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CORRECTION TO "THE FIRST EIGENVALUE OF *P*-MANIFOLDS"

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In our paper, "The First Eigenvalue of P-manifolds" [2] we proved the following

Theorem 1. *Let (M, g) be a $P_{2\pi}$-manifold of dimension $n \geq 2$ with Ricci curvature $\mathrm{Ric}_M \geq l$ and $\lambda_1 = (1/3)(2l + n + 2)$. Then*

1. (a) *$\lambda_1 = k(m+1)/2 = \lambda_1(\overline{M})$ and $l = \mathrm{Ric}_{\overline{M}}$ where $\overline{M}$ is a simply connected compact rank-1 symmetric space(CROSS) of dimension $n = km$ with sectional curvature $1/4 \leq K_{\overline{M}} \leq 1$ and $k = 1, 2, 4, 8$ or n is the degree of the generator of $H^*(M, \mathbb{Q}) = H^*(\overline{M}, \mathbb{Q})$ and $H^*(\widetilde{M}, \mathbb{Z}_2) = H^*(\overline{M}, \mathbb{Z}_2)$ where $\widetilde{M}$ is the simply connected cover of M.*
   (b) *If $k \geq 4$ then M is simply connected and the integral cohomology ring of M is same as that of $\overline{M}$.*
   (c) *If $k = 2$ then either M is simply connected or M is non-orientable and it has a two sheeted simply connected cover $\widetilde{M}$. Moreover $H^*(\widetilde{M}, \mathbb{Z}) = H^*(\overline{M}, \mathbb{Z})$.*
2. *If $k = 1$ then $(\widetilde{M}, \widetilde{g})$ is isometric to S^n with constant sectional curvature $1/4$.*
3. *If $k = n$ then (M, g) is isometric to S^n with constant sectional curvature* 1. *(Lichnerowicz-Obata Theorem)*
4. *If $k = 2$, 4 or 8 and if there is a first eigenfunction without saddle points then the universal cover $(\widetilde{M}, \widetilde{g})$ of (M, g) is isometric to $\overline{M}$ of dimension km.*

The proof of Theorem 1(4) above depends on Lemma 4.1 of [2] where we have discovered an error in the proof of the fact $E_{(1-\cos t)/2}$ is parallel along γ. The notations are as in [2]. We showed there that any Jacobi field J along γ such that $J(0) \in TD_{\max}$ and $J(\pi) = 0$ with the normalization $\|J'(\pi)\| = 1$, is of the form $J(t) = 2\cos((t/2)E(t))$, where $E(t) \in E_{(1-\cos t)/2}$ is a unit parallel field along γ. To show this we claimed that $\langle J', J\rangle/\|J\| \geq -(1/2)(\sin(t/2)/\cos(t/2))$ on $(-\pi, \pi)$. However, we can only say that this inequality is valid on $(0, \pi)$ and not on $(-\pi, \pi)$.

In this note we prove Lemma 4.1 and hence Theorem 1(4) of [2].

Assume to begin with that $\mathrm{Vol}(M) = \mathrm{Vol}(\mathbb{P}^m(k))$. Any Jacobi field J along γ such that $J(0) \in TD_{\max}$ and $J(\pi) = 0$ with the normalization $\|J'(\pi)\| = 1$ satisfy the inequality $\|J(t)\| \leq 2\cos(t/2)$ along γ on $(0, \pi)$. Hence $\mathrm{Vol}(D_{\max}) \leq \mathrm{Vol}(\mathbb{P}^a(k))$. Further equality holds iff the fibration $\Pi_{\min} : S(0,1) \to D_{\max}$ defined by $\Pi_{\min}(u) := \exp(\pi u)$ of the unit normal sphere $S(0,1)$ of $D_{\min}$ at x over $D_{\min}$ is congruent to the Hopf fibration and the Jacobi field $J(t)$ is of the form $J(t) = 2\cos((t/2)E(t))$ along γ on $(0,\pi)$ with $E(t) \in E_{(1-\cos t)/2}$ a parallel unit vector field along γ.

Similarly by starting with Jacobi fields J along γ from $D_{\min}$, with the same initial conditions as the Jacobi fields described above, we see that $\mathrm{Vol}(D_{\min}) \leq \mathrm{Vol}(\mathbb{P}^b(k))$. Further equality holds iff the fibration $\Pi_{\max} : S(0,1) \to D_{\min}$ of the unit normal sphere at x over $D_{\max}$ is congruent to Hopf fibration and the Jacobi fields are of the form $J(t) = 2\sin((t/2)E(t))$ along γ where $E(t) \in E_{-(1+\cos t)/2}$ is a parallel vector field along γ. Therefore, if we show that either $\mathrm{Vol}(D_{\max}) = \mathrm{Vol}(\mathbb{P}^a(k))$ or $\mathrm{Vol}(D_{\min}) = \mathrm{Vol}(\mathbb{P}^b(k))$ we will be through.

First we show that the volume density relative to $D_{\max}$ (respectively relative to $D_{\min}$) is same as in $\mathbb{P}^m(k)$ relative to $\mathbb{P}^a(k)$ (respectively relative to $\mathbb{P}^b(k)$).

