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# CONVERGENCE TOWARDS AN ELASTICA IN A RIEMANNIAN MANIFOLD

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## 0. Introduction

Consider a springy circle wire in a riemannian manifold $M$. We describe it as a closed curve $\gamma$ with unit line element and fixed length. For such a curve, its elastic energy is given by

$$E(\gamma) = \oint |D_x\gamma'|^2\, dx.$$

Solutions of the corresponding Euler-Lagrange equation are called *elastic curves*. We discuss a corresponding parabolic equation in this paper. We will see that the equation becomes an initial value problem:

$$\text{(EP)} \qquad \begin{cases} \partial_t \gamma = -D_x^3\gamma' + R(\gamma', D_x\gamma')\gamma' + D_x(w\gamma'), \\ -w'' + |D_x\gamma'|^2 w = 2\{|D_x\gamma'|^2\}'' - |D_x^2\gamma'|^2 - (R(\gamma', D_x\gamma')\gamma', D_x\gamma'), \\ \gamma(x,0) = \gamma_0(x), \end{cases}$$

where $w = w(x,t)$ is an unknown real valued function.

In [3], we treated the case of euclidean spaces and saw that the above equation has a unique long time solution and that the solution converges to an elastica. In this paper, we treat general riemannian manifolds, and get the following

**Theorem 5.6.** *Let $M$ be a compact real analytic riemannian manifold, and let $\gamma_0(x)$ be a closed curve with unit line element and length $L$. Suppose that there are no closed geodesics of length $L$ in $M$. Then* (EP) *has a unique solution $\gamma(x,t)$ for all time, and the solution $\gamma(*,t)$ converges to an elastica when $t \to \infty$.*

REMARK 0.1. Even if the metric of $M$ is not real analytic, there is a solution of (EP) which has a subsequence converging to an elastica (Theorem 4.1, 5.5). This proves the existence of an elastica. Existence of an elastica has been originally shown in [4] by using Palais-Smale's condition (C). Another proof has been given in [1] by using a direct method.

REMARK 0.2. The equation (EP) is *not* the so-called curve shortening equation. The principal part of (EP) is $(\partial/\partial t + \partial^4/\partial x^4)\gamma$ and $(\partial^2/\partial x^2)w$. Main difficulty of our equation comes from being coupled.

## 1. Preliminaries

By scaling, we may assume that the length of the initial curve $\gamma_0$ is 1. From now on, a closed curve means a map from $S^1 \equiv \mathbf{R}/\mathbf{Z}$ into a riemannian manifold $M$. The variable in $S^1$ is denoted by $x$, and differentiation with respect to $x$ is denoted by $*'$ or $*^{(n)}$. The covariant differentiation on $M$ is denoted by $D$.

For tensors on M, we use pointwise inner product $(*, *)$ and norm $|*|$. For functions on $S^1$ and vector fields along a closed curve $\gamma$, we use $L_2$ inner product $\langle *, * \rangle$ and $L_2$ norm $\| * \|$. Sobolev $H^s$ norm is denoted by $\| * \|_s$. For a tensor field $\xi$ along a closed curve $\gamma$, $H^s$ norm $\|\xi\|_s$ is defined by $\|\xi\|_s^2 = \sum_{i=0}^s \|D_x^i \xi\|^2$.

We recall basic lemmas from [3]. Some of them are extended to the case of tensor fields. We frequently use them to get estimation, but always makes no mention of them.

**Lemma 1.1** ([3, Lemma 3.1]). *For a tensor field $\xi$ along a closed curve $\gamma$,*

$$\max |\xi|^2 \leq 2\|\xi\| \cdot \{\|\xi\| + \|D_x \xi\|\}.$$

**Lemma 1.2** ([3, Lemma 3.2]). *For integers $0 \leq p \leq q \leq r$,*

$$\|D_x^q \xi\| \leq \|D_x^p \xi\|^{(r-q)/(r-p)} \cdot \|D_x^r \xi\|^{(q-p)/(r-p)}.$$

**Lemma 1.3** ([3, Lemma 4.1]). *Let $a$ and $b$ be $L_1$ functions on $S^1$ such that $a \geq 0$ and $\|a\|_{L_1} > 0$. Then, the ODE for a function $v$ on $S^1$:*

$$-v'' + av = b$$

*has a unique solution, and the solution is estimated as*

$$\max |v| \leq 2\{1 + \|a\|_{L_1}^{-1}\} \cdot \|b\|_{L_1},$$
$$\max |v'| \leq 2\{1 + \|a\|_{L_1}\} \cdot \|b\|_{L_1}.$$

We need also Hölder norms. The usual Hölder space for functions on $S^1$ is denoted by $C_x^{n+4\mu}$. The weighted Hölder space (time derivative is counted 4 times) for functions on $S^1 \times [0, T)$ is denoted by $C^{n+4\mu}$. See [3] for the detailed definition.

**Lemma 1.4** ([3, Proposition 5.6]). *Set $D = S^1 \times [0, T)$. Let $a : D \to \mathbf{R}$; $b_i, d_i, f : D \to \mathbf{R}^N$; $c_i : D \to \mathbf{R}^{N \times N}$ be $C^{4\mu}$ functions and $\phi : S^1 \to \mathbf{R}^N$ a $C_x^{4+4\mu}$ function. Suppose that $a$ is non-negative and $\|a\|_{L_1} \geq C > 0$. Then, the linear PDE for a $\mathbf{R}^N$*

*valued function $u$ and a function $v$:*

$$\begin{cases} \partial_t u + u^{(4)} + \sum_{i=0}^{3} c_i u^{(i)} + \sum_{i=0}^{1} d_i v^{(i)} = f, \\ -v'' + av = \sum_{i=0}^{3} b_i u^{(i)}, \\ u(x,0) = \phi(x) \end{cases}$$

*has a unique $C^{4+4\mu}$ solution on $D$, and the $C^{4+4\mu}$ norm of the solution is bounded by a constant depending on the $C^{4\mu}$ norms of $f$, $a$, $b_i$, $c_i$, $d_i$, the $C_x^{4+4\mu}$ norm of $\phi$, and $C^{-1}$.*

## 2. The equation

To derive the equation of motion governed by an energy, we perturb the curve $\gamma = \gamma(x)$ with a time parameter $t$: $\gamma = \gamma(x,t)$. Then the elastic energy changes at $t = 0$ as

$$\begin{aligned} \frac{d}{dt}E(\gamma) &= 2\langle D_x\gamma', D_t D_x\gamma'\rangle \\ &= 2\langle D_x\gamma', R(\partial_t\gamma, \gamma')\gamma' + D_x^2\partial_t\gamma\rangle \\ &= -2\langle \partial_t\gamma, R(\gamma', D_x\gamma')\gamma'\rangle + 2\langle \partial_t\gamma, D_x^3\gamma'\rangle \\ &= 2\langle \partial_t\gamma, D_x^3\gamma' - R(\gamma', D_x\gamma')\gamma'\rangle, \end{aligned}$$

where $\gamma(x,0) = \gamma(x)$. Therefore, $-D_x^3\gamma' + R(\gamma', D_x\gamma')\gamma'$ would be the most efficient direction to minimize the elastic energy. However, this direction does not preserve the condition $|\gamma'| \equiv 1$. To force to preserve the condition we have to add certain term. Let $V$ be the space of all directions satisfying the condition in the sense of first derivative. Namely,

$$V = \{\alpha \mid (\gamma', D_x\alpha) = 0\}.$$

We can check that a direction is $L_2$ orthogonal to $V$ if and only if it has a form $D_x(w\gamma')$ with some function $w(x)$. Therefore, the "true direction" should be

$$\partial_t\gamma = -D_x^3\gamma' + R(\gamma', D_x\gamma')\gamma' + D_x(w\gamma'),$$

where the function $w$ has to satisfy the condition

$$(\gamma', D_x\partial_t\gamma) = 0.$$

To simplify this relation, we use the following

**Lemma 2.1.** *For a curve $\gamma$ with $|\gamma'| \equiv 1$, we have the following identities.*

$$\begin{aligned}
(\gamma', D_x\gamma') &= 0,\\
(\gamma', D_x^2\gamma') &= -|D_x\gamma'|^2,\\
(\gamma', D_x^3\gamma') &= -\frac{3}{2}\{|D_x\gamma'|^2\}',\\
(\gamma', D_x^4\gamma') &= -2\{|D_x\gamma'|^2\}'' + |D_x^2\gamma'|^2.
\end{aligned}$$

