)_+^{q-1}\|_p}\right) \\
&\leq K_2 K_1^{1-q} + \|(S^*g - K)_+\|_q^{q/p} \psi_S\left(\frac{K_2}{\|S\|^{q/p}}\right),
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= \int_X S^*1 \cdot (S^*g - K)_+^{q-1} dm \\
&\leq K_3 \int_X (S^*g - K)_+^{q-1} dm + \int_X (S^*1 - K_3)_+ (S^*g)^{q-1} dm \\
&\leq K_3 K^{-1} \|S^*g\|_q^q + \|(S^*1 - K_3)_+\|_q \|(S^*g)^{q-1}\|_p \\
&\leq K_3 K^{-1} \|S\|^q + \|(S^*1 - K_3)_+\|_q \|S\|^{q/p}.
\end{aligned}$$

Therefore, taking $\overline{\lim}_{K_2\to\infty} \overline{\lim}_{K_1\to\infty} \overline{\lim}_{K_3\to\infty} \overline{\lim}_{K\to\infty} \sup_{g\in L_+^q,\, \|g\|_q\leq 1}$, we obtain that

$$\varlimsup_{K\to\infty} \psi_{S^*, L^q(m)}(K)^q \leq \varlimsup_{K\to\infty} \psi_{S^*, L^q(m)}(K)^{q/p} \varlimsup_{K_2\to\infty} \psi_S(K_2).$$

Since the left-hand side is finite, we conclude that

$$\varlimsup_{K\to\infty} \psi_{S^*, L^q(m)}(K) \leq \varlimsup_{K_2\to\infty} \psi_S(K_2) < 1.$$

(ii) We may assume that $\|\varphi\|_p = 1$. Define $\tilde{f} = \varphi f$ for $f \in L_+^p(m_\varphi)$. Then $\tilde{f} \in L_+^p$ and $\|\tilde{f}\|_p = \|f\|_{L^p(m_\varphi)}$. Taking $\overline{\lim}_{K_1\to\infty} \overline{\lim}_{K\to\infty} \sup_{f\in L_+^p(m_\varphi),\, \|f\|_{L^p(m_\varphi)}\leq 1}$ in the relation

$$\|(S_\varphi f - K)_+\|_{L^p(m_\varphi)} = \|(S\tilde{f} - K\varphi)_+\|_p \leq \|(S\tilde{f} - K_1)_+\|_p + \|(K_1 - K\varphi)_+\|_p,$$

we obtain that $\overline{\lim}_{K\to\infty} \psi_{S\varphi, L^p(m_\varphi)}(K) \le \overline{\lim}_{K_1\to\infty} \psi_S(K_1) < 1$. □

Proposition 2.5. *Let S be also sub-Markovian. Moreover, suppose that S satisfies* (I), *namely, $\psi_S(K) \le \gamma$ for some $K > 0$ and $\gamma \in (0, 1)$. Then for some $K' \ge K$, it holds that*

$$\psi_{S^n}(K') \le \gamma^n, \quad n \in \mathbb{N}. \tag{2.2}$$

In particular, $\{S^n\}_{n\in\mathbb{N}}$ are uniformly bounded.

Proof. Let $K' = K/(1-\gamma)$. For $f \in L^p_+$ with $\|f\|_p \le 1$, we have for $n \in \mathbb{N}$,

$$\begin{aligned}\|(S^n f - K')_+\|_p &= \|[S^{n-1}\{(Sf-K)_+ + (Sf \wedge K)\} - \{(K'-K)+K\}]_+\|_p \\ &\le \|[S^{n-1}\{(Sf-K)_+\} - (K'-K)]_+\|_p + \|\{S^{n-1}(Sf\wedge K) - K\}_+\|_p.\end{aligned}$$

Since S^{n-1} is sub-Markovian, the second term in the right-hand side vanishes. By applying (2.1) with using $\|(Sf-K)_+\|_p \le \gamma$, the first term is dominated by $\gamma\psi_{S^{n-1}}((K'-K)/\gamma) = \gamma\psi_{S^{n-1}}(K')$. Hence $\psi_{S^n}(K') \le \gamma\psi_{S^{n-1}}(K')$, which implies (2.2). The last claim follows from the domination $\|S^n\| \le \psi_{S^n}(K') + K'$. □

The following proposition characterizes (I) for semigroups in terms of the resolvent operators.

Proposition 2.6. *Suppose that $\{P_t\}$ is also sub-Markovian. Then*:
(i) *If P_{t_0} satisfies* (I) *for some $t_0 > 0$, then there exist some $\gamma \in (0, 1)$, $K > 0$, and $T > 0$ such that $\psi_{P_t}(K) \le \gamma^t$ for all $t \ge T$.*
(ii) *The following conditions are equivalent.*
(a) *P_{t_0} satisfies* (I) *for some $t_0 > 0$.*
(b) *There exist some $K > 0$, $M > 0$, $T > 0$, and $\delta > 0$ such that $\psi_{(\alpha R_\alpha)^n}(K) \le M(\alpha T)^n/n! + \{\alpha/(\alpha+\delta)\}^n$ for every $n \in \mathbb{N}$ and $\alpha > 0$.*
In particular, if P_{t_0} satisfies (I) *for some $t_0 > 0$, then for each $\alpha > 0$, there exists some $n \in \mathbb{N}$ such that $(\alpha R_\alpha)^n$ satisfies* (I).

Proof. (i) is easily deduced from Proposition 2.5.

(ii): Assume (a). From (i), $M := \sup_{t>0}\|P_t\| < \infty$. Take γ, K and T appearing in (i). Let $\Omega_1 = \{(t_1,\dots,t_n) \in [0,\infty)^n \mid t_1+\cdots+t_n \le T\}$ and $\Omega_2 = [0,\infty)^n \setminus \Omega_1$. For $f \in L^p_+$ with $\|f\|_p \le 1$,

$$\begin{aligned}\|\{(\alpha R_\alpha)^n f - K\}_+\|_p &= \left\| \left\{ \int_{[0,\infty)^n} \alpha^n e^{-\alpha(t_1+\cdots+t_n)} (P_{t_1+\cdots+t_n} f - K)\,dt_1\cdots dt_n \right\}_+ \right\|_p \\ &\le \int_{[0,\infty)^n} \alpha^n e^{-\alpha(t_1+\cdots+t_n)} \left\|(P_{t_1+\cdots+t_n} f - K)_+\right\|_p dt_1\cdots dt_n\end{aligned}$$

$$\leq M\int_{\Omega_1}\alpha^n dt_1\cdots dt_n+\int_{\Omega_2}\alpha^n e^{-\alpha(t_1+\cdots+t_n)}\gamma^{t_1+\cdots+t_n}dt_1\cdots dt_n$$
$$\leq \frac{M(\alpha T)^n}{n!}+\left(\frac{\alpha}{\alpha-\log\gamma}\right)^n.$$

Therefore, (b) holds.

Next assume (b). For $f\in L^p_+$ with $\|f\|_p\leq 1$, by using the expression $P_tf=\lim_{n\to\infty}e^{-nt}\sum_{m=0}^{\infty}(1/m!)(n^2tR_n)^m f$,

$$\begin{aligned}\|(P_tf-K)_+\|_p &\leq \varlimsup_{n\to\infty}e^{-nt}\sum_{m=0}^{\infty}\frac{1}{m!}(nt)^m\left\|\left\{(nR_n)^m f-K\right\}_+\right\|_p\\ &\leq \varlimsup_{n\to\infty}e^{-nt}\sum_{m=0}^{\infty}\frac{1}{m!}(nt)^m\left\{\frac{M(nT)^m}{m!}+\left(\frac{n}{n+\delta}\right)^m\right\}\\ &= \varlimsup_{n\to\infty}e^{-nt}\sum_{m=0}^{\infty}\left\{\frac{M(n^2tT)^m}{(m!)^2}+\frac{(n^2t/(n+\delta))^m}{m!}\right\}.\end{aligned}$$

Since

$$e^{-nt}\sum_{m=0}^{\infty}\frac{M(n^2tT)^m}{(m!)^2}\leq Me^{-nt}\sum_{m=0}^{\infty}\frac{(n\sqrt{tT})^{2m}}{(2m)!/2^{2m}}\leq Me^{-nt+2n\sqrt{tT}}$$

and

$$e^{-nt}\sum_{m=0}^{\infty}\frac{(n^2t/(n+\delta))^m}{m!}=e^{-nt}e^{n^2t/(n+\delta)}=e^{-nt\delta/(n+\delta)}\to e^{-t\delta}\quad\text{as } n\to\infty,$$

we conclude that $\psi_{P_t}(K)<1$ when $t>4T$. □

Now, in order to prove the existence of invariant measures, we prepare a lemma.

