

ON R -BORDISM OF MAPS AND OBSTRUCTION TO TOPOLOGICAL EMBEDDINGS

Dedicated to Professor Fuichi Uchida on his 60th birthday

CARLOS BIASI, JANEY DACCACH and OSAMU SAEKI

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1. Introduction

Let M and N be topological manifolds of dimensions m and n respectively and suppose $k = n - m > 0$ and that M is closed. For a continuous map $f : M \rightarrow N$, a homology class $\theta(f) \in H_{m-k}(M; \mathbf{Z}_2)$ has been defined in [3, Definition 2.5]. This is a *homotopy invariant* of f and has the property that, when M and N are smooth manifolds, if f is homotopic to a topological embedding,¹ then $\theta(f)$ vanishes (see [3]). Thus $\theta(f)$ can be regarded as a (primary) obstruction to topological embeddings.

In [2], *bordism invariance* of the obstruction $\theta(f)$ has been studied. Let f_1 and $f_2 : M \rightarrow N$ be two continuous maps between smooth manifolds, where their source manifolds coincide with each other. Suppose that $H^{m-k}(M; \mathbf{Z}_2)$ is generated by the cup products of the Stiefel-Whitney classes of M . Then it has been shown in [2] that if f_1 and f_2 are bordant [5], then $\theta(f_1) = \theta(f_2)$. This bordism invariance of the obstruction has played an important role in the study of topological embeddings of even dimensional projective spaces [2]. Note that the condition on $H^{m-k}(M; \mathbf{Z}_2)$ is essential in the above result as the examples in [2, §4] show.

In this paper, we first define the notion of R -bordism between two continuous maps. Note that the corresponding notion for manifolds has been defined and studied by Gibbs [7]. Suppose that $f_1 : M_1 \rightarrow N$ and $f_2 : M_2 \rightarrow N$ are continuous maps between topological manifolds, where M_1 and M_2 are closed and $\dim M_1 = \dim M_2 = m$. Note that their source manifolds may be distinct. Then f_1 and f_2 are R -bordant when there exists a bordism $F : W \rightarrow N$ between them such that the cobordism W between M_1 and M_2 admits retractions to the boundaries (for details, see §3). In such a case, as will be seen in §3, there exists a canonical isomorphism between the homologies of M_1 and M_2 . Our main result of this paper is that if f_1 and f_2 are R -bordant, then $\theta(f_1) \in H_{m-k}(M_1; \mathbf{Z}_2)$ and $\theta(f_2) \in H_{m-k}(M_2; \mathbf{Z}_2)$ correspond to each other under this canonical isomorphism (Corollary 4.3). In other words, the primary obstruction to topological embeddings is invariant under R -bordism. We will also show that

¹A continuous map is said to be a *topological embedding* if it is a homeomorphism onto its image. In particular, topological embeddings in this paper may not necessarily be locally flat.

$w_i(f_1)$ and $w_i(f_2)$ correspond to each other under the same isomorphism for all $i > k$, where $w_i(f_j)$ denotes the i -th Stiefel-Whitney class of the stable normal bundle of f_j ($j = 1, 2$). As an immediate corollary, we see that a continuous map $f : M \rightarrow N$ between smooth manifolds with $\theta(f) \neq 0$ (or $w_i(f) \neq 0$ for some $i > k$) cannot be R -bordant to a topological embedding.

The paper is organized as follows. In §2, we recall the definition of the obstruction $\theta(f)$. In §3 we define the notion of R -bordism between two continuous maps $f_1 : M_1 \rightarrow N$ and $f_2 : M_2 \rightarrow N$. Using the bordism theory due to [5], we give a homological characterization for two continuous maps to be R -bordant (Proposition 3.2). In §4, we prove our main theorem. We also give an example of a continuous map f such that $\theta(f) = 0$ and $w_i(f) = 0$ for all $i > k$, but that f is not R -bordant to a topological embedding. We also give various examples and discuss related topics.

Throughout the paper the homology and the cohomology groups have \mathbf{Z}_2 coefficients unless otherwise indicated. The symbol “ \cong ” denotes an appropriate isomorphism between algebraic objects.

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2. Primary obstruction to topological embeddings

Let us recall the definition of the primary obstruction for a continuous map to be homotopic to a topological embedding.

