Nagami, S.
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# EXISTENCE OF SPIN STRUCTURES ON DOUBLE BRANCHED COVERING SPACES OVER FOUR-MANIFOLDS

Dedicated to the memory of Professor Katsuo Kawakubo

SEIJI NAGAMI

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## 1. Introduction

In this paper, we shall consider whether a double branched covering space over a given 4-manifold is spin or not. It is a fact known to experts that the double branched covering space $\widetilde{X}_{S^4}$ over the 4-sphere $S^4$ branched along a smoothly embedded connected closed surface $F$ is spin if and only if $F$ is orientable. However, the proof of this fact has never been published before as far as the author knows.

Hitchin [7] has shown that the double branched covering space $\widetilde{X}^q_{\mathbf{C}P^2}$ over the complex projective plane $\mathbf{C}P^2$ branched along a non-singular algebraic curve $C$ of degree $2q$ is spin if and only if $q$ is odd. Note that $\widetilde{X}^q_{\mathbf{C}P^2}$ is simply-connected [7, p. 22] and that $\widetilde{X}^3_{\mathbf{C}P^2}$ is a *K3* surface [5, p. 28].

The base spaces $S^4$ and $\mathbf{C}P^2$ of $\widetilde{X}_{S^4}$ and $\widetilde{X}^q_{\mathbf{C}P^2}$ respectively are both simply connected. The former is spin, and the latter is not. The total space $\widetilde{X}_{S^4}$ is spin when $F$ is orientable, and the total space $\widetilde{X}^q_{\mathbf{C}P^2}$ is not necessarily spin, although the branch locus $C$ is always orientable. Whether the double branched covering space $\widetilde{X}^q_{\mathbf{C}P^2}$ is spin or not depends on the homology class represented by the branch locus $C$ in the base 4-manifold $\mathbf{C}P^2$.

Moreover, until now, there has been no example of a *spin* double branched covering space over a simply connected closed smooth 4-manifold branched along a smoothly embedded *non-orientable* surface.

In this paper, we consider a more general case. Let $X$ be an oriented connected closed smooth 4-manifold with $H_1(X;\mathbf{Z}_2) = 0$ and $\widetilde{X}$ the double branched covering space over $X$ branched along a connected closed surface $F$ smoothly embedded in $X$. We consider whether $\widetilde{X}$ is spin or not.

Our result for this problem is the following.

**Theorem 1.1.** *Let $X$ be a connected closed smooth 4-manifold with $H_1(X;\mathbf{Z}_2) = 0$ and $\widetilde{X}$ the double branched covering space over $X$ branched along a connected closed surface $F$ smoothly embedded in $X$. Then $\widetilde{X}$ is spin if and only if $F$ is orientable and the modulo 2 reduction of $[F]/2 \in H_2(X;\mathbf{Z})$ coincides with the Poincaré* *dual of the second Stiefel-Whitney class $w_2(X)$ of $X$ for a fixed orientation of $F$.*

Here, $[F]/2 \in H_2(X;\mathbf{Z})$ denotes the unique homology class such that $2([F]/2)$ coincides with the homology class $[F] \in H_2(X;\mathbf{Z})$ represented by $F$ with coefficient in $\mathbf{Z}$. (Since there exists a double branched covering space $\widetilde{X}$ branched along $F$, $[F]$ is divisible by 2. See §2 (II), Definition 2.6 and Corollary 2.10.)

Theorem 1.1 is a generalization of the above-mentioned fact for the case $X = S^4$ and the result of Hitchin for the case $X = \mathbf{C}P^2$.

REMARK 1.2. Edmonds has shown the following theorem in [4]: each component of the fixed point set of an involution on a smooth spin manifold which preserves a spin structure and orientation is orientable. Since $H_1(\widetilde{X};\mathbf{Z}_2) = 0$ in our situation (see Lemma 3.8), the above theorem of Edmonds gives another proof of the fact that $\widetilde{X}$ is not spin when $F$ is non-orientable.

This paper is organized as follows: In Section 2, we give some notation and definitions, and review the condition for a given closed surface $F$ embedded in a 4-manifold $X$ to admit a double branched covering space over $X$ branched along it; in Section 3, we give several lemmas for the proof of Theorem 1.1; in Sections 4 we prove Theorem 1.1 in the case where $X$ is spin (spin case); in Section 5, we prove Theorem 1.1 in the case where $X$ is not spin (nonspin case); in Section 6, we conclude this paper by giving some examples.

## 2. Preliminaries

Throughout this paper, let $X$ be an oriented connected closed smooth 4-manifold with $H_1(X;\mathbf{Z}_2) = 0$ and $F$ a connected closed surface smoothly embedded in $X$. In this section, first, we give some notation and definitions. Then we compute the first (co)homology group of the complement of the branch locus $F$ in the base manifold $X$. As a corollary, we obtain a necessary and sufficient condition for $F$ to admit a double branched covering space over $X$ branched along $F$.