Lemma 2.7. *Suppose that there exists $\varphi\in\operatorname{Ker}(1-S)$ such that $\varphi>0$ m-a.e. Then $\operatorname{Ker}(1-S^*)$ is a vector lattice.*

Proof. This is almost the same as in [5, Corollary 2.13], but we give a proof for completeness. It suffices to prove that $\rho\in\operatorname{Ker}(1-S^*)$ implies $\rho_+\in\operatorname{Ker}(1-S^*)$. Take $\rho\in\operatorname{Ker}(1-S^*)$. Since $S^*\rho_+\geq S^*\rho=\rho$ and $S^*\rho_+\geq 0$, we have $S^*\rho_+\geq\rho_+$. On the other hand, it holds that $\int_X(S^*\rho_+-\rho_+)\varphi\,dm=0$. Since $\varphi>0$ m-a.e., we conclude that $S^*\rho_+=\rho_+$. □

Theorem 2.8. *Suppose that S^n satisfies* (I) *for some $n\in\mathbb{N}$ and that there exists $\varphi\in\operatorname{Ker}(1-S)$ such that $\varphi>0$ m-a.e. Then there exists $\rho\in\operatorname{Ker}(1-S^*)$ such that $\rho\not\equiv 0$ and $\rho\geq 0$ m-a.e.*

Proof. First we consider the case that $\varphi \equiv 1$. Since L^p is assumed to be separable, there exists a sub σ-field $\mathcal{F}^0$ of $\mathcal{F}$ such that $\mathcal{F}^0$ is countably generated and $L^p(X, \mathcal{F}, m) = L^p(X, \mathcal{F}^0, m)$. Take a sequence of finitely generated sub σ-fields $\{\mathcal{F}_l\}_{l=1}^\infty$ of $\mathcal{F}^0$ satisfying $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$, and $\bigvee_{l=1}^\infty \mathcal{F}_l = \mathcal{F}^0$. For each l, $\iota_{r,l}$ denotes a natural injection operator from $L^r(\mathcal{F}_l) := L^r(X, \mathcal{F}_l, m)$ to L^r and $\pi_{r',l}$ its adjoint operator from $L^{r'}$ to $L^{r'}(\mathcal{F}_l)$ for $1 < r < \infty$, $1/r + 1/r' = 1$. Then $\pi_{p,l} S \iota_{p,l}$ is bounded on $L^p(\mathcal{F}_l)$. Since $\pi_{p,l}$ is expressed as a conditional expectation under m given $\mathcal{F}_l$, $\iota_{p,l}\pi_{p,l}$ converges strongly to the identity map as $l \to \infty$ from the martingale convergence theorem. We also note that $\pi_{p,l} S \iota_{p,l} 1 = 1$. Since $L^p(\mathcal{F}_l)$ is finite dimensional, $\pi_{p,l} S \iota_{p,l}$ is a compact operator and $(\pi_{p,l} S \iota_{p,l})^* = \pi_{q,l} S^* \iota_{q,l}$ has the same eigenvalues as $\pi_{p,l} S \iota_{p,l}$. Therefore, there exists an element ρ_l in $L^q(\mathcal{F}_l)$ such that $\|\rho_l\|_q = 1$ and $\pi_{q,l} S^* \iota_{q,l} \rho_l = \rho_l$. From Lemma 2.7, we may assume that $\rho_l \geq 0$ m-a.e. Since $\{\iota_{q,l}\rho_l\}_{l=1}^\infty$ is weakly relatively compact in L^q, we can take a divergent sequence $\{l_k\}$ such that $\iota_{q,l_k}\rho_{l_k}$ converges weakly to some $\rho \geq 0$ in L^q. For every $f \in L^p$, it holds that

$$\int_X S\iota_{p,l_k}\pi_{p,l_k} f \cdot \iota_{q,l_k}\rho_{l_k} dm = \int_X f \cdot \iota_{q,l_k}\rho_{l_k} dm.$$

Letting $k \to \infty$, we obtain that $\int_X Sf \cdot \rho \, dm = \int_X f \rho \, dm$. Hence $\rho \in \operatorname{Ker}(1 - S^*)$. The rest to be proved is that ρ is not identically zero. Assume that $\rho \equiv 0$. Then $\iota_{q,l_k}\rho_{l_k} \to 0$ in L^1 since ρ_{l_k} is nonnegative. In the following, we write ρ_k instead of $\iota_{q,l_k}\rho_{l_k}$ for simplicity. Take $K > 0$ so that $\psi_{S^n}(K) < 1$. Since $\|\rho_k^{q-1}\|_p = 1$, it holds that

$$\begin{aligned} 1 &= \int_X \rho_k^{q-1}\rho_k \, dm = \int_X (S^n \rho_k^{q-1})\rho_k \, dm \\ &\leq K\|\rho_k\|_1 + \|(S^n\rho_k^{q-1} - K)_+\|_p \|\rho_k\|_q \\ &\leq K\|\rho_k\|_1 + \psi_{S^n}(K). \end{aligned}$$

Taking $\lim_{k\to\infty}$ on both sides induces a contradiction.

Next we prove the claim in general cases. From Proposition 2.4 (ii), S_φ^n satisfies $(\mathrm{I})_{L^p(m_\varphi)}$. Moreover, $S_\varphi 1 = 1$. So we can use the first part of this proof to obtain that there exists some $\rho_\varphi \in \operatorname{Ker}(1 - S_\varphi^*)$ such that $\rho_\varphi \geq 0$, $\rho_\varphi \not\equiv 0$ m-a.e. Then it is easy to see that $\varphi^{p-1}\rho_\varphi \in \operatorname{Ker}(1 - S^*)$. This completes the proof. □

Theorem 2.9. *Suppose that there exists some $\varphi \in \operatorname{Ker} A$ such that $\varphi > 0$ m-a.e. If some P_{t_0} or $(\alpha R_\alpha)^n$ satisfies (I), then there exists $\rho \in \operatorname{Ker} A^*$ such that $\rho \not\equiv 0$ and $\rho \geq 0$ m-a.e. In particular, the measure $\rho \, dm$ is an invariant measure for $\{P_t\}$.*

Proof. From Proposition 2.4 (ii) and Proposition 2.6 (ii), we may assume that $(\alpha R_\alpha)^n$ satisfies (I) for some $\alpha > 0$, $n \in \mathbb{N}$. By noting that $\operatorname{Ker} A = \operatorname{Ker}(1 - \alpha R_\alpha)$ and $\operatorname{Ker} A^* = \operatorname{Ker}(1 - \alpha R_\alpha^*)$, Theorem 2.8 completes the proof. □

REMARK 2.10. (i) Unlike former works [19, 10, 5], the underlying space need not have a vector space structure.
(ii) We can also prove the claim without using Proposition 2.6 (ii) when the condition (I) for P_{t_0} is assumed. Indeed, from Theorem 2.8, $\mathrm{Ker}(1-P_{t_0}^*)$ has a nonzero and nonnegative function, say ρ'. Then $\int_0^{t_0} P_t^*\rho'\,dt$ belongs to $\bigcap_{t>0}\mathrm{Ker}(1-P_t^*)$ and is nonzero and nonnegative (cf. [18, Theorem C-III,1.1]).

3. Exponential decay

In this section, we investigate the conditions for exponential decay of $\{P_t\}$ in the L^p sense. The following definition is a slight modification of Aida's in [1].

DEFINITION 3.1. We set

$$\chi_S(\varepsilon) := \inf\left\{\int_X S1_{B_1}\cdot 1_{B_2}\,dm \,\middle|\, m(B_1)\ge\varepsilon, m(B_2)\ge\varepsilon\right\},\quad \varepsilon>0.$$

Here we consider that $\inf\emptyset=\infty$. We say that S satisfies the condition (E) if for each $\varepsilon>0$, there exists some $n\in\mathbb{N}$ such that $\chi_{S^n}(\varepsilon)>0$. We say that the semigroup $\{P_t\}$ satisfies (E) if for each $\varepsilon>0$, there exists some $t>0$ such that $\chi_{P_t}(\varepsilon)>0$.

We remark that we can also define this notion for positivity preserving bounded operators on L^∞. As stated in [1, Lemma 2.6], $\chi_S(\varepsilon)>0$ for every $\varepsilon>0$ if S has a strictly positive integral kernel.

We prove several properties on χ_S and (E).