Let $f : M \rightarrow N$ be a continuous map of a closed m -dimensional topological manifold M into an n -dimensional topological manifold N , where we suppose $k = n - m > 0$. Let $[M] \in H_m(M)$ denote the fundamental class of M . Denote by $U_f \in H^k(N)$ the Poincaré dual (for example, see [16, Théorème 0.3]) of $f_*[M] \in H_m(N)$, where H_*^c denotes the (singular) homology of the compatible family with respect to compact subsets (see [14, Chapter 6, §3]), or equivalently the (singular) homology based on infinite chains (see [10, §5 and §65]), or equivalently the (singular) homology with closed support (see [16, p. 118]). In other words, $f_*[M] = U_f \frown [N]$, where $[N] \in H_n(N)$ is the fundamental class of N . Let the total Stiefel-Whitney classes of M and N be denoted by $w(M) \in H^*(M)$ and $w(N) \in H^*(N)$ respectively and let $\bar{w}(M) \in H^*(M)$ denote the total dual Stiefel-Whitney class of M ; i.e., $\bar{w}(M) = w(M)^{-1}$. Set $w(f) = f^*w(N) \smile \bar{w}(M)$, which is called the *total Stiefel-Whitney class of the stable normal bundle of f* . We denote by $w_k(f) \in H^k(M)$ the degree k term of $w(f)$, which is the k -th Stiefel-Whitney class of the stable normal bundle of f .

DEFINITION 2.1. We define

$$\theta(f) = (f^*U_f - w_k(f)) \frown [M] \in H_{m-k}(M).$$

Note that this is a homotopy invariant of f .

The above homology class has originally been defined in [3] and denoted by $\theta_1(f)$. In this paper, we use the notation $\theta(f)$ instead of $\theta_1(f)$, which will cause no confusion.

Many important results about f^*U_f and $w_k(f) \in H^k(M)$ have been given in [3, §2]. The most important property of $\theta(f)$ is that if $f : M \rightarrow N$ is a continuous map between smooth manifolds which is homotopic to a topological embedding, then $\theta(f)$ vanishes (see [3, Theorem 5.1]). For this reason, we call the homology class $\theta(f)$ the *primary obstruction to topological embeddings*.

The reason why we use the homology class instead of the corresponding cohomology class is that when M and N are smooth manifolds, $\theta(f)$ coincides with the fundamental class carried by the closure of the self-intersection set of a generic map homotopic to f (see [12], [3]).

3. R-bordism of continuous maps

In this section we define the notion of R -bordism between two continuous maps, which plays a central role in this paper.

Let $f_1 : M_1 \rightarrow N$ and $f_2 : M_2 \rightarrow N$ be continuous maps between topological manifolds, where $\dim M_1 = \dim M_2 = m$ and M_1 and M_2 are closed.

DEFINITION 3.1. Two continuous maps $f_1 : M_1 \rightarrow N$ and $f_2 : M_2 \rightarrow N$ as above are said to be R -bordant if there exist a compact $(m + 1)$ -dimensional topological manifold W with boundary and a continuous map $F : W \rightarrow N$ such that

- (1) W is a cobordism between M_1 and M_2 ; i.e., ∂W is identified with the disjoint union of M_1 and M_2 ,
- (2) there exist retractions $r_1 : W \rightarrow M_1$ and $r_2 : W \rightarrow M_2$, and
- (3) $F|_{M_1} = f_1$ and $F|_{M_2} = f_2$.

We call W an R -cobordism between M_1 and M_2 and $F : W \rightarrow N$ an R -bordism between f_1 and f_2 .

For example, if $M_1 = M_2$ and f_1 and f_2 are homotopic, then they are R -bordant.

Note that if f_1 and f_2 are R -bordant, then there exists a canonical isomorphism between the homologies $H_*(M_1)$ and $H_*(M_2)$ (see [7, Theorem 1.2]). In fact, $(r_2 \circ i_1)_* : H_*(M_1) \rightarrow H_*(M_2)$ and $(r_1 \circ i_2)_* : H_*(M_2) \rightarrow H_*(M_1)$ are isomorphisms and are inverse of each other, where $i_1 : M_1 \rightarrow W$ and $i_2 : M_2 \rightarrow W$ denote the inclusion maps. Furthermore, we have $f_{1*} = f_{2*} \circ (r_2 \circ i_1)_* : H_*(M_1) \rightarrow H_*(N)$ and $f_{2*} = f_{1*} \circ (r_1 \circ i_2)_* : H_*(M_2) \rightarrow H_*(N)$, since

$$\begin{aligned}
 (3.1) \quad f_{2*} \circ (r_2 \circ i_1)_* &= F_* \circ i_{2*} \circ r_{2*} \circ i_{1*} \\
 (3.2) \quad &= F_* \circ i_{1*} \circ r_{1*} \circ i_{2*} \circ r_{2*} \circ i_{1*}
 \end{aligned}$$

$$(3.3) \quad = F_* \circ i_{1*}$$

$$(3.4) \quad = f_{1*},$$

where the equation (3.2) follows from [7, Lemma 1.1]. Note that we also have the corresponding results for cohomologies.