Proof. We can get these by a simple calculation. □

Therefore we have

$$\begin{aligned}
0 &= (D_x\{-D_x^3\gamma' + R(\gamma', D_x\gamma')\gamma' + D_x(w\gamma')\}, \gamma')\\
&= -(\gamma', D_x^4\gamma') + (\gamma', D_x\{R(\gamma', D_x\gamma')\gamma'\}) + (\gamma', D_x^2(w\gamma'))\\
&= 2\{|D_x\gamma'|^2\}'' - |D_x^2\gamma'|^2 + (R(\gamma', D_x\gamma')D_x\gamma', \gamma')\\
&\quad + (\gamma', w''\gamma' + 2w'D_x\gamma' + wD_x^2\gamma')\\
&= 2\{|D_x\gamma'|^2\}'' - |D_x^2\gamma'|^2 - (R(\gamma', D_x\gamma')\gamma', D_x\gamma') + w'' - |D_x\gamma'|^2 w.
\end{aligned}$$

Thus the equation for the function $w(x)$ becomes

$$-w'' + |D_x\gamma'|^2 w = 2\{|D_x\gamma'|^2\}'' - |D_x^2\gamma'|^2 - (R(\gamma', D_x\gamma')\gamma', D_x\gamma').$$

If we put

$$v = w + 2|D_x\gamma'|^2,$$

then we have

$$-v'' + |D_x\gamma'|^2 v = -|D_x^2\gamma'|^2 + 2|D_x\gamma'|^4 - (R(\gamma', D_x\gamma')\gamma', D_x\gamma').$$

Therefore our equation becomes

$$\text{(EP)} \qquad \begin{cases} \partial_t\gamma = -D_x^3\gamma' + R(\gamma', D_x\gamma')\gamma' + D_x(w\gamma'),\\ -w'' + |D_x\gamma'|^2 w = 2\{|D_x\gamma'|^2\}'' - |D_x^2\gamma'|^2 - (R(\gamma', D_x\gamma')\gamma', D_x\gamma'),\\ \gamma(x,0) = \gamma_0(x). \end{cases}$$

Or, equivalently,

$$\text{(EP}_v\text{)} \qquad \begin{cases} \partial_t\gamma = -D_x^3\gamma' + R(\gamma', D_x\gamma')\gamma' + D_x\{(v - 2|D_x\gamma'|^2)\gamma'\},\\ -v'' + |D_x\gamma'|^2 v = -|D_x^2\gamma'|^2 + 2|D_x\gamma'|^4 - (R(\gamma', D_x\gamma')\gamma', D_x\gamma'),\\ \gamma(x,0) = \gamma_0(x). \end{cases}$$

Note that both $\gamma$ and $w$ (or $v$) are unknown functions on $S^1 \times \mathbf{R}_+$.

## 3. Short time existence

In this section, we consider a modified equation for an $\mathbf{R}^N$ valued function $\gamma$ and a function $v$:

$$\text{(ST)} \qquad \begin{cases} \partial_t\gamma = -\gamma^{(4)} + F(x, \gamma, \gamma', \gamma'', \gamma^{(3)}, v, v'), \\ -v'' + G(x, \gamma, \gamma', \gamma'', \gamma^{(3)}) \cdot v = H(x, \gamma, \gamma', \gamma'', \gamma^{(3)}), \end{cases}$$

where $F$, $G$ and $H$ are given $C^\infty$ functions on $S^1 \times (\mathbf{R}^N)^6$, $S^1 \times (\mathbf{R}^N)^4$ and $S^1 \times (\mathbf{R}^N)^4$, respectively, and the function $G$ is non-negative. For functions $\gamma$ and $v$, we take their jets and use abbreviated notations such as $F(x, j_3\gamma, j_1 v)$, $G(x, j_3\gamma)$ and $H(x, j_3\gamma)$.

**Theorem 3.1.** *For any $C^\infty$ initial data $\gamma_0$ with $G(x, j_3\gamma_0) > 0$ at some point $x \in S^1$, there is a positive time $T$ so that* (ST) *has a unique $C^\infty$ solution on the time interval* $[0, T)$.

To prove this, we need "cut off" functions for $F$, $G$ and $H$. Let $\rho_a(y)$ be a $C^\infty$ function of $y$ such that $\rho_a(y) = 1$ for $|y| \le a$, $\rho_a(y) = 0$ for $|y| \ge 2a$, and $0 \le \rho_a(y) \le 1$ for all $y$. Let $v_0$ be the solution of the ODE: $-v'' + G(x, j_3\gamma_0) \cdot v = H(x, j_3\gamma_0)$ and put $A = \max(|j_3\gamma_0|^2 + |j_1 v_0|^2)$. Set

$$\begin{aligned} \tilde{F}(x, j_3\gamma, j_1 v) &= \rho_{2A}(|j_3\gamma|^2 + |j_1 v|^2) \cdot F(x, j_3\gamma, j_1 v), \\ \tilde{H}(x, j_3\gamma) &= \rho_{2A}(|j_3\gamma|^2) \cdot H(x, j_3\gamma). \end{aligned}$$

For the function $G$, we take a point $x_0 \in S^1$ and positive numbers $B \le 1$ and $C$ so that $G(x, j_3\gamma) \ge C$ for all 3-jets $\{x, \gamma\}$ with $|x - x_0|$, $|j_3(\gamma - \gamma_0)|^2 \le B$. Set

$$\tilde{G}(x, j_3\gamma) = \rho_{B/2}(|j_3(\gamma - \gamma_0)|^2) \cdot G(x, j_3\gamma) + 1 - \rho_{B/2}(|j_3(\gamma - \gamma_0)|^2).$$

Take any point $x$ with $|x - x_0| \le B$. If $|j_3(\gamma - \gamma_0)|^2 \le B$, then $\tilde{G}(x, j_3\gamma) \ge \min\{G(x, j_3\gamma), 1\} \ge C$. If $|j_3(\gamma - \gamma_0)|^2 \ge B$, then $\tilde{G}(x, j_3\gamma) = 1$. In particular, for any function $\gamma$, we have

$$\oint \tilde{G}(x, j_3\gamma)\, dx \ge BC.$$

Note that if $\gamma$ is sufficiently close to $\gamma_0$ in $C^3$ topology, then $\tilde{G}(x, j_3\gamma) = G(x, j_3\gamma)$ and $\tilde{H}(x, j_3\gamma) = H(x, j_3\gamma)$. It also implies that the solution $\tilde{v}$ of the ODE: $-\tilde{v}'' + \tilde{G}(x, j_3\gamma) \cdot \tilde{v} = \tilde{H}(x, j_3\gamma)$ coincides with $v$. Therefore, if we have a solution for the equation

$$(\widetilde{\text{ST}}) \qquad \begin{cases} \partial_t\gamma = -\gamma^{(4)} + \tilde{F}(x, j_3\gamma, j_1 v), \\ -v'' + \tilde{G}(x, j_3\gamma) \cdot v = \tilde{H}(x, j_3\gamma), \end{cases}$$

then it is a solution for the original equation for some short time.

Now we consider the equation

$$(\widetilde{\mathrm{ST}}_\lambda) \qquad \begin{cases} \partial_t \gamma = -\gamma^{(4)} + \lambda \tilde{F}(x, j_3\gamma, j_1 v), \\ -v'' + \tilde{G}(x, j_3\gamma) \cdot v = \tilde{H}(x, j_3\gamma), \end{cases}$$

where $\lambda$ is a constant in $[0, 1]$.