Lemma 3.2. *For $g_1, h_2\in L^p_{>0}$ and $g_2, h_1\in L^q_{>0}$, define*

$$\chi_{S,g_1,h_1,g_2,h_2}(\varepsilon)$$
$$=\inf\left\{\int_X Sf_1\cdot f_2\,dm\,\middle|\, 0\le f_i\le g_i\ m\text{-a.e.},\ \int_X f_ih_i\,dm\ge\varepsilon,\ i=1,2\right\},\quad \varepsilon>0.$$

Then for each $\varepsilon>0$, there exists some constant $\delta>0$ independent of S such that

$$\delta\chi_{S,g_1,h_1,g_2,h_2}(\delta)\le\chi_S(\varepsilon)\quad\text{and}\quad \delta\chi_S(\delta)\le\chi_{S,g_1,h_1,g_2,h_2}(\varepsilon).$$

In particular, S satisfies (E) if and only if for each $\varepsilon>0$, there exists some $n\in\mathbb{N}$ such that $\chi_{S^n,g_1,h_1,g_2,h_2}(\varepsilon)>0$.

Proof. It is enough to prove the following for $g\in L^p_{>0}$ and $h\in L^q_{>0}$ (or $g\in L^q_{>0}$ and $h\in L^p_{>0}$):
(i) For each $\varepsilon>0$, there exist some $\varepsilon'>0$ and $a>0$ such that for any measurable set B of X with $m(B)\ge\varepsilon$, there exists a subset B' of B with $a1_{B'}\le g$ m-a.e. and $\int_X 1_{B'}h\,dm\ge\varepsilon'$.

(ii) For each $\varepsilon > 0$, there exist some $\varepsilon' > 0$ and $a > 0$ such that $m(\{f \geq a\}) \geq \varepsilon'$ for all f satisfying $0 \leq f \leq g$ m-a.e. and $\int_X fh\,dm \geq \varepsilon$.

First we prove (i). Since g and h are strictly positive, there exists $a > 0$ such that the measure of $Y := \{g < a\} \cup \{h < a\}$ is less than $\varepsilon/2$. Define $B' = B\backslash Y$. Then $0 \leq a1_{B'} \leq g$ m-a.e. and $\int_X 1_{B'}h\,dm \geq am(B') \geq a\varepsilon/2$.

Next we prove (ii). Define $b = \int_X gh\,dm > 0$, $a_0 = \varepsilon/(3b)$, and $B_0 = \{f \geq a_0 g\}$. Then

$$\varepsilon \leq \int_X fh\,dm \leq \int_{B_0} gh\,dm + a_0 \int_{X\backslash B_0} gh\,dm \leq \int_{B_0} gh\,dm + a_0 b.$$

Therefore, $\int_{B_0} gh\,dm \geq \varepsilon - a_0 b = 2\varepsilon/3$. Take $a_1 > 0$ small and $M > 1$ large so that $\int_{X\backslash B_1} gh\,dm \leq \varepsilon/3$ with $B_1 := \{g \geq a_1\} \cap \{gh \leq M\}$. Then $f \geq a_0 a_1$ on $B_0 \cap B_1$ and $m(B_0 \cap B_1) \geq \varepsilon/(3M)$. □

Proposition 3.3. *Let S satisfy* (E). *Then the following assertions hold.*

(i) S^* *satisfies* (E).

(ii) *For any $\varphi \in L^p_{>0}$, S_φ satisfies* (E) *with respect to the measure m_φ.*

(iii) *If there exists $\varphi \in \mathrm{Ker}(1-S)$ that is nonzero and nonnegative, then φ is strictly positive and* $\dim \mathrm{Ker}(1-S^*) \leq 1$.

Proof. (i) is obvious. (ii): Apply Lemma 3.2 twice using

$$\chi_{S,\|\varphi\|_p^{-1},\|\varphi\|_p^{1-p},\|\varphi\|_p^{1-p},\|\varphi\|_p^{-1}}(\varepsilon) = \chi_{S_\varphi,\varphi^{-1},\varphi^{1-p},\varphi^{1-p},\varphi^{-1}}(\varepsilon),$$

where the quantity in the right-hand side is with respect to m_φ. (iii): For some $a > 0$, $m(\{\varphi \geq a\}) > 0$. For every $n \in \mathbb{N}$, $\int_X S^n 1_{\{\varphi \geq a\}} \cdot 1_{\{\varphi = 0\}}\,dm \leq (1/a)\int_X S^n\varphi \cdot 1_{\{\varphi=0\}}\,dm = 0$. So $m(\{\varphi = 0\})$ has to be 0. If ρ_1, $\rho_2 \in \mathrm{Ker}(1-S^*)$ are linearly independent, then $\alpha\rho_1 + \beta\rho_2$ has both positive part and negative part for some α, $\beta \in \mathbb{R}$. But then, $(\alpha\rho_1 + \beta\rho_2)_+$ belongs to $\mathrm{Ker}(1-S^*)$ from Lemma 2.7 and is nonzero but not strictly positive. This is contradictory to the first part of (iii) since S^* also satisfies (E). □

Proposition 3.4. (i) *Suppose that S satisfies* (E) *and both* $\mathrm{Ker}(1-S)$ *and* $\mathrm{Ker}(1-S^*)$ *have nonzero and nonnegative elements. Then $\underline{\lim}_{n\to\infty} \chi_{S^n}(\varepsilon) > 0$ for every $\varepsilon > 0$.*

(ii) *Suppose that $\{P_t\}$ satisfies* (E) *and both* $\mathrm{Ker}\,A$ *and* $\mathrm{Ker}\,A^*$ *have nonzero and nonnegative elements. Then $\underline{\lim}_{t\to\infty} \chi_{P_t}(\varepsilon) > 0$ and $\chi_{R_\alpha}(\varepsilon) > 0$ for every $\varepsilon > 0$ and α.*

Proof. (i): According to Proposition 3.3, both $\mathrm{Ker}(1-S)$ and $\mathrm{Ker}(1-S^*)$ have strictly positive elements, say, φ and ρ, respectively. If $0 \leq f \leq \varphi$ m-a.e. and $\int_X f\rho\,dm \geq \varepsilon$, then for $n \in \mathbb{N}$, $0 \leq S^n f \leq S^n\varphi = \varphi$ m-a.e. and $\int_X S^n f \cdot \rho\,dm = \int_X f\rho\,dm \geq \varepsilon$. Therefore for fixed $\varepsilon > 0$, $\chi_{S^n,\varphi,\rho,1,1}(\varepsilon)$ is nondecreasing with respect to n. So the assertion follows from Lemma 3.2.

(ii): The first part is similarly proved to (i). The second assertion follows from the first part and the expression of R_α by the Laplace transform of P_t. $\square$

Proposition 3.5. *Suppose that* $\operatorname{Ker}(1-S)$ *and* $\operatorname{Ker}(1-S^*)$ *have nonzero and non-negative elements,* φ *and* ρ*, respectively. Then* $\varphi > 0$, $\rho > 0$ m-a.e. *and*

$$\sup_{|f|\le\varphi\ m\text{-a.e.}} \left\| S^n f - \frac{\int_X f\rho\,dm}{\int_X \varphi\rho\,dm}\varphi \right\|_p \to 0 \quad \text{as } n\to\infty, \tag{3.1}$$

if and only if S *satisfies* (E).