When the manifolds M_1 and M_2 admit differentiable structures, we can characterize the above relation as follows.

Proposition 3.2. *Let M_1 and M_2 be closed smooth manifolds with the same dimension and N a topological manifold. Then two continuous maps $f_1 : M_1 \rightarrow N$ and $f_2 : M_2 \rightarrow N$ are R -bordant if and only if there exist two continuous maps $g_1 : M_1 \rightarrow M_2$ and $g_2 : M_2 \rightarrow M_1$ such that $g_{1*} : H_*(M_1) \rightarrow H_*(M_2)$ and $g_{2*} : H_*(M_2) \rightarrow H_*(M_1)$ are inverse isomorphisms and that $f_{2*} \circ g_{1*} = f_{1*} : H_*(M_1) \rightarrow H_*(N)$ and $f_{1*} \circ g_{2*} = f_{2*} : H_*(M_2) \rightarrow H_*(N)$.*

Proof. If f_1 and f_2 are R -bordant, then the continuous maps $g_1 = r_2 \circ i_1 : M_1 \rightarrow M_2$ and $g_2 = r_1 \circ i_2 : M_2 \rightarrow M_1$ satisfy the required conditions as has been seen above.

Suppose conversely the existence of g_1 and g_2 . Then by the proof of [7, Theorem 1.7], we see that $[M_1, \text{id} \times g_1] = [M_2, g_2 \times \text{id}] \in \mathcal{N}_m(M_1 \times M_2)$, where \mathcal{N}_m denotes the m -th bordism group with $m = \dim M_1 = \dim M_2$ (see [5]) and “id” denotes the identity map. Consider the images under the homomorphism

$$\mathcal{N}_m(M_1 \times M_2) \xrightarrow{(\text{id} \times (f_1 \circ g_1))_*} \mathcal{N}_m((M_1 \times M_2) \times N),$$

where $p_1 : M_1 \times M_2 \rightarrow M_1$ denotes the projection to the first factor. Then we have $[M_1, \text{id} \times g_1 \times f_1] = [M_2, g_2 \times \text{id} \times (f_1 \circ g_2)] \in \mathcal{N}_m(M_1 \times M_2 \times N)$. Since $(f_1 \circ g_2)_* = f_{2*}$ by our assumption, by an argument using the method of Stiefel-Whitney numbers (see [5, (17.2) Theorem]), we see that $[M_2, g_2 \times \text{id} \times (f_1 \circ g_2)] = [M_2, g_2 \times \text{id} \times f_2] \in \mathcal{N}_m(M_1 \times M_2 \times N)$. Hence we have $[M_1, \text{id} \times g_1 \times f_1] = [M_2, g_2 \times \text{id} \times f_2]$. Thus there exist a (smooth) cobordism W between M_1 and M_2 and a continuous map $\tilde{F} : W \rightarrow M_1 \times M_2 \times N$ such that $\tilde{F}|_{M_1} = \text{id} \times g_1 \times f_1$ and $\tilde{F}|_{M_2} = g_2 \times \text{id} \times f_2$. Then $r_1 = \pi_1 \circ \tilde{F} : W \rightarrow M_1$ and $r_2 = \pi_2 \circ \tilde{F} : W \rightarrow M_2$ give retractions which restrict to g_2 and g_1 on M_2 and M_1 respectively, and $F = \pi_3 \circ \tilde{F} : W \rightarrow N$ gives a continuous map such that $F|_{M_1} = f_1$ and $F|_{M_2} = f_2$, where $\pi_1 : M_1 \times M_2 \times N \rightarrow M_1$, $\pi_2 : M_1 \times M_2 \times N \rightarrow M_2$ and $\pi_3 : M_1 \times M_2 \times N \rightarrow N$ denote the respective projections. Thus f_1 and f_2 are R -bordant. This completes the proof. \square

Corollary 3.3. *Let M be a closed smooth manifold and N a topological manifold. Then two continuous maps $f_1 : M \rightarrow N$ and $f_2 : M \rightarrow N$ are R -bordant if $f_{1*} = f_{2*} : H_*(M) \rightarrow H_*(N)$.*

4. R-bordism invariance of $\theta(f)$

We begin by the following lemma.