Since f is an eigenfunction of Δ, we have $(\Delta f)(x) = (k(m+1)/2)(\cos t + C)$ for $x \in S(t)$, the level set of radius t around $D_{\max}$. We write

$$\Delta = -\frac{\partial^2}{\partial t^2} - (km-1)H\frac{\partial}{\partial t} + \Delta_{S(t)}$$

where H is the mean curvature of the hypersurface $S(t)$ and $\Delta_{S(t)}$ is the Laplacian on $S(t)$ with respect to the induced metric from (M, g). Since the function f is constant on $S(t)$, $\Delta_{S(t)} f = 0$ on $S(t)$. Therefore

$$\begin{aligned}\frac{k(m+1)}{2}(\cos t + C) &= \Delta f \\ &= \cos t + (km-1)H\sin t\end{aligned}$$

Hence $(km-1)H\sin t = ((k(m+1)/2) - 1)\cos t + (k(m+1)/2)C$. The constant C is computed as follows: Along $D_{\max}$,

$$\begin{aligned}\frac{k(m+1)}{2}(1 + C) &= (\Delta f)(x) \\ &= -\mathrm{Tr}(\nabla^2 f(x)) \\ &= k(m-a)\end{aligned}$$

and therefore $(k(m+1)/2)C = k(b+1) - (k(a+b+2)/2) = (k(b-a)/2)$. We substitute this value of C in the equation above to get

$$H = \frac{1}{km-1}\left[-\frac{ka}{2}\tan\frac{t}{2} + \frac{kb}{2}\cot\frac{t}{2} + (k-1)\cot t\right]$$

Hence the volume density relative $D_{\max}$ is $\cos^{ka}(t/2)\sin^{kb}(t/2)\sin^{k-1} t$ which is same as the volume density of $\mathbb{P}^m(k)$ relative to $\mathbb{P}^a(k)$ ($\mathbb{P}^b(k)$). This proves that

$\text{Vol}(M, g)/\text{Vol}(\mathbb{P}^m(k)) = \text{Vol}(D_{\max})/\text{Vol}(\mathbb{P}^a(k)) = \text{Vol}(D_{\min})/\text{Vol}(\mathbb{P}^b(k))$. Since we assume that $\text{Vol}(M, g) = \text{Vol}(\mathbb{P}^m(k))$, this completes the proof of the Lemma 4.1 under the assumption on the volume of M. We now proceed to justify the assumption. We consider two cases:

(a) Either $D_{\max}$ or $D_{\min}$ is a point.

(b) None of them is a point.

Proof in case (a). Let, without loss of generality, $D_{\max} = \{p\}$ and J be a Jacobi field describing the variation of a geodesic γ starting at p with the initial conditions $J(0) = 0$, $J(\pi) \in TD_{\min}$. We normalize J so that $\|J'(0)\| = 1$. Then using the inequality $\langle \nabla^2 f(E), E\rangle \geq -((1 + \cos t)/2)\|E\|^2$ for every $E \in TM$, we get that $\|J(t)\| \leq 2\sin(t/2)$ along γ; in particular $\|J(\pi)\| \leq 2$. Hence $\text{Vol}(D_{\min}) \leq \text{Vol}(\mathbb{P}^{m-1}(k))$. More over equality holds iff the fibration $\Pi_{\max} : S(0, 1) \to D_{\min}$ of the unit sphere $S(0, 1)$ in T_pM is congruent to Hopf fibration and $J(t) = 2\sin((t/2)E(t))$ along γ where $E(t) \in E_{-(1+\cos t)/2}$ is a parallel unit vector field along γ.

We know that the relative density is same as in the standard $\mathbb{P}^m(k)$. From this it follows that $\text{Vol}(M) = \text{Vol}(\mathbb{P}^m(k))$ and hence $\text{Vol}(D_{\min}) = \text{Vol}(\mathbb{P}^{m-1}(k))$. Since $(1 - \cos t)/2$ is not an eigenvalue of $\nabla^2 f$, the proof of Lemma 4.1 and hence the proof of the theorem is complete. □

Proof in case (b). The first thing to note is that in this situation k cannot be equal to 8. It is either 2 or 4, i.e. our manifold is either a homology complex or a homology quaternionic projective space. The reason being that the dimension of M has to be $k(a + b + 1)$ and in case $k = 8$, it can only be 8 or 16 forcing at least one of a or b to vanish.

Let $S_p(0, 1)$ denote the set of unit normal vectors at p, whenever p is in either $D_{\max}$ or $D_{\min}$. For $u \in S_p(0, 1)$ and $v \perp u$, let J_v denote the Jacobi field along γ_u with $J_v(0) = 0$, $J'_v(0) = v$. □

Lemma 1. *Let $p \in D_{\max}$, $u \in S_p(0, 1)$ and further v be a unit vector at p tangential to $D_{\max}$. Let $q = \gamma_u(\pi)$ be the point in $D_{\min}$ where γ_u meets it and $J_v^N(\pi)$ be the component of $J_v(\pi)$ normal to $D_{\min}$ at q. Then*

(i) $\|J_v^N(\pi)\| = 2$.