Proof. First, we prove the if part. From Proposition 3.3, the strict positivity of φ and ρ holds. By considering S_φ instead of S and normalization, we may assume that $\varphi\equiv 1$ and $\int_X \rho\,dm = 1$ to prove (3.1). Before proving (3.1), we shall prove that

$$\sup_{\|f\|_\infty\le 1} \int_X \left| S^n f - \int_X f\rho\,dm \right| \rho\,dm \to 0 \quad \text{as } n\to\infty. \tag{3.2}$$

Take $\varepsilon\in(0,1/2)$. Since $\rho>0$ m-a.e., $\delta := \inf\{\int_B \rho\,dm \mid B\subset X,\ m(B)\ge 1-2\varepsilon\} > 0$. Since S satisfies (E), for some $n_0\in\mathbb{N}$, $c := \chi_{S^{n_0}}(\varepsilon) > 0$. Let $\varepsilon' = 2\varepsilon^{1/p}\|\rho\|_q$. For $f\in L^\infty$ with $\|f\|_\infty\le 1$, let $g = f - \int_X f\rho\,dm$. It holds that $\|g\|_\infty\le 2$ and $\int_X g\rho\,dm = 0$. Define $s = \int_X |g|\rho\,dm$. Then $s\le 2$, $\int_X g_+\rho\,dm = \int_X g_-\rho\,dm = s/2$ and

$$\int_X |S^n g|\rho\,dm \le \int_X S|S^{n-1}g|\cdot\rho\,dm = \int_X |S^{n-1}g|\rho\,dm \le\cdots\le s \tag{3.3}$$

for every $n\in\mathbb{N}$. In particular, if $s\le 2\varepsilon'$, then $\int_X |S^{n_0}g|\rho\,dm \le 2\varepsilon'$. Assume that $s>2\varepsilon'$. We set $B_0 = \{g_+ > s/2-\varepsilon'\}$. Then $s/2 = \int_X g_+\rho\,dm \le 2\int_{B_0}\rho\,dm + (s/2-\varepsilon')\int_{X\setminus B_0}\rho\,dm \le 2m(B_0)^{1/p}\|\rho\|_q + s/2-\varepsilon'$, therefore $m(B_0)\ge\varepsilon$. Let $B_1 = \{S^{n_0}1_{B_0} < c\}$. Since $\int_X S^{n_0}1_{B_0}\cdot 1_{B_1}\,dm < c$, $m(B_1)$ is less than ε from the definition of $\chi_{S^{n_0}}$. Hence

$$m(\{S^{n_0}g_+ \ge (s/2-\varepsilon')c\}) \ge m(\{S^{n_0}1_{B_0}\ge c\}) \ge 1-\varepsilon.$$

In the same way, $m(\{S^{n_0}g_- \ge (s/2-\varepsilon')c\}) \ge 1-\varepsilon$. Therefore, $m(B)\ge 1-2\varepsilon$ for $B = \{S^{n_0}g_+\wedge S^{n_0}g_- \ge (s/2-\varepsilon')c\}$, and

$$\begin{aligned}
\int_X |S^{n_0}g|\rho\,dm &\le \int_X \left\{\left|S^{n_0}g_+ - \left(\frac{s}{2}-\varepsilon'\right)c1_B\right| + \left|S^{n_0}g_- - \left(\frac{s}{2}-\varepsilon'\right)c1_B\right|\right\}\rho\,dm\\
&= \int_X \left\{S^{n_0}|g| - 2\left(\frac{s}{2}-\varepsilon'\right)c1_B\right\}\rho\,dm\\
&\le s - 2\left(\frac{s}{2}-\varepsilon'\right)c\delta = (1-c\delta)(s-2\varepsilon') + 2\varepsilon'.
\end{aligned}$$

Thus, in either case, $\int_X |S^{n_0}g|\rho\,dm \le \{(1-c\delta)(s-2\varepsilon')+2\varepsilon'\}\vee 2\varepsilon'$. Since $\|S^{n_0}g\|_\infty\le 2$

and $\int_X S^{n_0} g \cdot \rho \, dm = 0$, we can repeat the argument above to obtain that

$$\int_X |S^{kn_0} g| \rho \, dm \leq \{(1 - c\delta)^k (s - 2\varepsilon') + 2\varepsilon'\} \vee 2\varepsilon', \quad k \in \mathbb{N}.$$

Together with (3.3), we conclude that

$$\overline{\lim_{n\to\infty}} \sup_{\|f\|_\infty \leq 1} \int_X \left| S^n f - \int_X f \rho \, dm \right| \rho \, dm \leq 2\varepsilon'.$$

Since ε' can be taken to be arbitrarily small, we obtain (3.2).

Then for $f \in L^\infty$ with $\|f\|_\infty \leq 1$,

$$\begin{aligned} \left\| S^n f - \int_X f \rho \, dm \right\|_p^p &= \int_X \left| S^n f - \int_X f \rho \, dm \right|^p \left(1_{\{\rho < \varepsilon\}} + 1_{\{\rho \geq \varepsilon\}} \right) dm \\ &\leq 2^p m(\{\rho < \varepsilon\}) + \int_X \left| S^n f - \int_X f \rho \, dm \right| 2^{p-1} \varepsilon^{-1} \rho \, dm, \quad \varepsilon > 0. \end{aligned}$$

Taking $\lim_{\varepsilon\to 0} \overline{\lim}_{n\to\infty} \sup_{\|f\|_\infty \leq 1}$ on both sides completes the proof of the if part.

Next, we prove the only if part. Again, we may assume that $\varphi \equiv 1$ and $\int_X \rho \, dm = 1$. Fix $\varepsilon > 0$. For some $a > 0$, $m(Y) < \varepsilon/2$ with $Y := \{\rho < a\}$. Let B be a subset of X with $m(B) \geq \varepsilon$. Then, $\int_X 1_B \rho \, dm \geq a m(B \backslash Y) \geq a\varepsilon/2$. Set $Z_{n,B} = \{S^n 1_B < a\varepsilon/4\}$. Then

$$\left\| \int_X 1_B \rho \, dm - S^n 1_B \right\|_p \geq \left\| 1_{Z_{n,B}} \left(\frac{a\varepsilon}{2} - \frac{a\varepsilon}{4} \right) \right\|_p = \left(\frac{a\varepsilon}{4} \right) m(Z_{n,B})^{1/p}.$$

Since (3.1) holds, $\sup_{m(B)\geq\varepsilon} m(Z_{n,B}) \leq \varepsilon/2$ when n is taken to be sufficiently large. Hence we conclude that $\chi_{S^n}(\varepsilon) \geq a\varepsilon/4 \cdot \varepsilon/2 > 0$. □

Now we give a characterization for the exponential decay.

Theorem 3.6. (i) *Suppose that there exists some $\varphi \in \mathrm{Ker}(1 - S)$ with $\varphi \not\equiv 0$, $\varphi \geq 0$ m-a.e. Then the following are equivalent.*

(a) *S^n satisfies* (I) *for some $n \in \mathbb{N}$ and S satisfies* (E).

(b) *$\varphi > 0$ m-a.e. and there exist some $M > 0$, $\delta > 0$ and $\rho \in L^q_{>0}$ such that*

$$\left\| S^n f - \left(\int_X f \rho \, dm \right) \varphi \right\|_p \leq M e^{-\delta n} \|f\|_p, \quad f \in L^p, \ n \in \mathbb{N}. \tag{3.4}$$

(ii) *Suppose that there exists some $\varphi \in \mathrm{Ker}\, A$ with $\varphi \not\equiv 0$, $\varphi \geq 0$ m-a.e. Then the following are equivalent.*

(a) *P_{t_0} satisfies* (I) *for some $t_0 > 0$ and $\{P_t\}$ satisfies* (E).

(b) *$\varphi > 0$ m-a.e. and there exist some $M > 0$, $\delta > 0$ and $\rho \in L^q_{>0}$ such that*

$$\left\| P_t f - \left(\int_X f\rho\, dm \right) \varphi \right\|_p \le M e^{-\delta t} \|f\|_p, \quad f \in L^p,\ t > 0.$$

In either case, ρ is uniquely determined and $\rho \in \operatorname{Ker}(1 - S^*)$ (resp. $\rho \in \operatorname{Ker} A^*$).

Proof. It suffices to prove (i). Let us assume (a). From Proposition 3.3, $\varphi > 0$ m-a.e. From Theorem 2.8 and Proposition 3.3 again, we can take $\rho \in \operatorname{Ker}(1 - S^*)$ with $\rho > 0$ m-a.e. In order to prove (3.4), we may assume that $\varphi \equiv 1$, $\int_X \rho\, dm = 1$, and S itself satisfies (I), and it suffices to prove that

$$\lim_{n\to\infty} \sup_{\|f\|_p \le 1} \left\| S^n f - \int_X f\rho\, dm \right\|_p = 0.$$

Let $M = \sup_n \|S^n\|$, which is finite from Proposition 2.5. Take K' as in the same proposition. Let $f \in L^p$ with $\|f\|_p \le 1$. For $n_1 \in \mathbb{N}$, let $g_1 = (S^{n_1} f_+ \wedge K') - (S^{n_1} f_- \wedge K')$ and $g_2 = (S^{n_1} f_+ - K')_+ - (S^{n_1} f_- - K')_+$. Then $S^{n_1} f = g_1 + g_2$ and

$$\begin{aligned} \left\| S^{n_0+n_1} f - \int_X f\rho\, dm \right\|_p &\le \left\| S^{n_0} g_1 - \int_X g_1 \rho\, dm \right\|_p + \|S^{n_0} g_2\|_p + \left| \int_X g_2 \rho\, dm \right| \\ &\le \left\| S^{n_0} g_1 - \int_X g_1 \rho\, dm \right\|_p + (M + \|\rho\|_q)\|g_2\|_p. \end{aligned}$$

Keeping that $\|g_1\|_\infty \le K'$ in mind, we have

$$\varlimsup_{n_0, n_1 \to \infty} \sup_{\|f\|_p \le 1} \left\| S^{n_0+n_1} f - \int_X f\rho\, dm \right\|_p = 0$$

from Proposition 3.5 and Proposition 2.5.