Lemma 4.1. *Let $f_1 : M_1 \rightarrow N$ and $f_2 : M_2 \rightarrow N$ be continuous maps of closed m -dimensional topological manifolds M_1 and M_2 into an $(m+k)$ -dimensional topological manifold N with $k > 0$. If there exists a continuous map $g : M_1 \rightarrow M_2$ such that $g_*[M_1] = [M_2]$ and $g^*w(M_2) = w(M_1)$, then we have $g_*\theta(f_2 \circ g) = \theta(f_2)$. Furthermore, if $f_{1*} = f_{2*} \circ g_*$, then we have $g_*\theta(f_1) = \theta(f_2)$.*

Proof. Let ξ be an arbitrary element of $H^{m-k}(M_2)$. Then we have

$$\begin{aligned}
 (4.1) \quad \langle \xi, g_*\theta(f_2 \circ g) \rangle &= \langle g^*\xi, ((f_2 \circ g)^*U_{f_2 \circ g} - w_k(f_2 \circ g)) \frown [M_1] \rangle \\
 (4.2) \quad &= \langle g^*\xi, (g^*f_2^*U_{f_2} - \bar{w}(M_1) \smile (f_2 \circ g)^*w(N)) \frown [M_1] \rangle \\
 (4.3) \quad &= \langle g^*\xi, g^*(f_2^*U_{f_2} - \bar{w}(M_2) \smile f_2^*w(N)) \frown [M_1] \rangle \\
 (4.4) \quad &= \langle g^*\xi \smile g^*(f_2^*U_{f_2} - w(f_2)), [M_1] \rangle \\
 (4.5) \quad &= \langle \xi \smile (f_2^*U_{f_2} - w(f_2)), [M_2] \rangle \\
 (4.6) \quad &= \langle \xi, (f_2^*U_{f_2} - w(f_2)) \frown [M_2] \rangle \\
 (4.7) \quad &= \langle \xi, \theta(f_2) \rangle,
 \end{aligned}$$

where the equations (4.2) and (4.5) follow from our assumption that $g_*[M_1] = [M_2]$ and the equation (4.3) from $g^*w(M_2) = w(M_1)$ (and hence $g^*\bar{w}(M_2) = \bar{w}(M_1)$). Thus the first half of the lemma has been proved by the universal coefficient theorem. Furthermore, if $f_{1*} = f_{2*} \circ g_*$, then we see easily that $\theta(f_2 \circ g) = \theta(f_1)$ by Definition 2.1 and the universal coefficient theorem. This completes the proof. \square

By the above lemma, we get the following.²

Theorem 4.2. *Let $f_1 : M_1 \rightarrow N$ and $f_2 : M_2 \rightarrow N$ be continuous maps of closed m -dimensional topological manifolds M_1 and M_2 into an $(m+k)$ -dimensional topological manifold N with $k > 0$. If there exists a continuous map $g : M_1 \rightarrow M_2$ such that $g_* : H_*(M_1) \rightarrow H_*(M_2)$ is an isomorphism and that $f_{2*} \circ g_* = f_{1*}$, then we have $g_*\theta(f_1) = \theta(f_2)$ and $g^*w(f_2) = w(f_1)$.*

Proof. By [7, Lemma 1.4], we have

$$(4.8) \quad g^*w(M_2) = w(M_1).$$

Thus by Lemma 4.1, we have $g_*\theta(f_1) = \theta(f_2)$. The second equality $g^*w(f_2) = w(f_1)$

²Theorem 4.2 has been suggested by the referee.

follows from the definition of $w(f_1)$ and $w(f_2)$, (4.8), and the universal coefficient theorem. This completes the proof. \square

As a corollary, we have the following main result of this paper.

Corollary 4.3. *Let $f_1 : M_1 \rightarrow N$ and $f_2 : M_2 \rightarrow N$ be continuous maps of closed m -dimensional topological manifolds M_1 and M_2 into an $(m+k)$ -dimensional topological manifold N with $k > 0$. If f_1 and f_2 are R -bordant, then $\theta(f_1) \in H_{m-k}(M_1)$ and $w(f_1) \in H^*(M_1)$ correspond to $\theta(f_2) \in H_{m-k}(M_2)$ and $w(f_2) \in H^*(M_2)$ respectively by the canonical isomorphisms.*

Combining Corollary 4.3 with [3, Theorem 5.1] and [8, Théorème 5.2], we have the following.