(ii) *$J_v^N(\pi)$ is orthogonal to the fibres of $\Pi_{\min} : S_q(0, 1) \longrightarrow D_{\max}$*

Proof. Set $u_\theta = \cos\theta u + \sin\theta v$, $\gamma_\theta = \gamma_{u_\theta}$. Then $A_{u_\theta} = \cos^2\theta$ and consequently $f(\gamma_\theta(t)) = \cos^2\theta(\cos t - 1) + C + 1$. As $\gamma_u(\pi)$ is a critical point (an absolute minimum) of f, we conclude that

$$\langle(\nabla^2 f)(J_v(\pi)), J_v(\pi)\rangle = \left.\frac{\partial^2 f(\gamma_\theta(\pi))}{\partial\theta^2}\right|_{\theta=0} = 4$$

Since $\nabla^2 f$ has eigenvalues 0 and 1 along tangent and normal directions respectively, we get the proof of (i).

For the proof of (ii), let w be a vector tangent to the fibre contained in $S_p(0, 1)$ based at u. Then $J_w(\pi)$ will vanish at q and $J'_w(\pi)$ will be tangent to the fibre inside $S_q(0, 1)$. (J_w is tangent to the 'link sphere' between p and q.) Now let ω be the symplectic form on the space of normal Jacobi fields along γ_u. $\omega(J_v, J_w)$ vanishes as $J_v(0) = J_w(0) = 0$. On the other hand at $t = \pi$, $\omega(J_v, J_w) = \langle J_v(\pi), J'_w(\pi)\rangle$. This proves that the normal component of $J_v(\pi)$ is a 'horizontal' vector, i.e. orthogonal to the fibres of the fibration of $S_q(0, 1)$ over $D_{\max}$. □

Lemma 2. $\Pi_{\max} : S_p(0, 1) \longrightarrow D_{\min}$ *is a Riemannian submersion* (*up to a constant scale factor of* 2).

Proof. Let $q \in D_{\min}$ and $u \in \Pi_{\max}^{-1}(q)$. This means that $\gamma_u(0) = p$ and $\gamma_u(\pi) = q$. Let $w \in T_q D_{\min}$ and $w^* \in T_u S_p(0, 1)$ be such that $\Pi_{\max\,*}(w^*) = w$. In other words, there is a Jacobi field $J = J_{w^*}$ along γ_u satisfying $J(0) = 0$, $J'(0) = w^*$, $J(\pi) = w$. Also given any $v \in T_q D_{\min}$ there is a Jacobi field L such that $L(\pi) = 0$, $L'(\pi) = v$. Since L is just the J_v of lemma 1 when viewed from q to p, we see that $L(0)^N$ is a horizontal vector of length equal to $2\|v\|$. Now at $t = 0$, $\omega(J, L) = -\langle w^*, L(0)\rangle = -\langle w^*, L(0)^N\rangle$, and at $t = \pi$, $\omega(J, L) = \langle w, v\rangle$. It follows then that w^{*h}, the horizontal component of w^*, has length $(1/2)\|w\|$. This proves the lemma. □

Corollary 1. *Volume of M is equal to that of its model CROSS.*

Proof. Since the exponential map from the unit normal spheres to the opposite critical set is a Riemannian submersion with fibres of dimension 1 or 3, by the classification theorem in [1] they are all standard Hopf fibrations. Thus critical sets are isometric to their respective model CROSSes and in particular have the correct volumes. Consequently, so does M. □

Now the proof of Lemma 4.1 of [2] is complete in all cases.

REMARKS. 1. In the more general situation when there are more critical sets of f other than just a maximum and a minimum, the computations done here can be repeated to see that each critical set is a CROSS. One just has to obsereve that if we take critical sets D_α and D_β and study the geodesics between them, then lemma 2.8 from [2] ensures that J_v^N is tangential to $D_\alpha * D_\beta$. (For explanation of the notation, refer to its definition just prior to lemma 2.13 in [2]) Note that classification of Riemannian submersions of S^{15} with 7-dimensional fibres is not needed since it can occur only when M is a homology CaP^2 and there are only two critical sets: a situation already taken care of.

2. Further, one can try to show that there can be no short geodesics in M by observing that any short geodesic will perforce stay in a level set of f and those geodesics that start tangentially to a regular level set but do not lie in it must stay on only one side of it. Moreover, a generic geodesic starting tangentially to a regular level set will have to leave it. (A regular level set cannot be totally geodesic.) Since a nearby long geodesic must go 'around' a short one, we can see that there can be no short geodesics in regular level sets. This can be seen more clearly perhaps by lifting f to UM. Thus short geodesics will only be confined to critical levels. Within critical sets there are no short geodesics. The problem lies with regular points in a critical level. If it could be resolved it will follow that M is actually a C-manifold and hence its volume is same as that of its model CROSS. See [3] and [4].

3. These observations make one believe that ultimately one should be able to drop the condition that f be without saddle points.

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