Next let us assume (b). The condition (E) for S is proved in the same way as in Proposition 3.5. For $f \in L^p_+$ with $\|f\|_p \le 1$,

$$\begin{aligned} \|(S^n f - K)_+\|_p &\le \left\| S^n f - \left(\int_X f\rho\, dm \right) \varphi \right\|_p + \left\| \left\{ \left(\int_X f\rho\, dm \right) \varphi - K \right\}_+ \right\|_p \\ &\le M e^{-\delta n} + \|(\|\rho\|_q \varphi - K)_+\|_p. \end{aligned}$$

Therefore $\psi_{S^n}(K) < 1$ when n and K are large enough; namely S^n satisfies (I). □

We close this section by remarking the symmetric case. Suppose that $\{P_t\}$ is a symmetric Markovian semigroup on L^2 which has an associated Dirichlet form $\mathcal{E}$ with $1 \in \operatorname{Dom}(\mathcal{E})$ and $\mathcal{E}(1,1) = 0$. Then from Proposition 3.5, [16, Proposition 2], and [13, Lemma 6.13], the following statements are mutually equivalent. (See also [1].)

(i) If $\{f_n\}_{n=1}^\infty \subset \operatorname{Dom}(\mathcal{E})$ satisfies that $\int_X f_n\, dm = 0$, $\|f_n\|_2 \le 1$ and $\mathcal{E}(f_n, f_n) \to 0$, then $f_n \to 0$ in probability.

(ii) $\sup_{\|f\|_2 \le 1} \left\| P_t f - \int_X f\,dm \right\|_1 \to 0$ as $t \to \infty$.
(iii) $\{P_t\}$ satisfies (E).

In [1, 17], they prove a spectral gap of the generator (that is, exponential decay of $\{P_t\}$ in the L^2 sense) under (i) above and the hypothesis in Lemma 2.2 (ii) with $p = 2$ in our language. From Proposition 3.4, Theorem 3.6 and the spectral decomposition theorem, the latter can be weakened to the condition that some $(\alpha R_\alpha)^n$ satisfies (I), which is also a necessary condition.

4. Criteria for (E)

In this section, we investigate sufficient conditions for (E). First, we consider the case that we are given a bilinear form $\mathcal{E}$ on L^2 with a kind of square field operators. To say more precisely, we assume the following.

(F1) $\mathcal{E}$ is a bilinear form with domain $\mathrm{Dom}(\mathcal{E})$ satisfying a sector condition in the wide sense: there exist some $\lambda \ge 0$ and $K \ge 1$ such that $\mathcal{E}_\lambda(f, f) \ge 0$ and $|\mathcal{E}(f, g)| \le K\mathcal{E}_{\lambda+1}(f, f)^{1/2}\mathcal{E}_{\lambda+1}(g, g)^{1/2}$ for $f, g \in \mathrm{Dom}(\mathcal{E})$. Here $\mathcal{E}_\alpha(f, g) = \mathcal{E}(f, g) + \alpha \int_X fg\,dm$. Also, $\mathrm{Dom}(\mathcal{E})$ is dense in L^2 and closed under the norm $\mathcal{E}_{\lambda+1}(\cdot, \cdot)^{1/2}$.

(F2) The associated semigroup is positivity preserving. One necessary and sufficient condition in terms of $\mathcal{E}$ is the following: for every $f \in \mathrm{Dom}(\mathcal{E})$, f_+ also belongs to $\mathrm{Dom}(\mathcal{E})$ and $\mathcal{E}_\lambda(f_+, f) \ge 0$.

(F3) There exists a (not necessarily symmetric) bilinear map $\Gamma : \mathrm{Dom}(\mathcal{E}) \times \mathrm{Dom}(\mathcal{E}) \to L^1$ such that $\mathcal{E}(f, g) = \int_X \Gamma(f, g)dm$, $f, g \in \mathrm{Dom}(\mathcal{E})$.

(F4) Γ has a derivation property with respect to the *first* component: for any $f_i \in \mathrm{Dom}(\mathcal{E})\,(i = 1, \dots, n)$ and any $\varphi \in C_b^\infty(\mathbb{R}^n)$, $\varphi(f_1, \dots, f_n)$ belongs to $\mathrm{Dom}(\mathcal{E})$ and

$$\Gamma(\varphi(f_1, \dots, f_n), g) = \sum_{i=1}^n \frac{\partial \varphi}{\partial x_i}(f_1, \dots, f_n)\Gamma(f_i, g), \tag{4.1}$$

for every $g \in \mathrm{Dom}(\mathcal{E})$.

Let $\{P_t\}$, $\{R_\alpha\}$ and A denote the semigroup, the resolvent, and the generator on L^2 associated with $\mathcal{E}$, respectively. From (F4), $1 \in \mathrm{Dom}(\mathcal{E})$ and $\Gamma(1, g) = 0$ for every $g \in \mathrm{Dom}(\mathcal{E})$. Therefore, $A1 = 0$ and $\{P_t\}$ is conservative and Markovian. Let $U := \{p > 1 \mid \{P_t\}$ is extended (or restricted) to a strongly continuous semigroup on $L^p\}$. By the Riesz-Thorin interpolation theorem, U is an interval including $[2, \infty)$. In particular, $U = (1, \infty)$ if $\mathcal{E}$ is symmetric.

Now, to state a criterion for (E), we further suppose the following: there exist another bilinear map $\Gamma_0 : \mathrm{Dom}(\mathcal{E}) \times \mathrm{Dom}(\mathcal{E}) \to L^1$, functions $\eta \in \mathrm{Dom}(\mathcal{E})$, $\chi \in L^2$, $\sigma > 0$, $\xi \ge 0$ m-a.e., and constants $r > 0$, $\kappa > 0$ such that

(F5) Γ_0 has a derivation property just like (F4) with $n = 1$ with respect to *both* components, (F6) $\|\eta\|_2 = 1$, $\eta \in \bigcup_{s\in U} L^{2s}$, and $\eta^2\xi \in L^1$,

(F7) (Poincaré-type inequality) for every $f \in \mathrm{Dom}(\mathcal{E}) \cap L^\infty$,

$$\int_X \Gamma_0(f, f)\eta^2 dm \geq \left\{ \int_X \left| f - \int_X f\eta^2 dm \right|^r \sigma dm \right\}^{2/r},$$

(F8) for every $f \in \mathrm{Dom}(\mathcal{E})$, we have $\Gamma_0(f, f) + \xi f^2 \geq 0$ m-a.e.,

$$\Gamma(\eta, f) \leq \chi(\Gamma_0(f, f) + \xi f^2)^{1/2} \quad m\text{-a.e.},$$

and

$$\Gamma(f, f) \geq \kappa\Gamma_0(f, f) - \xi f^2 \quad m\text{-a.e.}$$

Typical examples satisfying from (F1) to (F8) are given in the section 5. We just note here that if $\mathcal{E}$ satisfies (F1)–(F4) and Γ in (F3) is symmetric, and $\mathcal{E}$ satisfies the Poincaré inequality: $\mathcal{E}(f, f) \geq c\|f - \int_X f\,dm\|_2^2$ for some $c > 0$, then (F5)–(F8) are clearly satisfied by setting $\Gamma_0 = \Gamma$, $\eta \equiv 1$, $\chi \equiv 0$, $\sigma \equiv c$, $\xi \equiv 0$, $r = 2$, and $\kappa = 1$.

Lemma 4.1. *Let $\varepsilon \in (0, 1]$ and $T > 0$. Then there exists some constant $C(\varepsilon, T) \geq 0$ such that*

$$\int_X (\log P_T^* f)_- \eta^2 dm \leq C(\varepsilon, T)$$

for any $f \in L^\infty$ with $0 \leq f \leq 1$ m-a.e. and $\|f\|_1 \geq \varepsilon$.