**Lemma 3.2.** *Let $\gamma = \gamma(t, x)$ be a $C^{4+4\mu}$ solution of $(\widetilde{\mathrm{ST}}_\lambda)$ with a $C^\infty$ initial data $\gamma_0(x)$. Then $\gamma$ is $C^\infty$.*

Proof. If $\gamma$ belongs in the class $C^{n+4+4\mu}$, then the functions $\tilde{G}(x, j_3\gamma)$ and $\tilde{H}(x, j_3\gamma)$ belong to the class $C^{n+1+4\mu}$. Hence Lemma 1.4 implies that $v$ and $v'$ belong to $C^{n+1+4\mu}$, therefore also $\tilde{F}(x, j_3\gamma, j_1 v)$ belongs to $C^{n+1+4\mu}$. Thus we see that $\gamma$ belongs to $C^{n+5+4\mu}$. By induction, we see the smoothness of the solution $\gamma$. $\square$

**Lemma 3.3.** *Consider the ODE: $-v'' + \tilde{G}(x, j_3\gamma) \cdot v = \tilde{H}(x, j_3\gamma)$. For any non-negative integer $n$ and a positive number $C$, there is a positive number $K$ with the following property:*

*If $\|\gamma\|_n \le C$, then $\|v\|_n \le K \cdot \{1 + \|\gamma^{(n+1)}\|\}$.*

Proof. Since $|v|$ and $|v'|$ are bounded by Lemma 1.3, the claim holds for $n = 0, 1$. Suppose that the claim holds for an integer $n$ $(\ge 1)$ and that $\|\gamma\|_{n+1} \le C$. Then, by Lemmas 1.1 and 1.2, we have

$$\begin{aligned} \|v\|_{n+1} &\le \|v\|_n + \|v^{(n+1)}\| \\ &\le C + \|\tilde{G}(x, j_3\gamma) \cdot v\|_{n-1} + \|\tilde{H}(x, j_3\gamma)\|_{n-1} \\ &\le C + C_1 \cdot \|\tilde{G}(x, j_3\gamma)\|_{n-1} + \|\tilde{H}(x, j_3\gamma)\|_{n-1}. \end{aligned}$$

The last expression involves the derivatives of $\gamma$ up to $\gamma^{(n+2)}$. Counting the fact that $|\gamma^{(n)}|$ is bounded, we see

$$\begin{aligned} \|v\|_{n+1} &\le C_2 \cdot \{1 + \|\gamma^{(n+2)}\| + \||\gamma^{(n+1)}| \cdot |\gamma^{(4)}\|_{(\#3)}\} \\ &\le C_2 \cdot \{1 + \|\gamma^{(n+2)}\| + \|\gamma^{(n+1)}\| \cdot \max|\gamma^{(4)}|_{(\#3)}\} \\ &\le C_3 \cdot \{1 + \|\gamma^{(n+2)}\| + \max|\gamma^{(n+1)}|_{(\#3)}\} \\ &\le C_4 \cdot \{1 + \|\gamma^{(n+2)}\|\}, \end{aligned}$$

where (#3) means that the indicated term appears only if $n \ge 3$. $\square$

**Lemma 3.4.** *Let $\gamma$ be a solution of $(\widetilde{\mathrm{ST}}_\lambda)$ on a finite time interval $[0, T)$. For any non-negative integer $n$, the norm $\|\gamma^{(n)}\|$ is uniformly bounded with respect to $\lambda \in [0, 1]$.*

Proof. First of all, for $n \leq 2$, we have

$$\begin{aligned}
\frac{d}{dt}\|\gamma^{(n)}\|^2 &= 2\langle\gamma^{(n)}, \partial_t\gamma^{(n)}\rangle \\
&= 2\langle\gamma^{(n)}, -\gamma^{(n+4)} + \lambda\tilde{F}(x, j_3\gamma, j_1v)^{(n)}\rangle \\
&= -2\|\gamma^{(n+2)}\|^2 \pm 2\lambda\langle\gamma^{(2n)}, \tilde{F}(x, j_3\gamma, j_1v)\rangle \\
&\leq -2\|\gamma^{(n+2)}\|^2 + 2\|\tilde{F}(x, j_3\gamma, j_1v)\| \cdot \|\gamma^{(2n)}\|.
\end{aligned}$$

Thus for $n = 0$, we have

$$\frac{d}{dt}\|\gamma\|^2 \leq 2C_1 \cdot \|\gamma\|,$$

hence $(d/dt)\|\gamma\|$ is bounded. Also, for $n = 2$, we have

$$\frac{d}{dt}\|\gamma''\|^2 \leq -2\|\gamma^{(4)}\|^2 + 2C_2\|\gamma^{(4)}\| \leq C_3.$$

Therefore, the norm $\|\gamma\|_2$ increases at most linear order.

Suppose that we know estimation of $\|\gamma\|_{n+1}$ for an integer $n$ ($\geq 1$). By Lemma 3.3, we have

$$\begin{aligned}
\|v\|_n &\leq C_4, \\
\|v\|_{n+1} &\leq C_5 \cdot \{1 + \|\gamma^{(n+2)}\|\}.
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{d}{dt}\|\gamma^{(n+2)}\|^2 &= 2\langle\gamma^{(n+2)}, -\gamma^{(n+6)} + \lambda\tilde{F}(x, j_3\gamma, j_1v)^{(n+2)}\rangle \\
&\leq -2\|\gamma^{(n+4)}\|^2 + 2\|\gamma^{(n+4)}\| \cdot \|\tilde{F}(x, j_3\gamma, j_1v)^{(n)}\| \\
&\leq -\|\gamma^{(n+4)}\|^2 + \|\tilde{F}(x, j_3\gamma, j_1v)^{(n)}\|^2.
\end{aligned}$$

Here, the term $\tilde{F}(x, j_3\gamma, j_1v)^{(n)}$ contains the derivatives of $\gamma$ and $v$ up to $\gamma^{(n+3)}$ and $v^{(n+1)}$, and $|\gamma^{(n)}|$ and $|v^{(n-1)}|$ are bounded. Therefore we have to estimate the following terms:

$$\begin{aligned}
&\|\gamma^{(n+3)}\|, \ \| |\gamma^{(n+2)}| \cdot |\gamma^{(4)}| \|, \ \| |\gamma^{(n+2)}| \cdot |v''| \|, \\
&\| |\gamma^{(n+1)}| \cdot |\gamma^{(5)}| \|, \ \| |\gamma^{(n+1)}| \cdot |\gamma^{(4)}| \cdot |\gamma^{(4)}| \|, \ \| |\gamma^{(n+1)}| \cdot |v^{(3)}| \|, \\
&\| |\gamma^{(n+1)}| \cdot |v''| \cdot |v''|, \ \| |\gamma^{(n+1)}| \cdot |v^{(4)}| \cdot |v''| \\
&\|v^{(n+1)}\|, \ \| |v^{(n)}| \cdot |\gamma^{(4)}| \|, \ \| |v^{(n)}| \cdot |v''| \|.
\end{aligned}$$

Note that terms with multiple factors appear only if $n \geq$ (their number of factors). By Lemma 1.2, we can estimate each factor as:

$$\|\gamma^{(n+3)}\| \leq C_6 \cdot \|\gamma^{(n+4)}\|^{2/3},$$

$$
\begin{aligned}
\max|\gamma^{(n+2)}| &\le C_7\cdot\{1+\|\gamma^{(n+2)}\|^{1/2}\cdot\|\gamma^{(n+3)}\|^{1/2}\}\\
&\le C_8\cdot\{1+\|\gamma^{(n+4)}\|^{1/2}\},\\
\max|\gamma^{(n+1)}| &\le C_9\cdot\{1+\|\gamma^{(n+2)}\|^{1/2}\}\le C_{10}\cdot\{1+\|\gamma^{(n+4)}\|^{1/6}\},\\
\|v^{(n+1)}\| &\le C_{11}\cdot\{1+\|\gamma^{(n+2)}\|\}\le C_{12}\cdot\{1+\|\gamma^{(n+4)}\|^{1/3}\},\\
\max|v^{(n)}| &\le C_{13}\cdot\{1+\|v^{(n+1)}\|^{1/2}\}\\
&\le C_{14}\cdot\{1+\|\gamma^{(n+2)}\|^{1/2}\}\le C_{15}\cdot\{1+\|\gamma^{(n+4)}\|^{1/6}\}.
\end{aligned}
$$