Corollary 4.4. *Let $f : M \rightarrow N$ be a continuous map of a closed m -dimensional topological manifold M into an $(m+k)$ -dimensional smooth manifold N with $k > 0$. If f is R -bordant to a topological embedding of a smooth manifold, then $\theta(f) = 0$ in $H_{m-k}(M)$ and $w_i(f) = 0$ in $H^i(M)$ for all $i > k$.*

REMARK 4.5. In [8], the target manifold N is always assumed to be closed. However, for the result that we need here, this condition is redundant, which can be proved as follows. Let $f : M \rightarrow N$ be as in the above theorem and we further assume that f is a topological embedding. Note that there exists a codimension 0 compact submanifold V (possibly with boundary) of N such that $f(M) \subset \text{Int } V$, since M is compact. This is a well-known fact if N is a smooth manifold. If N is only a topological manifold, then by [6, Chapter IV, §8], N can be embedded in some Euclidean space as a closed subset. By considering the Euclidean distance function from a point, we see that there exists a proper continuous map $g : N \rightarrow \mathbf{R}$. Since $f(M)$ is compact, there exist a and $b \in \mathbf{R}$ with $a < b$ such that $g(f(M)) \subset (a, b)$. Then by [9, §8, p. 320] and [11], we can homotope g a little bit so that g is topologically transverse to a and $b \in \mathbf{R}$. Then $V = g^{-1}([a, b])$ is the desired submanifold. Let us denote the map f by $f_1 : M \rightarrow V$ when we regard V as the target manifold. Furthermore, let us consider the double $D(V) = V \cup V$ of V , which is a closed manifold. We denote by $j_1 : V \rightarrow N$ and $j_2 : V \rightarrow D(V)$ the respective inclusion maps. Then we see easily that $w(f) = w(j_1 \circ f_1) = w(f_1) = w(j_2 \circ f_1) \in H^*(M)$. Since the target manifold $D(V)$ of $j_2 \circ f_1$ is closed, Haefliger's result implies that $w_i(j_2 \circ f_1)$ vanishes for all $i > k$. Thus we have $w_i(f) = 0$ for all $i > k$.

EXAMPLE 4.6. Consider a continuous map $f : M \rightarrow S^{n+1}$, where M is a closed 1-connected n -dimensional smooth manifold with $w_2(M) \neq 0$ (for example, $n = 4$ and $M = \mathbf{C}P^2$). Since $H_{n-1}(M) = 0$, $\theta(f)$ always vanishes. However, $w_2(f) = \bar{w}_2(M) =$

$w_2(M)$ does not vanish by our assumption. Hence, f is not R -bordant to a topological embedding.

Corollary 4.4 suggests the following problem.

Problem 4.7. Let $f : M \rightarrow N$ be a continuous map of a closed m -dimensional topological manifold M into an $(m + k)$ -dimensional topological manifold N with $k > 0$. If $\theta(f) = 0$ in $H_{m-k}(M)$ and $w_i(f) = 0$ in $H^i(M)$ for all $i > k$, then is f R -bordant to a topological embedding?

We have a positive answer in the following situation, for example.

EXAMPLE 4.8. Consider an arbitrary continuous map $f : S^1 \rightarrow N$, where N is an arbitrary 2-dimensional topological manifold. Then it is not difficult to show that $\theta(f) \in H_0(S^1)$ always vanishes (for example, we may assume that N is a smooth surface and can use the generic map argument as in [12], [3]). Furthermore, obviously we have $w_i(f) = 0$ for all $i > 1$. On the other hand, it is not difficult to show that every element of $H_1(N)$ can be represented by a topological embedding of S^1 into N . Thus, by Corollary 3.3, we see that every continuous map $f : S^1 \rightarrow N$ is R -bordant to a topological embedding. In fact, when N is smooth, every continuous map f is R -bordant to a smooth embedding.

The answer to Problem 4.7 is negative in general as the following example shows.