Proof. The proof is a modification of that of Lemma 3.3.2 in [7]. Take $q > 1$ such that $q/(q-1) \in U$ and $\eta \in L^{2q/(q-1)}$. $\{P_t^*\}$ is a positivity preserving, strongly continuous semigroup on L^q, as well as on L^2. So, for some α_0, we can define $\alpha R_\alpha^* f =: f_\alpha$ and $f_\alpha \in L^q \cap L_+^2$ for $\alpha \geq \alpha_0$. We consider only α larger than α_0. For $t \in (0, T]$, let $u_{t,\alpha} = P_t^* f_\alpha$. Henceforth we suppress α for notational convenience. Take $\delta \in (0, 1)$. In the sequel, C_i denotes a constant independent of f, α, t, and δ. It holds that $u_t \geq 0$ m-a.e., $\|u_t\|_1 = \|f\|_1 \geq \varepsilon$ and $\|u_t\|_2 \leq C_1 \exp(C_2 t)$ for some $C_1 \geq 1$ and $C_2 \geq 0$. Define $h_t = \log(u_t + \delta) + C_3 t - C_4$ and $G(t) = \int_X h_t \eta^2 dm$, $0 < t \leq T$, where C_3 and C_4 are chosen so that $C_3 \geq (1+\kappa)\|\xi\eta^2\|_1 + (2/\kappa)\|\chi\|_2^2$, $\{C_1 \exp(C_2 T) + 1\}^2 \exp(C_3 T - C_4) \leq \varepsilon/2$, and $\sup\{\int_X (\log(|g| + 1))\eta^2 dm + C_3 T - C_4 \mid \|g\|_2 \leq C_1 \exp(C_2 T)\} < 0$. Then, $G(T) < 0$ holds. We first prove that $G'(t) = \int_X \{A^* u_t/(u_t + \delta)\}\eta^2 dm + C_3$. Fix t and let $\Phi(a) = \log(a + \delta)$, $a \geq 0$. It is enough to prove that

$$\frac{1}{s}(\Phi(u_{t+s}) - \Phi(u_t)) \to \frac{A^* u_t}{u_t + \delta} \quad \text{in } L^q \text{ as } s \to 0.$$

By the mean value theorem, there exists a measurable function $v_s(x)$ such that

$$\Phi(u_{t+s}) - \Phi(u_t) = \Phi'(v_s)(u_{t+s} - u_t) \quad m\text{-a.e.},$$

and $v_s(x)$ lies in the interval between $u_t(x)$ and $u_{t+s}(x)$ for m-a.e.x. Recall that u_t belongs to the domain of A^* on L^q. Then

$$(u_{t+s} - u_t)/s \to A^* u_t \quad \text{in } L^q \text{ as } s \to 0.$$

Also, since $u_{t+s} \to u_t$ in L^q, every sequence $\{s_k\}$ converging to 0 has a subsequence $\{s_{k(i)}\}$ such that $v_{s_{k(i)}}(x) \to u_t(x)$ m-a.e. as $i \to \infty$. So we conclude that $(\Phi(u_{t+s}) - \Phi(u_t))/s \to \Phi'(u_t)A^*u_t = A^*u_t/(u_t + \delta)$ in L^q as $s \to 0$ from the dominated convergence theorem. Note that Φ' is a bounded function.

To finish the proof of the lemma, define $\Psi_n \in C_b^\infty(\mathbb{R})$, $n \in \mathbb{N}$ such that $\Psi_n(a) = a^2$ on $[-n, n]$, $\Psi_n(a) \nearrow a^2$ as $n \to \infty$, and $\sqrt{5\Psi_n} \geq \Psi_n' \geq 0$ everywhere. Then by (F3) and (F4),

$$\begin{aligned}
(4.2) \qquad \int_X \frac{A^*u_t}{u_t+\delta}\Psi_n(\eta)dm &= -\mathcal{E}\left(\frac{\Psi_n(\eta)}{u_t+\delta}, u_t\right) = -\int_X \Gamma\left(\frac{\Psi_n(\eta)}{u_t+\delta}, u_t\right) dm \\
&= -\int_X \left\{\frac{\Psi_n'(\eta)}{u_t+\delta}\Gamma(\eta, u_t) - \frac{\Psi_n(\eta)}{(u_t+\delta)^2}\Gamma(u_t, u_t)\right\} dm.
\end{aligned}$$

From the assumption (F8) and the inequality $xy \leq (x^2 + y^2)/2$, we have

$$\begin{aligned}
\frac{\Psi_n'(\eta)}{u_t+\delta}\Gamma(\eta, u_t) &\leq \frac{\Psi_n'(\eta)}{u_t+\delta}\chi(\Gamma_0(u_t, u_t) + \xi u_t^2)^{1/2} \\
&\leq \frac{2}{\kappa}\chi^2 + \frac{\kappa\Psi_n'(\eta)^2}{8(u_t+\delta)^2}(\Gamma_0(u_t, u_t) + \xi u_t^2) \\
&\leq \frac{2}{\kappa}\chi^2 + \frac{5\kappa\Psi_n(\eta)}{8(u_t+\delta)^2}(\Gamma_0(u_t, u_t) + \xi u_t^2)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\Psi_n(\eta)}{(u_t+\delta)^2}\Gamma(u_t, u_t) &\geq \frac{\Psi_n(\eta)}{(u_t+\delta)^2}(\kappa\Gamma_0(u_t, u_t) - \xi u_t^2) \\
&\geq \frac{\kappa\Psi_n(\eta)}{(u_t+\delta)^2}(\Gamma_0(u_t, u_t) + \xi u_t^2) - (1+\kappa)\xi\Psi_n(\eta).
\end{aligned}$$

Therefore, (4.2) exceeds

$$\int_X \left\{\frac{3\kappa\Psi_n(\eta)}{8(u_t+\delta)^2}(\Gamma_0(u_t, u_t) + \xi u_t^2) - (1+\kappa)\xi\Psi_n(\eta) - \frac{2}{\kappa}\chi^2\right\} dm.$$

By letting $n \to \infty$, we obtain that

$$\begin{aligned}
G'(t) &\geq \int_X \left\{\frac{3\kappa\eta^2}{8(u_t+\delta)^2}(\Gamma_0(u_t, u_t) + \xi u_t^2) - (1+\kappa)\xi\eta^2 - \frac{2}{\kappa}\chi^2\right\} dm + C_3 \\
&\geq \frac{3\kappa}{8}\int_X \frac{\eta^2}{(u_t+\delta)^2}(\Gamma_0(u_t, u_t) + \xi u_t^2)dm.
\end{aligned}$$

For $n \in \mathbb{N}$, let φ_n be a nonnegative function in $C_0^\infty(\mathbb{R})$ such that $\varphi_n(a) \le 1/(a+\delta)$ on $[0,\infty)$ and the equality holds on $[0,n]$. Define $\Phi_n(a) = \log\delta + \int_0^a \varphi_n(\tau)d\tau$. Then $\Phi_n \in C_b^\infty(\mathbb{R})$ and

$$\begin{aligned} G'(t) &\ge \frac{3\kappa}{8}\int_X \eta^2\varphi_n(u_t)^2(\Gamma_0(u_t,u_t)+\xi u_t^2)dm \\ &\ge \frac{3\kappa}{8}\int_X \Gamma_0(\Phi_n(u_t),\Phi_n(u_t))\eta^2 dm \\ &\ge \frac{3\kappa}{8}\left\{\int_X \left|\Phi_n(u_t) - \int_X \Phi_n(u_t)\eta^2\,dm\right|^r (\sigma\wedge 1)dm\right\}^{2/r}, \end{aligned}$$

by the assumptions (F5) and (F7). Letting $n\to\infty$, we have

$$\begin{aligned} G'(t) &\ge \frac{3\kappa}{8}\left\{\int_X \left|\log(u_t+\delta) - \int_X \log(u_t+\delta)\eta^2 dm\right|^r (\sigma\wedge 1)dm\right\}^{2/r} \\ &= \frac{3\kappa}{8}\left\{\int_X |h_t - G(t)|^r(\sigma\wedge 1)dm\right\}^{2/r}. \end{aligned}$$

Now, if $x\in X$ satisfies that $r + G(t) \le h_t(x) \le 0$, then

$$1 - \frac{h_t(x)}{G(t)} \ge 1 - \frac{h_t(x)}{h_t(x)-r} = \frac{1}{1-h_t(x)/r} \ge \exp\left(\frac{h_t(x)}{r}\right),$$

hence $h_t(x) - G(t) \ge -\exp(h_t(x)/r)G(t)$. Take a constant $\gamma\in(0,1]$ and a measurable subset Y of X such that $m(X\backslash Y) \le \varepsilon e^{-C_4}/4$ and $\sigma \ge \gamma$ on Y. Then

$$\begin{aligned} G'(t) &\ge \frac{3\kappa}{8}\left\{\int_{\{r+G(t)\le h_t\le 0\}\cap Y} e^{h_t}|G(t)|^r\gamma dm\right\}^{2/r} \\ &= \frac{3\kappa\gamma^{2/r}}{8}\left\{\int_{\{r+G(t)\le h_t\le 0\}\cap Y} e^{h_t}dm\right\}^{2/r} G(t)^2. \end{aligned} \tag{4.3}$$