When $n\ge 2$, we have

$$
\begin{aligned}
\|\gamma^{(4)}\| &\le C_{16}\cdot\{1+\|\gamma^{(n+2)}\|\}\le C_{17}\cdot\{1+\|\gamma^{(n+4)}\|^{1/3}\},\\
\|v''\| &\le C_{18}\cdot\{1+\|v^{(n)}\|\}\le C_{19},\\
\|\gamma^{(5)}\| &\le C_{20}\cdot\{1+\|\gamma^{(n+3)}\|\}\le C_{21}\cdot\{1+\|\gamma^{(n+4)}\|^{2/3}\},\\
\|v^{(3)}\| &\le C_{22}\cdot\{1+\|v^{(n+1)}\|\}\le C_{23}\cdot\{1+\|\gamma^{(n+4)}\|^{1/3}\}.
\end{aligned}
$$

When $n\ge 3$, we have

$$
\begin{aligned}
\max|v''| &\le C_{24}\cdot\{1+\max|v^{(n-1)}|\}\le C_{25},\\
\max|\gamma^{(4)}| &\le C_{26}\cdot\{1+\max|\gamma^{(n+1)}|\}\le C_{27}\cdot\{1+\|\gamma^{(n+2)}\|^{1/2}\}\\
&\le C_{28}\cdot\{1+\|\gamma^{(n+4)}\|^{1/6}\}.
\end{aligned}
$$

Combining all, we conclude

$$\|\tilde F(x, j_3\gamma, j_1 v)^{(n)}\|\le C_{29}\cdot\{1+\|\gamma^{(n+4)}\|^{5/6}\},$$

and

$$\frac{d}{dt}\|\gamma^{(n+2)}\|^2\le C_{30}. \qquad \square$$

Proof (of Theorem 3.1). We use the so-called open closed method. Take any positive time $T$. By the implicit function theorem with Lemma 1.4, the set $\Lambda$ of $\lambda$ which has a solution $\gamma$ of $(\widetilde{\mathrm{ST}}_\lambda)$ on $[0, T)$ is open in the interval $[0, 1]$. On the other hand, by Lemma 3.4, $\Lambda$ is closed in $[0, 1]$. Since $\Lambda$ contains 0, it should coincide with $[0, 1]$. By definition, the solution of $(\widetilde{\mathrm{ST}}_\lambda)$ with $\lambda = 1$ is a solution of $(\widetilde{\mathrm{ST}})$, which gives a short time solution of (ST). For detailed discussion, see [3, Proof of Theorem 6.5]. $\square$

**Theorem 3.5.** *The equation* (EP) *with non-geodesic initial data of unit line element has a unique short time solution $\gamma(x, t)$. Moreover, every closed curve $\gamma(*, t)$ has unit line element.*

Proof. We may assume that the induced tangent bundle of the initial data $\gamma_0$ is orientable, taking a double cover if necessary. Then, using a tubular neighbourhood of $\gamma_0$, ($\mathrm{EP}_v$) is expressed as (ST), hence has a short time solution. Let $\{\gamma, v\}$ be a solution. Since $\partial_t|\gamma'|^2 = 2(\gamma', D_t\gamma') = 2(\gamma', D_x\partial_t\gamma) = 0$, we have $|\gamma'|^2 \equiv 1$. Let $\{\gamma+\zeta, v+u\}$ be another solution of (ST) in the tubular neighbourhood of $\gamma_0$. Then $\{\zeta, u\}$ satisfies the equation:

$$\begin{cases} \partial_t\zeta = -\zeta^{(4)} + f(x, t, \zeta, \zeta', \zeta'', \zeta^{(3)}, u, u'), \\ -u'' + G(x, \gamma, \gamma', \gamma'', \gamma^{(3)}) \cdot u = h(x, t, \zeta, \zeta', \zeta'', \zeta^{(3)}, u). \end{cases}$$

Here, $|f|$ and $|h|$ are bounded by $C\{\sum_{i=0}^3 |\zeta^{(i)}| + |u| + |u'|\}$, because $\{\gamma + \zeta, v + u\}$ is bounded. Therefore, we have $\|u\|_1 \le C_1\|\zeta\|_3$, and

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\zeta\|_1^2 &= \langle\zeta, \partial_t\zeta\rangle + \langle\zeta', \partial_t\zeta'\rangle = \langle\zeta - \zeta'', -\zeta^{(4)} + f\rangle \\ &= -\|\zeta''\|^2 - \|\zeta^{(3)}\|^2 + \langle\zeta, f\rangle \\ &\le -\|\zeta''\|^2 - \|\zeta^{(3)}\|^2 + C_2 \cdot \|\zeta\| \cdot (\|\zeta\|_3 + \|u\|_1) \\ &\le C_3 \cdot \|\zeta\|_1^2. \end{aligned}$$

Since $\zeta = 0$ at $t = 0$, we have $\zeta \equiv 0$. Replacing $t = 0$ to arbitrary $t = t_0$, we see that the set of all $t$ such that two solutions coincide is open. Hence the solutions coincide for all time. □

## 4. Long time existence

In this section, we consider the original equation:

$$\text{(EP)} \qquad \begin{cases} \partial_t\gamma = -D_x^3\gamma' + R(\gamma', D_x\gamma')\gamma' + D_x(w\gamma'), \\ -w'' + |D_x\gamma'|^2 w = 2\{|D_x\gamma'|^2\}'' - |D_x^2\gamma'|^2 - (R(\gamma', D_x\gamma')\gamma', D_x\gamma'), \\ \gamma(x, 0) = \gamma_0(x), \end{cases}$$

where $\gamma_0$ is a closed curve of unit line element.

**Theorem 4.1.** *Let $M$ be a compact riemannian manifold and $\gamma_0$ a closed curve of unit line element. Then* (EP) *has a unique solution for a time interval* $[0, T)$ *and one of the followings holds.*

1) *There is a sequence of times $t_i \to T$ such that $\gamma(*, t_i)$ converges to a closed geodesic in $C^1$ topology.*
2) $T = \infty$.

To prove this, we need some preparation. For a closed curve $\gamma$, let $v$ and $w$ be solutions of the ODE:

$$-v'' + |D_x\gamma'|^2 v = -|D_x^2\gamma'|^2 + 2|D_x\gamma'|^4 - (R(\gamma', D_x\gamma')\gamma', D_x\gamma'),$$
$$-w'' + |D_x\gamma'|^2 w = 2\{|D_x\gamma'|^2\}'' - |D_x^2\gamma'|^2 - (R(\gamma', D_x\gamma')\gamma', D_x\gamma'),$$

and put

$$\delta = -D_x^3\gamma' + R(\gamma', D_x\gamma')\gamma' + D_x(w\gamma').$$

In Lemmas 4.2–4.6, we consider this ODE and estimate $v$, $w$ and $\delta$ by $\gamma'$. They will be applied to the PDE (EP) later.

**Lemma 4.2.** *For any non-negative integer $n$ and any positive real number $C$, there is a positive number $K$ with the following property:*
*If $\|D_x\gamma'\| \geq C^{-1}$, $\|\gamma'\|_1 \leq C$ and $\|\gamma'\|_n \leq C$, then*

$$\|w\|_{n+1} \leq K \cdot \{1 + \|D_x^2\gamma'\| \cdot \|D_x^{n+2}\gamma'\|\}.$$

Proof. The assumption and Lemma 1.3 imply that

$$\|v\|_{C^1} \leq \|D_x^2\gamma'\|^2 + 2\||D_x\gamma'|^2\|^2 + \|D_x\gamma'\|^2.$$

But we know that $\max|D_x\gamma'| \leq C_1 \cdot \{1 + \|D_x^2\gamma'\|^{1/2}\}$. Therefore,

$$\|v\|_{C^1} \leq C_2 \cdot \{1 + \|D_x^2\gamma'\|^2\}.$$

Moreover,

$$\||D_x\gamma'|^2\| \leq C_3 \cdot \{1 + \|D_x^2\gamma'\|^{1/2}\},$$
$$\|\{|D_x\gamma'|^2\}'\| \leq 2\||D_x\gamma'| \cdot |D_x^2\gamma'|\| \leq C_4 \cdot \{1 + \|D_x^2\gamma'\|^{3/2}\}.$$