EXAMPLE 4.9. Let W' be the interior of the compact smooth 4-manifold W constructed by Silver [13, Theorem]. Recall that W is homotopy equivalent to $W_0 \sharp \mathbb{C}P^2$, where W_0 is a compact smooth 4-manifold with $\tilde{H}_*(W_0; \mathbf{Z}) = 0$, and that no nontrivial element of $H_2(W; \mathbf{Z}) \cong \mathbf{Z}$ can be represented by a topologically embedded 2-sphere. Let $f : S^2 \rightarrow W'$ be a continuous map which represents an odd multiple of a generator of $H_2(W'; \mathbf{Z}) \cong \mathbf{Z}$. Such a continuous map exists, since W is homotopy equivalent to $W_0 \sharp \mathbb{C}P^2$. Note that $\theta(f) = 0$ and that $w_i(f) = 0$ for all $i > 2$. Suppose that f is R -bordant to a topological embedding $g : F \rightarrow W'$, where F is a closed surface. Then by Proposition 3.2, we see that F is homeomorphic to S^2 and that $f_*[S^2] = g_*[F]$ in $H_2(W'; \mathbf{Z}_2) \cong \mathbf{Z}_2$. In particular, g represents a nontrivial element of $H_2(W'; \mathbf{Z})$. This contradicts the result of [13]. Hence f is not R -bordant to a topological embedding, although $\theta(f) = 0$ and $w_i(f) = 0$ for all $i > 2$.

Note that the above continuous map f is bordant (in the sense of Conner and Floyd [5]) to a topological embedding, which follows from results obtained in [5] and [17].

REMARK 4.10. Let us consider the following bordism relation introduced by Stong [15]. Two continuous maps $f_j : M_j \rightarrow N_j$ ($j = 1, 2$) between closed manifolds ($\dim M_j = m, \dim N_j = n$) are said to be *bordant in the sense of Stong* if there exists a continuous map $F : V \rightarrow W$ between compact manifolds ($\dim V = m+1, \dim W = n+1$) such that $\partial V = M_1 \cup M_2, \partial W = N_1 \cup N_2$ and $F|M_j = f_j$ ($j = 1, 2$). Brown [4] and Aguilar, Pastor [1] have considered the problem of finding an embedding in a given bordism class in the above sense. Using the results obtained in [1], we see the following: *a continuous map $f : M \rightarrow N$ with $\dim N = \dim M + k$ ($k = \dim M - 1, \dim M - 2$) is bordant in the sense of Stong to a smooth embedding if $\theta(f) = 0$ and $w_i(f) = 0$ for all $i > k$.*

Finally, we have a small remark about the “invariance” of the obstruction class. Recall that, when we say that the obstruction is “invariant”, the source manifolds of the relevant maps coincide with each other in [3], while they may be different in the present paper. In fact, the invariance in the sense of the present paper is weaker than that considered in [2] as is seen in the following example.

EXAMPLE 4.11. Consider two continuous maps f_1 and $f_2 : T^2 \rightarrow \mathbf{R}P^2 \times \mathbf{R}$ such that

- (1) $f_{1*}(a) = f_{2*}(b) = u$, and
- (2) $f_{1*}(b) = f_{2*}(a) = 0$,

where $T^2 = S^1 \times S^1$ is the 2-dimensional torus, $a = [S^1 \times \{*\}]$ and $b = [\{*\} \times S^1]$ are the generators of $H_1(T^2) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$, and u is the generator of $H_1(\mathbf{R}P^2 \times \mathbf{R}) \cong \mathbf{Z}_2$. Then we have that $\theta(f_1) = b$ and $\theta(f_2) = a$. Let $h : T^2 \rightarrow T^2$ be the involution given by $h(x, y) = (y, x)$ for $(x, y) \in S^1 \times S^1 = T^2$. Obviously we have that $f_{1*} \circ h_* = f_{2*}$. Hence the maps f_1 and f_2 are R -bordant by Proposition 3.2, and thus $\theta(f_1)$ corresponds to $\theta(f_2)$ by the isomorphism $h_* : H_1(T^2) \rightarrow H_1(T^2)$. This example shows the following: even in the case that the source manifolds are the same, the fact that f_1 and f_2 are R -bordant does not necessarily imply that the obstructions $\theta(f_1)$ and $\theta(f_2)$ coincide with each other, but implies that they only correspond to each other by the canonical isomorphism.

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C. Biasi

Departamento de Matemática, ICMC-USP
 Caixa Postal 668
 13560-970, São Carlos, SP
 Brazil
 E-mail: biasi@icmc.sc.usp.br

J. Daccach

Departamento de Matemática
 Universidade Estadual de Maringá
 Avenida Colombo 5790
 87020-900, Maringá, PR
 Brazil
 E-mail: janey@gauss.dma.uem.br

O. Saeki

Department of Mathematics
 Graduate School of Science
 Hiroshima University
 Higashi-Hiroshima 739-8526
 Japan
 E-mail: saeki@math.sci.hiroshima-u.ac.jp