We evaluate the integral above, which is denoted by $I(t)$. We have

$$\begin{aligned} I(t) &\ge \int_{\{r+G(t)\le h_t\le 0\}} e^{h_t}dm - m(X\backslash Y) \\ &\ge \int_X e^{h_t}dm - \int_{\{h_t>0\}} e^{h_t}dm - \int_{\{h_t<r+G(t)\}} e^{h_t}dm - \frac{\varepsilon e^{-C_4}}{4}, \\ \int_X e^{h_t}dm &= \int_X (u_t+\delta)e^{C_3t-C_4}dm \ge \varepsilon e^{C_3t-C_4}, \\ \int_{\{h_t>0\}} e^{h_t}dm &= \int_{\{(u_t+\delta)e^{C_3t-C_4}>1\}} (u_t+\delta)e^{C_3t-C_4}dm \\ &\le \|u_t+\delta\|_2^2 e^{2C_3t-2C_4} \le (C_1e^{C_2t}+1)^2e^{2C_3t-2C_4} \end{aligned}$$

$$\leq \frac{\varepsilon e^{C_3 t - C_4}}{2},$$

and

$$\int_{\{h_t < r+G(t)\}} e^{h_t} dm \leq e^{r+G(t)}.$$

Therefore

$$I(t) \geq \varepsilon e^{C_3 t - C_4} - \frac{\varepsilon e^{C_3 t - C_4}}{2} - e^{r+G(t)} - \frac{\varepsilon e^{-C_4}}{4} \geq \frac{\varepsilon e^{-C_4}}{4} - e^{r+G(t)}.$$

If $e^{r+G(T)} \geq \varepsilon e^{-C_4}/8$, then $G(T) \geq \log(\varepsilon/8) - C_4 - r$. Otherwise, $I(t) \geq \varepsilon e^{-C_4}/8$ for every $t \in (0, T]$ since $G(t)$ is nondecreasing by (4.3). Then

$$G'(t) \geq \frac{3\kappa\gamma^{2/r}}{8} \left(\frac{\varepsilon e^{-C_4}}{8}\right)^{2/r} G(t)^2 =: C_5 G(t)^2, \quad 0 < t \leq T.$$

This yields that $(d/dt)\{-G(t)^{-1}\} \geq C_5$ for $0 < t \leq T$. Combining with the fact that $G(t) < 0$, we obtain that $-G(T)^{-1} \geq -G(T)^{-1} + G(T/2)^{-1} = \int_{T/2}^{T} (d/dt)\{-G(t)^{-1}\} dt \geq C_5 T/2$, so $G(T) \geq -2/(C_5 T)$. Hence, in either case, $G(T) \geq -C_6$ for some $C_6 \geq 0$. Since $\int_X (\log(u_T + \delta))_+ \eta^2 dm \leq \int_X (\log(u_T + 1))\eta^2 dm \leq C_7$ for some $C_7 \geq 0$, we get that

$$\int_X (\log(u_{T,\alpha} + \delta))_- \eta^2 dm \leq C_6 + C_3 T - C_4 + C_7.$$

By letting $\alpha \uparrow \infty$, then $\delta \downarrow 0$, the proof is completed. □

REMARK 4.2. As is seen from the proof, we use (F4) only for $n = 2$. When $\eta \equiv 1$, we need (F4) with only $n = 1$.

Proposition 4.3. *Suppose moreover that $\mathcal{E}$ is symmetric or $\eta > 0$ m-a.e. Then $\chi_{P_t}(\varepsilon) > 0$ for all $\varepsilon > 0$ and $t > 0$. In particular, $\{P_t\}$ satisfies* (E).

Proof. Fix $\varepsilon > 0$ and $t > 0$. Let $f, g \in L^\infty$ with $0 \leq f \leq 1$, $0 \leq g \leq 1$ m-a.e. and $\|f\|_1 \geq \varepsilon$, $\|g\|_1 \geq \varepsilon$. Define $\hat{\eta} = \eta \wedge 1$.

Suppose that $\mathcal{E}$ is symmetric. Let $K = \int_X \hat{\eta}^2 dm$. Then

$$\int_X P_t f \cdot g \, dm = \int_X P_{t/2} f \cdot P_{t/2} g \, dm \geq K \int_X P_{t/2} f \cdot P_{t/2} g \cdot K^{-1} \hat{\eta}^2 \, dm.$$

Since $K^{-1}\hat{\eta}^2 \, dm$ is a probability measure, Jensen's inequality yields that

$$\log \int_X P_t f \cdot g \, dm \geq \log K + \int_X \log(P_{t/2} f) K^{-1} \hat{\eta}^2 \, dm + \int_X \log(P_{t/2} g) K^{-1} \hat{\eta}^2 \, dm$$

$$\geq \log K - \frac{2C(\varepsilon, t/2)}{K}.$$

Hence

$$\int_X P_t f \cdot g\, dm \geq K \exp\left(-\frac{2C(\varepsilon, t/2)}{K}\right).$$

Next, suppose that $\eta > 0$ m-a.e. There exists $a > 0$ such that the measure of $Z := \{\hat{\eta} < a\}$ is less than $\varepsilon/2$. Let $K_f = \int_X f\hat{\eta}^2\, dm$. Then $K_f \geq \int_{X\setminus Z} f a^2\, dm \geq a^2\varepsilon/2 =: \varepsilon'$. Since $\int_X P_t f \cdot g\, dm \geq K_f \int_X P_t^* g \cdot K_f^{-1} f\hat{\eta}^2\, dm$, we have

$$\begin{aligned}
\log \int_X P_t f \cdot g\, dm &\geq \log K_f + \int_X \log(P_t^* g) K_f^{-1} f\hat{\eta}^2\, dm \\
&\geq \log \varepsilon' - \int_X (\log(P_t^* g))_- \varepsilon'^{-1} \eta^2\, dm \\
&\geq \log \varepsilon' - \frac{C(\varepsilon, t)}{\varepsilon'}.
\end{aligned}$$

Hence

$$\int_X P_t f \cdot g\, dm \geq \varepsilon' \exp\left(-\frac{C(\varepsilon, t)}{\varepsilon'}\right). \qquad \square$$

REMARK 4.4. We also conclude that for every $t > 0$, P_t has the uniform positivity improving property, defined in [1].

Next, we state another criterion of (E) for semigroups obtained by the Girsanov transform, which has been already noted essentially in [1, 13].

Proposition 4.5. *Suppose that $\{P_t\}$ is a sub-Markovian semigroup on L^∞ and has an expression*

$$P_t f(x) = \int_\Omega f(X_t)\, dP_x,$$

where $(\Omega, X_t, P_x; x \in X)$ be a Markov process on X. Let another Markovian semigroup $\{Q_t\}$ be obtained by the Girsanov transform: for some Girsanov density Z_t, $\{Q_t\}$ is expressed as $Q_t f(x) = \int_\Omega f(X_t) Z_t\, dP_x$. If $\chi_{P_t}(\varepsilon) > 0$ and $Z_t > 0$ P_m-a.e., then $\chi_{Q_t}(\varepsilon) > 0$. Here $P_m(\cdot) = \int_X P_x(\cdot) dm$.