Thus we proved the claim for $n = 0$:

$$\|w\|_1 \leq C_5 \cdot \{1 + \|D_x^2\gamma'\|^2\}.$$

Suppose that the claim holds for a non-negative integer $n$ and that $\|\gamma'\|_{n+1} \leq C$. Then, we know $\|w\|_{n+1} \leq C_6 \cdot \{1 + \|D_x^2\gamma'\| \cdot \|D_x^{n+2}\gamma'\|\}$. Therefore,

$$\begin{aligned}
\|w^{(n+2)}\| \leq\ & \|\{|D_x\gamma'|^2 \cdot w\}^{(n)}\| + 2\|\{|D_x\gamma'|^2\}^{(n+2)}\| \\
& + \|\{|D_x^2\gamma'|^2\}^{(n)}\| + \|(R(\gamma', D_x\gamma')\gamma', D_x\gamma')^{(n)}\| \\
\leq\ & C_7 \cdot \{1 + \||D_x^{n+1}\gamma'| \cdot |D_x\gamma'| \cdot |w|\| + \|w\|_n\} \\
& + C_8 \cdot \{1 + \||D_x^{n+3}\gamma'| \cdot |D_x\gamma'|\| + \||D_x^{n+2}\gamma'| \cdot |D_x^2\gamma'|\| \\
& \quad + \||D_x^{n+1}\gamma'| \cdot |D_x^3\gamma'|\|_{(\#2)}\} \\
& + C_9 \cdot \{1 + \||D_x^{n+1}\gamma'| \cdot |D_x\gamma'|\|\},
\end{aligned}$$

where (#2) means that the indicated term appears only when $n \geq 2$.

Here, we know that

$$\begin{aligned}
\max |D_x^{n+1}\gamma'| &\leq C_{10} \cdot \{1 + \|D_x^{n+2}\gamma'\|^{1/2}\} \leq C_{11} \cdot \{1 + \|D_x^{n+3}\gamma'\|^{1/4}\}, \\
\max |w| &\leq \|w\|_1 \leq C_{12} \cdot \{1 + \|D_x^2\gamma'\|^2\} \\
&\leq C_{13} \cdot \{1 + \|D_x^2\gamma'\| \cdot \|D_x^{n+3}\gamma'\|^{1/2}\}, \\
\max |D_x\gamma'| &\leq C_{14} \cdot \{1 + \|D_x^2\gamma'\|\}, \\
\max |D_x^{n+2}\gamma'| &\leq C_{15} \cdot \{1 + \|D_x^{n+2}\gamma'\|^{1/2} \cdot \|D_x^{n+3}\gamma'\|^{1/2}\} \\
&\leq C_{16} \cdot \{1 + \|D_x^{n+3}\gamma'\|^{3/4}\}, \\
\|D_x^3\gamma'\|_{(\#2)} &\leq C_{17}.
\end{aligned}$$

Thus we have

$$\|w\|_{n+2} \leq C_{18} \cdot \{1 + \|D_x^2\gamma'\| \cdot \|D_x^{n+3}\gamma'\|\},$$

and the induction completes the proof. □

**Lemma 4.3.** *Set*

$$\phi = R(\gamma', D_x\gamma')\gamma' + D_x(w\gamma').$$

*For any non-negative integer $n$ and any positive real number $C$, there is a positive number $K$ with the following property:*

*If $\|D_x\gamma'\| \geq C^{-1}$, $\|\gamma'\|_1 \leq C$ and $\|\gamma'\|_n \leq C$, then*

$$\|\phi\|_n \leq K \cdot \{1 + \|D_x^2\gamma'\| \cdot \|D_x^{n+2}\gamma'\|\}.$$

Proof. The assumption and Lemma 4.2 imply that

$$\begin{aligned}
\|\phi\| &\leq C_1 \cdot \{1 + \|w'\| + \max |w|\} \\
&\leq C_2 \cdot \{1 + \|w\|_1\} \leq C_3 \cdot \{1 + \|D_x^2\gamma'\|^2\}.
\end{aligned}$$

Thus the claim holds for $n = 0$.

Suppose that the claim holds for a non-negative integer $n$ and that $\|\gamma'\|_{n+1} \leq C$. Then, we know $\|\phi\|_n \leq C_4 \cdot \{1 + \|D_x^2\gamma'\| \cdot \|D_x^{n+2}\gamma'\|\}$. Therefore,

$$\begin{aligned}
\|\phi^{(n+1)}\| &\leq \|D_x^{n+1}(R(\gamma', D_x\gamma')\gamma')\| + \|D_x^{n+2}(w\gamma')\| \\
&\leq C_5 \cdot \{1 + \|D_x^{n+2}\gamma'\| + \||D_x^{n+1}\gamma'| \cdot |D_x\gamma'|\| \\
&\quad + \||w| \cdot |D_x^{n+2}\gamma'|\| + \||w'| \cdot |D_x^{n+1}\gamma'|\| + \|w\|_{n+2}\}.
\end{aligned}$$

Here, by Lemma 4.2, the terms except 4th and 5th are estimated linearly by $\|D_x^2\gamma'\| \cdot \|D_x^{n+3}\gamma'\|$. For the excepted terms, Lemma 4.2 also implies that

$$\begin{aligned}
\||w| \cdot |D_x^{n+2}\gamma'|\| &\le C_6 \cdot \{1 + \|D_x^2\gamma'\|^2 \cdot \|D_x^{n+2}\gamma'\|\} \\
&\le C_7 \cdot \{1 + \|D_x^2\gamma'\|^2 \cdot \|D_x^{n+3}\gamma'\|^{1/2}\} \\
&\le C_8 \cdot \{1 + \|D_x^2\gamma'\| \cdot \|D_x^{n+2}\gamma'\| \cdot \|D_x^{n+3}\gamma'\|^{1/2}\} \\
&\le C_9 \cdot \{1 + \|D_x^2\gamma'\| \cdot \|D_x^{n+3}\gamma'\|\}, \\
\||w'| \cdot |D_x^{n+1}\gamma'|\| &\le C_{10} \cdot \{1 + \|D_x^2\gamma'\|^2 \cdot \max |D_x^{n+1}\gamma'|\} \\
&\le C_{11} \cdot \{1 + \|D_x^2\gamma'\| \cdot \|D_x^{n+3}\gamma'\|\}. \qquad \square
\end{aligned}$$

**Lemma 4.4.** *For any non-negative integer $n$ and any positive real number $C$, there is a positive number $K$ with the following property:*
*If $\|D_x\gamma'\| \ge C^{-1}$ and $\|\gamma'\|_{n+1} \le C$, then*

$$\|D_x^n\delta\| \le K \cdot \{1 + \|D_x^{n+3}\gamma'\|\},$$

*where $\delta$ is defined below* Theorem 4.1.

Proof. Lemma 4.3 implies that

$$\begin{aligned}
\|D_x^n\delta\| &\le \|D_x^{n+3}\gamma'\| + \|D_x^n\phi\| \\
&\le C_1 \cdot \{1 + \|D_x^{n+3}\gamma'\| + \|D_x^2\gamma'\| \cdot \|D_x^{n+2}\gamma'\|\}.
\end{aligned}$$

Here, we know

$$\begin{aligned}
\|D_x^2\gamma'\| &\le C_2 \cdot \|D_x^3\gamma'\|^{1/2} \le C_3 \cdot \{1 + \|D_x^{n+3}\gamma'\|^{1/2}\}, \\
\|D_x^{n+2}\gamma'\| &\le C_4 \cdot \|D_x^{n+3}\gamma'\|^{1/2},
\end{aligned}$$

which completes the proof. $\square$

**Lemma 4.5.** *Let $\gamma$ be the solution of* (EP). *For any non-negative integer $n$ and any positive real number $C$, there is a positive number $K$ with the following property:*
*If $\|D_x\gamma'\| \ge C^{-1}$ and $\|\gamma'\|_{n+1} \le C$, then*

$$\frac{d}{dt}\|D_x^{n+2}\gamma'\|^2 \le K \cdot \{1 + \|D_x^2\gamma'\|^2 \cdot \|D_x^{n+3}\gamma'\|^2\} - \|D_x^{n+4}\gamma'\|^2.$$

Proof.

$$\begin{aligned}
\frac{d}{dt}\|D_x^{n+2}\gamma'\|^2 &= 2\langle D_x^{n+2}\gamma', D_t D_x^{n+2}\gamma'\rangle \\
&= 2\left\langle D_x^{n+2}\gamma', \sum_{i=0}^{n+1} D_x^i(R(\delta, \gamma')D_x^{n+1-i}\gamma') + D_x^{n+3}\delta\right\rangle \\
&= 2\langle D_x^{n+2}\gamma', R(\delta, \gamma')D_x^{n+1}\gamma'\rangle
\end{aligned}$$

$$
\begin{aligned}
&-2\langle D_x^{n+3}\gamma', R(\delta,\gamma')D_x^n\gamma'\rangle \\
&+2\sum_{i=2}^{n+1}\langle D_x^{n+4}\gamma', D_x^{i-2}(R(\delta,\gamma')D_x^{n+1-i}\gamma')\rangle \\
&+2\langle D_x^{n+4}\gamma', D_x^{n+1}\{-D_x^3\gamma'+\phi\}\rangle \\
\le\ & C_1\cdot\{\|D_x^{n+2}\gamma'\|\cdot\||\delta|\cdot|D_x^{n+1}\gamma'|\| + \|D_x^{n+3}\gamma'\|\cdot\|\delta\| \\
&+\|D_x^{n+4}\gamma'\|\cdot\{\|\delta\|+\|D_x^{n-1}\delta\|\}_{(\#1)}\} \\
&-2\|D_x^{n+4}\gamma'\|^2+2\|D_x^{n+4}\gamma'\|\cdot\|D_x^{n+1}\phi\|,
\end{aligned}
$$

where (#1) means that the indicated term appears only when $n \ge 1$.

Here, we know that

$$
\begin{aligned}
\|D_x^{n+2}\gamma'\| &\le C_2\cdot\|D_x^{n+4}\gamma'\|^{1/3}, \\
\max|D_x^{n+1}\gamma'| &\le C_3\cdot\{1+\|D_x^{n+2}\gamma'\|^{1/2}\} \le C_4\cdot\{1+\|D_x^{n+4}\gamma'\|^{1/6}\}, \\
\|D_x^{n+3}\gamma'\| &\le C_5\cdot\|D_x^{n+4}\gamma'\|^{2/3}.
\end{aligned}
$$

Moreover, by Lemma 4.4,

$$
\begin{aligned}
\|\delta\| &\le C_6\cdot\{1+\|D_x^3\gamma'\|\} \\
&\le C_7\cdot\{1+\|D_x^{n+3}\gamma'\|\} \le C_8\cdot\{1+\|D_x^{n+4}\gamma'\|^{2/3}\}, \\
\|D_x^{n-1}\delta\| &\le C_9\cdot\{1+\|D_x^{n+2}\gamma'\|\} \quad (\text{when } n\ge 1),
\end{aligned}
$$

and by Lemma 4.3,

$$
\|D_x^{n+1}\phi\| \le C_{10}\cdot\{1+\|D_x^2\gamma'\|\cdot\|D_x^{n+3}\gamma'\|\}.
$$

Combining all gives the result. □

**Lemma 4.6.** *For any positive real number $C$ and a $C^1$ neighbourhood $U$ of the set of all closed geodesics of unit line element, there is a positive number $K$ with the following property:*

*If $\gamma$ is a closed curve of unit line element not in the set $U$ and if $\|D_x\gamma'\| \le C$, then*

$$
\|D_x^3\gamma'\| \le K\cdot\{1+\|\delta\|\}.
$$

Proof. Since

$$
(\gamma',\delta) = -(\gamma', D_x^3\gamma') + (\gamma', w'\gamma' + wD_x\gamma') = \frac{3}{2}\{|D_x\gamma'|^2\}' + w',
$$

we see

$$
\|w'\| \le \|\delta\| + 3\|(D_x\gamma', D_x^2\gamma')\|
$$

$$
\begin{aligned}
&\le C_1 \cdot \{\|\delta\| + \max |D_x\gamma'| \cdot \|D_x^2\gamma'\|\} \\
&\le C_2 \cdot \{1 + \|\delta\| + \|D_x\gamma'\|^{1/2} \cdot \|D_x^2\gamma'\|^{3/2}\} \\
&\le C_3 \cdot \{1 + \|\delta\| + \|D_x^3\gamma'\|^{3/4}\}.
\end{aligned}
$$

Put

$$\varphi = -D_x^2\gamma' + w\gamma'.$$

Then we have

$$(\gamma', \varphi) = -(\gamma', D_x^2\gamma') + w = |D_x\gamma'|^2 + w.$$

Therefore,

$$\oint w\,dx = \langle\gamma', \varphi\rangle - \|D_x\gamma'\|^2.$$

Let $\alpha$ be a vector field along $\gamma$ such that $D_x\alpha = \gamma'$ on $0 \le x \le 1$ and $\alpha(0) = 0$. Then,

$$
\begin{aligned}
\langle\gamma', \varphi\rangle &= \langle D_x\alpha, \varphi\rangle = \int_0^1 (D_x\alpha, \varphi)\,dx \\
&= [(\alpha, \varphi)]_0^1 - \int_0^1 (\alpha, D_x\varphi)\,dx \\
&= (\alpha(1), \varphi(0)) - \int_0^1 (\alpha, \delta - R(\gamma', D_x\gamma')\gamma')\,dx \\
&= -(\alpha(1), D_x^2\gamma'(0)) + w(0) \cdot (\alpha(1), \gamma'(0)) \\
&\quad - \int_0^1 (\alpha, \delta)\,dx + \int_0^1 (\alpha, R(\gamma', D_x\gamma')\gamma')\,dx.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\oint w\,dx - (\alpha(1), \gamma'(0)) \cdot w(0) \\
&= -(\alpha(1), D_x^2\gamma'(0)) - \int_0^1 (\alpha, \delta)\,dx \\
&\quad + \int_0^1 (\alpha, R(\gamma', D_x\gamma')\gamma')\,dx - \|D_x\gamma'\|^2.
\end{aligned}
$$

Here,

$$\{|\alpha|^2\}' = 2(\alpha, D_x\alpha) = 2(\alpha, \gamma') \le 2|\alpha|,$$

and so

$$|\alpha|' \leq 1 \quad \text{and} \quad |\alpha| \leq 1 \quad \text{on} \quad 0 \leq x \leq 1.$$

Thus,

$$\begin{aligned}
&\left|\oint w\,dx - (\alpha(1), \gamma'(0)) \cdot w(0)\right| \\
&\leq C_4 \cdot \{1 + \max|D_x^2\gamma'| + \|\delta\| + \|D_x\gamma'\| + \|D_x\gamma'\|^2\} \\
&\leq C_5 \cdot \{1 + \|\delta\| + \|D_x^2\gamma'\|^{1/2} \cdot \|D_x^3\gamma'\|^{1/2}\} \\
&\leq C_6 \cdot \{1 + \|\delta\| + \|D_x^3\gamma'\|^{3/4}\}.
\end{aligned}$$

We know that $(\alpha(1), \gamma'(0)) \leq 1$ and the equality holds if and only if the curve $\gamma$ is a closed geodesic. If there is a sequence $\gamma_i$ of closed curves such that $(\alpha_i(1), \gamma_i'(0)) \to 1$ for the corresponding vector field $\alpha_i$, then the sequence has a $C^1$ convergent subsequence, because the curves are $H^2$ bounded. Since the limiting curve is a closed geodesic, this contradicts the assumption. Therefore we have a positive number $C_0 < 1$ such that

$$(\alpha(1), \gamma'(0)) \leq 1 - C_0$$

for all closed curves satisfying the condition.