Proof. There exists $a > 0$ such that $P_m(Z_t < a) \leq \chi_{P_t}(\varepsilon)/2$. Let B_1 and B_2 be measurable subsets of X with $m(B_1) \geq \varepsilon$ and $m(B_2) \geq \varepsilon$. Then

$$\int_X Q_t 1_{B_1} \cdot 1_{B_2}\, dm = \int_\Omega 1_{B_1}(X_t) 1_{B_2}(X_0) Z_t\, dP_m$$

$$\geq a\left(\int_\Omega 1_{B_1}(X_t)1_{B_2}(X_0)\,dP_m - \frac{\chi_{P_t}(\varepsilon)}{2}\right)$$
$$\geq \frac{a\chi_{P_t}(\varepsilon)}{2}. \qquad \square$$

5. Examples

Let X be a separable Banach space and H a separable Hilbert space which is densely and continuously embedded in X. We take the Borel σ-field $\mathcal{F}$ and assume that the probability measure m on X has a full support for simplicity. Let $\mathcal{F}C_b^\infty$ be the space of smooth cylindrical functions on X;

$$\mathcal{F}C_b^\infty = \{f = F({}_X(\cdot, e_1)_{X^*}, \ldots, {}_X(\cdot, e_n)_{X^*}) \mid F \in C_b^\infty(\mathbb{R}^n),\ e_1, \ldots, e_n \in X^*\}.$$

The H-valued gradient operator ∇ on $\mathcal{F}C_b^\infty$ is defined by

$$\nabla f(h) = \sum_{i=1}^n \frac{\partial F}{\partial x_i}({}_X(\cdot, e_1)_{X^*}, \ldots, {}_X(\cdot, e_n)_{X^*})\,{}_X(h, e_i)_{X^*}, \quad h \in H \hookrightarrow X.$$

We assume that $(\nabla, \mathcal{F}C_b^\infty)$ is closable as an operator from $L^2(X)$ to $L^2(X \to H)$, and denote the closure by the same symbol. Let a be a measurable function on X taking values in the space of symmetric bounded operators on H such that for some $\varepsilon > 0$, $a(x) \geq \varepsilon\,\mathrm{Id}$ m-a.e.x, and $\|a\|_{\mathrm{op}} \in L^1$. Then the bilinear form $(\mathcal{E}^0, \mathcal{F}C_b^\infty)$ defined by

$$\mathcal{E}^0(f, g) = \int_X (a\nabla f, \nabla g)_H\,dm, \quad f, g \in \mathcal{F}C_b^\infty$$

is proved to be closable. Let the closure be denoted by $(\mathcal{E}^0, \mathrm{Dom}(\mathcal{E}^0))$. We also assume the following logarithmic Sobolev inequality: for some $\alpha > 0$,

$$\int_X f^2 \log\left(\frac{f^2}{\|f\|_2^2}\right) dm \leq \alpha\mathcal{E}^0(f, f), \quad f \in \mathrm{Dom}(\mathcal{E}^0).$$

Recall that this implies the Poincaré inequality:

$$\frac{2}{\alpha}\left\|f - \int_X f\,dm\right\|_2^2 \leq \mathcal{E}^0(f, f), \quad f \in \mathrm{Dom}(\mathcal{E}^0)$$

and the associated semigroup decays exponentially in the L^2 sense.

Let b be an H-valued measurable function. This is considered as a drift coefficient.

EXAMPLE 5.1 (cf. [5]). Suppose that $|a^{-1/2}b|_H \in L^2$ and there exist some $c \in [0, 1)$ and $k_1, k_2 \geq 0$ such that

$$\int_X (b, \nabla f)_H g\, dm \le k_1 \left(\mathcal{E}^0(f, f) + \|f\|_2^2\right)^{1/2} \left(\mathcal{E}^0(g, g) + \|g\|_2^2\right)^{1/2}, \quad f, g \in \mathrm{Dom}(\mathcal{E}^0),$$

and

$$\int_X (b, \nabla f)_H f\, dm \le c\mathcal{E}(f, f) + k_2 \|f\|_2^2, \quad f \in \mathrm{Dom}(\mathcal{E}^0).$$

For example, this is satisfied when $\exp(\theta |a^{-1/2}b|_H^2) \in L^1$ for some $\theta > \alpha$ (cf. [10, Proposition 3.4]).

Then a bilinear form $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$ defined by

$$\mathcal{E}(f, g) = \int_X \{(a\nabla f, \nabla g)_H - (b, \nabla f)_H g\}\, dm, \quad f, g \in \mathrm{Dom}(\mathcal{E}) = \mathrm{Dom}(\mathcal{E}^0)$$

satisfies (F1)–(F8) in the section 4 with $\Gamma(f, g) = (a\nabla f, \nabla g)_H - (b, \nabla f)_H g$, $\Gamma_0(f, g) = (a\nabla f, \nabla g)_H$, $\eta \equiv 1$, $\chi \equiv 0$, $\sigma \equiv 2/\alpha$, $\xi = |a^{-1/2}b|_H^2/2$, $r = 2$, and $\kappa = 1/2$. From Proposition 4.3, the associated semigroup $\{P_t\}$ satisfies (E). Moreover, since the inequality

$$\int_X f^2 \log\left(\frac{f^2}{\|f\|_2^2}\right) dm \le \frac{\alpha}{1-c}\{\mathcal{E}(f, f) + k_2\|f\|_2^2\}, \quad f \in \mathrm{Dom}(\mathcal{E})$$

holds, the resolvent operators are bounded ones from L^2 to $L^2 \log L$. Since the semigroup is analytic (see [14, Corollary I.2.21]), we can apply Lemma 2.2 to conclude that P_t satisfies (I) for every $t > 0$. Hence, from Theorem 3.6, the following inequality holds: for some $M > 0$, $\delta > 0$, and $\rho \in L^2$,

$$\left\| P_t f - \int_X f\rho\, dm \right\|_2 \le M e^{-\delta t} \|f\|_2, \quad f \in L^2, \ t > 0,$$

and $\rho\, dm$ is an invariant probability measure for $\{P_t\}$.

EXAMPLE 5.2. Suppose that $(\mathcal{E}^0, \mathrm{Dom}(\mathcal{E}^0))$ is quasi-regular and $\exp(\theta |a^{-1/2}b|_H^2) \in L^1$ for some $\theta > \alpha/4$. Fix $p > 1$ such that $\theta > \alpha/4 \cdot p^2/(p-1)^2$. In the same way as in [10, sections 2, 3], we can construct a conservative Markovian semigroup $\{P_t\}$ associated with a formal generator $-\nabla^* a \nabla + (b, \nabla\cdot)_H$ by using the Girsanov transformation. $\{P_t\}$ turns out to be strongly continuous on L^p and satisfies (I) for every $t > 0$ from a similar argument to that in [10, Proposition 3.1] and Lemma 2.2. Also, $\{P_t\}$ satisfies (E) from Theorem 3.6 and Proposition 4.5. So, from Theorem 3.6, there exist some $M > 0$, $\delta > 0$ and $\rho \in L^{p/(p-1)}$ such that

$$\left\| P_t f - \int_X f\rho\, dm \right\|_p \le M e^{-\delta t} \|f\|_p, \quad f \in L^p, \ t > 0,$$

and $\rho\, dm$ is an invariant probability measure for $\{P_t\}$.

Lastly, we remark some symmetric cases.

EXAMPLE 5.3. Let M be a compact, simply connected Riemannian manifold. Fix $x \in M$. The path space and the based loop space over M are defined by $P_x(M) = \{\gamma \in C([0,1] \to M) \mid \gamma(0) = x\}$ and $L_x(M) = \{\gamma \in C([0,1] \to M) \mid \gamma(0) = \gamma(1) = x\}$. We can define the Brownian motion measure μ_x on $P_x(M)$, the pinned Brownian motion measure ν_x on $L_x(M)$, the gradient operators $\nabla^{(P)}$ and $\nabla^{(L)}$ on each space, and the natural Dirichlet forms $\mathcal{E}^{(P)}$ and $\mathcal{E}^{(L)}$ by

$$\mathcal{E}^{(P)}(f,g) = \int_{P_x(M)} (\nabla^{(P)} f, \nabla^{(P)} g)\, d\mu_x, \quad \mathcal{E}^{(L)}(f,g) = \int_{L_x(M)} (\nabla^{(L)} f, \nabla^{(L)} g)\, d\nu_x.$$

See e.g. [1] for the detail. Let the associated semigroups and the resolvents be denoted by $P_t^{(P)}$, $R_\alpha^{(P)}$, etc. It is known that $\mathcal{E}^{(P)}$ satisfies the Poincaré inequality (and moreover, the logarithmic Sobolev inequality; see [8, 2, 11, 12]). So $\chi_{P_t^{(P)}}(\varepsilon) > 0$ holds for all $t > 0$ and $\varepsilon > 0$ from Proposition 3.4. Note that a slightly weaker assertion is mentioned in [1]. From Theorem 5.2 in [1], $\mathcal{E}^{(L)}$ satisfies (i) in the remark in the end of the section 3. Hence its generator has a spectral gap at 0 if and only if some $P_t^{(L)}$ (or $(\alpha R_\alpha^{(L)})^n$) satisfies (I). But whether this property holds or not is yet to be investigated.

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Department of Applied Analysis
and Complex Dynamical Systems
Graduate School of Informatics
Kyoto University
Kyoto, 606-8501, Japan
e-mail: hino@i.kyoto-u.ac.jp