We choose the origin 0 so that $\oint w\,dx = w(0)$. Then

$$\begin{aligned}
&\left|\oint w\,dx - (\alpha(1), \gamma'(0)) \cdot w(0)\right| \\
&= |\{1 - (\alpha(1), \gamma'(0))\} \cdot w(0)| \\
&\geq C_0|w(0)|.
\end{aligned}$$

Thus, we see

$$|w(0)| \leq C_8 \cdot \{1 + \|\delta\| + \|D_x^3\gamma'\|^{3/4}\},$$

hence

$$\max|w| \leq |w(0)| + \|w'\| \leq C_9 \cdot \{1 + \|\delta\| + \|D_x^3\gamma'\|^{3/4}\}.$$

Therefore, we have

$$\begin{aligned}
\|D_x^3\gamma'\| &= \|\delta - R(\gamma', D_x\gamma')\gamma' - D_x(w\gamma')\| \\
&\leq C_{10} \cdot \{\|\delta\| + \|D_x\gamma'\| + \max|w| \cdot \|D_x\gamma'\| + \|w'\|\} \\
&\leq C_{11} \cdot \{1 + \|\delta\| + \|D_x^3\gamma'\|^{3/4}\},
\end{aligned}$$

and

$$\|D_x^3\gamma'\| \le C_{12}\cdot\{1+\|\delta\|\}. \qquad \square$$

Let $\gamma$ be a solution of (EP). Since $|\gamma'|\equiv 1$, we have

$$\begin{aligned}\frac{d}{dt}\|D_x\gamma'\|^2 &= 2\langle\delta, -\delta + D_x(w\gamma')\rangle = -2\|\delta\|^2 - 2\langle D_x\delta, w\gamma'\rangle \\ &= -2\|\delta\|^2 - 2\langle D_t\gamma', w\gamma'\rangle = -2\|\delta\|^2.\end{aligned}$$

Thus we have the following

**Lemma 4.7.** *For a solution $\gamma$ of* (EP), *$\|D_x\gamma'\|^2$ is non-increasing.*

**Lemma 4.8.** *For any positive real numbers $C$, $T$ and any non-negative integer $n$, there is a positive number $K$ with the following property:*
*If $\gamma$ is a solution of* (EP) *on $[0,T)$ and if $\|D_x^3\gamma'\| \le C\cdot\{1+\|\delta\|\}$, then $\|\gamma\|_n' \le K$.*

Proof. We know that $\|D_x\gamma'\| \le C_1$. From Lemma 4.5, we have

$$\frac{d}{dt}\|D_x^2\gamma'\|^2 \le C_2\cdot\{1+\|D_x^2\gamma'\|^2\cdot\|D_x^3\gamma'\|^2\} - \|D_x^4\gamma'\|^2.$$

It implies that

$$\frac{d}{dt}\log\|D_x^2\gamma'\|^2 \le C_3\cdot\{1+\|D_x^3\gamma'\|^2\}.$$

Combining it with inequality

$$\frac{d}{dt}\|D_x\gamma'\|^2 = -2\|\delta\|^2 \le -C_4\|D_x^3\gamma'\|^2 + C_5$$

which follows from the assumption, we have

$$\frac{d}{dt}(\log\|D_x^2\gamma'\|^2 + C_6\cdot\|D_x\gamma'\|^2) \le C_7.$$

Hence,

$$\|D_x^2\gamma'\| \le C_8.$$

Suppose that $\|\gamma'\|_{n+1} \le C$ for an integer $n$ ($\ge 1$). Then, Lemma 4.5 implies that

$$\frac{d}{dt}\|D_x^{n+2}\gamma'\|^2 \le C_9\cdot\{1+\|D_x^{n+3}\gamma'\|^2\} - \|D_x^{n+4}\gamma'\|^2 \le C_{10}.$$

Thus the induction completes the proof. $\square$

Proof (of Theorem 4.1). Suppose that no sequences $\gamma(*, t_i)$ converge to closed geodesics. By Lemmas 4.7 and 4.6, the assumption of Lemma 4.8 is satisfied. Therefore, for any finite time interval $[0, T)$, the solution $\gamma$ is bounded in $C^\infty$ norm. Thus the solution in Theorem 3.1 can be continued onto $[0, \infty)$. □

## 5. Convergence

In this section, we assume that the solution $\gamma$ of (EP) does not have the property (1) of Theorem 4.1. In particular, $\|D_x\gamma'\| \geq C^{-1}$ and the solution is defined for all time interval $[0, \infty)$. To show the convergence of the solution $\gamma$, we need some preparation.

**Lemma 5.1.** *For any non-negative integer $n$ and a positive real number $C$, there is a positive number $K$ with the following property*:
*If $\|\delta\|_n \leq C$, then $\|\gamma'\|_{n+3} \leq K$.*

Proof. For $n = 0$, the claim holds by Lemma 4.6. Suppose that the claim holds for $n$ and that $\|\delta\|_{n+1} \leq C$. Then we know that $\|\gamma'\|_{n+3} \leq C_1$. Thus, from Lemma 4.3, we have

$$\begin{aligned}\|D_x^{n+4}\gamma'\| &\leq C_2 \cdot \{\|D_x^{n+1}\delta\| + \|D_x^{n+1}\phi\|\} \\ &\leq C_3 \cdot \{1 + \|D_x^2\gamma'\| \cdot \|D_x^{n+3}\gamma'\|\}.\end{aligned}$$

□

**Proposition 5.2.** *For any non-negative integer $n$ and any positive number $C$, there is a positive number $K$ with the following property*:
*If $\gamma$ is a solution of* (EP) *and if $\|\delta\|_n \leq C$, then*

$$\|\partial_t w\|_{n+1} \leq K \cdot \{\|\delta\| + \|D_x^{n+3}\delta\|\}.$$

Proof. From the defining equation of $v$:

$$-v'' + |D_x\gamma'|^2 \cdot v = 2|D_x\gamma'|^4 - |D_x^2\gamma'|^2 - (R(\gamma', D_x\gamma')\gamma', D_x\gamma'),$$

we have

$$\begin{aligned}&-\partial_t v'' + |D_x\gamma'|^2 \cdot \partial_t v \\ &= -\partial_t\{|D_x\gamma'|^2\} \cdot v + \partial_t\{2|D_x\gamma'|^4 - |D_x^2\gamma'|^2 - (R(\gamma', D_x\gamma')\gamma', D_x\gamma')\} \\ &= -2(D_x\gamma', R(\delta, \gamma')\gamma' + D_x^2\delta) \cdot v \\ &\quad +8(D_x\gamma', R(\delta, \gamma')\gamma' + D_x^2\delta) \cdot |D_x\gamma'|^2 \\ &\quad -2(D_x^2\gamma', R(\delta, \gamma')D_x\gamma' + D_x\{R(\delta, \gamma')\gamma'\} + D_x^3\delta) \\ &\quad -((D_\delta R)(\gamma', D_x\gamma')\gamma', D_x\gamma') - 2(R(D_x\delta, D_x\gamma')\gamma', D_x\gamma') \\ &\quad -2(R(\gamma', R(\delta, \gamma')\gamma' + D_x^2\delta)\gamma', D_x\gamma').\end{aligned}$$

By Lemma 5.1, the assumption implies that $\|\gamma'\|_{n+3} \leq C_1$. Hence,

$$\text{(the } H^n \text{ norm of the last expression)} \leq C_2 \cdot \{\|\delta\| + \|D_x^{n+3}\delta\|\}.$$

Therefore, Lemma 1.3 implies that

$$\begin{aligned} \|\partial_t v\|_1 &\leq C_3 \cdot \{\|\delta\| + \|D_x^3\delta\|\}, \\ \|\partial_t v\|_{n+1} &\leq C_4 \cdot \{\|\delta\| + \|D_x^{n+2}\delta\|\} \quad (\text{when } n \geq 1). \end{aligned}$$

Moreover, from

$$\partial_t\{|D_x\gamma'|^2\} = 2(D_x\gamma', R(\delta, \gamma')\gamma' + D_x^2\delta),$$

we have

$$\|\partial_t\{|D_x\gamma'|^2\}\|_{n+1} \leq C_5 \cdot \{\|\delta\| + \|D_x^{n+3}\delta\|\}.$$

Thus the claim holds for any non-negative integer $n$. □

**Lemma 5.3.** *The norm $\|\delta\|$ tends to 0 when $t \to \infty$. The integrals*

$$\int_0^\infty \|\delta\|^2\, dt, \quad \int_0^\infty \|D_x^2\delta\|^2\, dt$$

*are finite.*

Proof. We have

$$\int_0^\infty \|\delta\|^2\, dt = \int_0^\infty -\frac{1}{2}\frac{d}{dt}\|D_x\gamma'\|^2\, dt = -\frac{1}{2}\left[\|D_x\gamma'\|^2\right]_0^\infty < \infty.$$

Moreover,

$$\begin{aligned} \frac{d}{dt}\|\delta\|^2 &= 2\langle\delta, D_t\delta\rangle \\ &= 2\langle\delta, -D_tD_x^3\gamma' + D_t(R(\gamma', D_x\gamma')\gamma') + D_tD_x(w\gamma')\rangle \\ &= 2\langle\delta, -R(\delta, \gamma')D_x^2\gamma' - D_x(R(\delta, \gamma')D_x\gamma') \\ &\qquad -D_x^2(R(\delta, \gamma')\gamma') - D_x^4\delta \\ &\qquad +(D_\delta R)(\gamma', D_x\gamma')\gamma' - R(D_x\delta, D_x\gamma')\gamma' \\ &\qquad +R(\gamma', R(\delta, \gamma')\gamma' + D_x^2\delta)\gamma' \\ &\qquad +R(\gamma', D_x\gamma')D_x\delta \\ &\qquad +R(\delta, \gamma')(w\gamma') + D_x\{-\partial_t w \cdot \gamma' + w \cdot D_x\delta\}\rangle. \end{aligned}$$

Here, from Lemmas 4.6 and 4.2, we know

$$\begin{aligned}\|D_x^3\gamma'\| &\le C_1\cdot\{1+\|\delta\|\},\\ \|w\|_1 &\le C_2\cdot\{1+\|D_x^2\gamma'\|^2\}\le C_3\cdot\{1+\|D_x^3\gamma'\|\}\le C_4\cdot\{1+\|\delta\|\}.\end{aligned}$$

Thus, using equation: $(D_x\delta, \gamma') = 0$,

$$\begin{aligned}\frac{d}{dt}\|\delta\|^2 &\le -2\|D_x^2\delta\|^2 - 2\langle D_x\delta, \partial_t w\cdot\gamma'\rangle\\ &\quad + C_5\cdot\|\delta\|\cdot\{1+\|\delta\|^N\}\cdot\{\|\delta\|+\|D_x^2\delta\|\}\\ &\le -\|D_x^2\delta\|^2 + C_6\cdot\|\delta\|^2\cdot\{1+\|\delta\|^N\},\end{aligned}$$

where $N$ is an absolute constant.

Thus $\|\delta\|$ tends to 0. In particular,

$$\frac{d}{dt}\|\delta\|^2 \le -\|D_x^2\delta\|^2 + C_7\cdot\|\delta\|^2.$$

Therefore,

$$\begin{aligned}\int_0^\infty \|D_x^2\delta\|^2\,dt &\le C_7\int_0^\infty \|\delta\|^2\,dt - \int_0^\infty \frac{d}{dt}\|\delta\|^2\,dt\\ &< \infty.\end{aligned}$$

□

**Lemma 5.4.** *For any non-negative even integer $n$, $\|D_x^n\delta\|$ tends to 0 when $t\to\infty$.*

Proof. Suppose

$$\|D_x^n\delta\|\to 0,\qquad \int_0^\infty \|D_x^{n+2}\delta\|^2\,dt < \infty$$

for a non-negative even integer $n$. This holds for $n = 0$ by Lemma 5.3.

As in the proof of Lemma 5.3, we have

$$\begin{aligned}\frac{d}{dt}\|D_x^{n+2}\delta\|^2 &= 2\langle D_x^{n+2}\delta, D_t D_x^{n+2}\delta\rangle\\ &= 2\left\langle D_x^{n+2}\delta, \sum_{i=0}^{n+1} D_x^i(R(\delta,\gamma')D_x^{n+1-i}\delta) + D_x^{n+2}D_t\delta\right\rangle\\ &= 2\langle D_x^{n+2}\delta, R(\delta,\gamma')D_x^{n+1}\delta\rangle - 2\langle D_x^{n+3}\delta, R(\delta,\gamma')D_x^n\delta\rangle\\ &\quad +2\sum_{i=2}^{n+1}\langle D_x^{n+4}\delta, D_x^{i-2}(R(\delta,\gamma')D_x^{n+1-i}\delta)\rangle_{(\#2)} + 2\langle D_x^{n+4}\delta, D_x^n\delta\rangle,\end{aligned}$$

and $D_x^n\delta$ in the last term is expanded as

$$
\begin{aligned}
&D_x^n\{-R(\delta,\gamma')D_x^2\gamma' - D_x(R(\delta,\gamma')D_x\gamma') - D_x^2(R(\delta,\gamma')\gamma') - D_x^4\delta \\
&\quad +(D_\delta R)(\gamma', D_x\gamma')\gamma' + R(D_x\delta, D_x\gamma')\gamma' + R(\gamma', R(\delta,\gamma')\gamma' + D_x^2\delta)\gamma' \\
&\quad +R(\gamma', D_x\gamma')D_x\delta + R(\delta,\gamma')(w\gamma') + D_x\{\partial_t w\cdot\gamma' + wD_x\delta\}\}.
\end{aligned}
$$

Form the assumption, Lemma 5.1 implies that $\|\gamma'\|_{n+3} \le C_1$. Therefore, Lemma 4.2 implies that $\|w\|_{n+2} \le C_2$, and Lemma 5.2 implies that $\|\partial_t w\|_{n+1} \le C_3\cdot\{\|D_x^{n+3}\delta\| + \|\delta\|\}$. Moreover, we know that $\max|\delta| \le C_4\cdot\{1+\|D_x\delta\|^{1/2}\}$ and $\|D_x^{n+1}\delta\| \le C_5\cdot\{1+\|D_x^{n+2}\delta\|^{1/2}\}$. Thus all terms in the last expression except the term

$$2\langle D_x^{n+4}\delta, -D_x^n D_x^4\delta\rangle = -2\|D_x^{n+4}\delta\|^2$$

are bounded by the form $C_6\cdot\|D_x^{n+4}\delta\|\cdot\{\|\delta\| + \|D_x^{n+3}\delta\|\}$. Therefore,

$$
\begin{aligned}
\frac{d}{dt}\|D_x^{n+2}\delta\|^2 &\le -\|D_x^{n+4}\delta\|^2 + C_7\cdot\{\|\delta\|^2 + \|D_x^{n+3}\delta\|^2\} \\
&\le -\frac{1}{2}\|D_x^{n+4}\delta\|^2 + C_8\cdot\|\delta\|^2.
\end{aligned}
$$

Thus we have $\|D_x^{n+2}\delta\| \to 0$ and $\int_0^\infty \|D_x^{n+4}\delta\|^2\,dt$ is finite. □

Note that Lemma 5.4 holds on any compact $C^\infty$ riemannian manifold satisfying the assumption of this section. In particular, $\delta$ converges to 0 in $C^\infty$ topology when $t$ tends to $\infty$. Combining it with Lemma 5.1, we have the boundedness of the solution $\gamma$.

**Theorem 5.5.** *Let $M$ be a compact riemannian manifold, and let $\gamma_0(x)$ be a closed curve with unit line element and length $L$. If there are no closed geodesics of length $L$ in the manifold $M$, then* (EP) *has a unique solution $\gamma(x,t)$ for all time, and the solution has a subsequence converging to an elastica.*

If the metric is real analytic, we have the main result.

**Theorem 5.6.** *Let $M$ be a compact real analytic riemannian manifold, and let $\gamma_0(x)$ be a closed curve with $|\gamma_0'| = 1$ and length $L$. If there are no geodesics of length $L$ in the manifold $M$, then* (EP) *has a unique solution $\gamma(x,t)$ for all time, and the solution converges to an elastica when $t \to \infty$.*

Proof. The proof of Theorem 8.6 of [3] remains valid. We use Simon's real analytic implicit function theorem. For detail, see [3]. □

REMARK 5.7. We have an example of almost oscillate solution on a $C^{\infty}$ riemannian manifold. See [2].

## References


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