**2.1. Notation and Definitions** (I) Let $L$ be a closed tubular neighborhood of $F$ in $X$. Let $W$ denote the closure of the complement of $L$ in $X$. Let $\widetilde{X}$ be the double branched covering space over $X$ branched along $F$. Let $\widetilde{F}$ denote the inverse image of $F$ under $\pi$, where $\pi : \widetilde{X} \to X$ denotes the covering projection. Let $\widetilde{L}$ (resp. $\widetilde{W}$) denote the inverse image of $L$ (resp. $W$) under $\pi$. Let $\tau : \widetilde{X} \to \widetilde{X}$ denote the covering transformation map. Note that $p = \pi|\widetilde{W} : \widetilde{W} \to W$ is a regular 2-fold covering map and that $\widetilde{L}$ is a closed tubular neighborhood of $\widetilde{F}$.

$$\begin{array}{c}\widetilde{F} \subset \widetilde{L} \cup \widetilde{W} = \widetilde{X} \\ \Big\downarrow \pi \\ F \subset L \cup W = X\end{array}$$

DEFINITION 2.1. $\partial L$ (resp. $\partial \widetilde{L}$) has an $S^1$-bundle structure over $F$ (resp. $\widetilde{F}$). We call each of their fibers a *meridian*. If $F$ is oriented, then we choose an orientation for each of the meridians in $\partial L$ as follows. $L$ has an orientable $D^2$-bundle structure over $F$. Let each of the fibers of the $D^2$-bundle be oriented so that the product orientation of $T_xF \oplus T_xD_x^2$ coincides with the orientation of $T_xX$ for each $x \in F$, where $D_x^2$ denotes the fiber over $x$ and $T_xF$, $T_xD_x^2$ and $T_xX$ denote the respective tangent spaces. Since the boundary of each fiber is a meridian, it has the induced orientation. We call a meridian with this orientation an *oriented meridian*. If $F$ is non-orientable, then we call a meridian with an (arbitrary) orientation fixed an *oriented meridian*.

(II) Let $Q$ be an oriented closed smooth manifold and $G$ a closed $n$-dimensional manifold smoothly embedded in $Q$. We denote by $[G]_2 \in H_n(Q;\mathbf{Z}_2)$ the homology class represented by $G$ with coefficients in $\mathbf{Z}_2$. When $G$ is oriented, we denote by $[G] \in H_n(Q;\mathbf{Z})$ the homology class represented by $G$ with coefficients in $\mathbf{Z}$.

For two homology classes $x \in H_r(Q;\mathbf{Z})$ and $y \in H_{n-r}(Q;\mathbf{Z})$ (resp. $x_2 \in H_r(Q;\mathbf{Z}_2)$ and $y_2 \in H_{n-r}(Q;\mathbf{Z}_2)$), we denote by $x \cdot y \in \mathbf{Z}$ (resp. $x_2 \cdot y_2 \in \mathbf{Z}_2$) the intersection number of $x$ and $y$ (resp. $x_2$ and $y_2$) in $Q$.

REMARK 2.2. The normal Euler number of $F$ in $X$ is twice the normal Euler number of $\widetilde{F}$ in $\widetilde{X}$ (see [12]); moreover, when $F$ is oriented, we have $\pi_*(d) \cdot [F] = 2d \cdot [\widetilde{F}]$ for all $d \in H_2(\widetilde{X};\mathbf{Z})$.

DEFINITION 2.3. Let $Q$ be an oriented closed smooth 4-manifold. Then a $\mathbf{Z}_2$-homology class $c_2 \in H_2(Q;\mathbf{Z}_2)$ is *characteristic* if $c_2$ is equal to the Poincaré dual of the second Stiefel-Whitney class $w_2(Q)$ of $Q$. A $\mathbf{Z}$-homology class $c \in H_2(Q;\mathbf{Z})$ is *characteristic* if its modulo 2 reduction coincides with the Poincaré dual of $w_2(Q)$. A closed surface $G$ embedded in $Q$ is a *characteristic surface* if $[G]_2 \in H_2(Q;\mathbf{Z}_2)$ is characteristic.

REMARK 2.4. (1) The characteristic homology class $c_2 \in H_2(X;\mathbf{Z}_2)$ is the unique class which satisfies $c_2 \cdot x_2 = x_2 \cdot x_2$ for all $x_2 \in H_2(X;\mathbf{Z}_2)$ [9, p. 27].

(2) For any $x \in H_2(Q;\mathbf{Z})$ (resp. $x_2 \in H_2(Q;\mathbf{Z}_2)$), there exists an oriented closed surface (resp. a closed surface) $G$ smoothly embedded in $Q$ such that $[G] = x$ (resp. $[G]_2 = x_2$) (see [9, p. 20]). In particular, a characteristic surface of $Q$ always exists.

(III) Let $H$ be a finitely generated $\mathbf{Z}$-module. We denote by $\mathrm{Tor}(H)$ the torsion subgroup of $H$.

DEFINITION 2.5. Let $H$ be a finitely generated free $\mathbf{Z}$-module. Then a nonzero element $x \in H$ is *primitive* if it is not divisible by any integer other than $\pm 1$. Let $H'$ be a finitely generated $\mathbf{Z}$-module. Then a nonzero element $x \in H'$ is *primitive* if $[x] \in H'/\mathrm{Tor}(H')$ is primitive, where $H'/\mathrm{Tor}(H')$ denotes the quotient module and $[x]$ the equivalence class represented by $x$.

DEFINITION 2.6. Suppose $F$ is oriented. When $[F] \in H_2(X;\mathbf{Z})$ is of infinite order, there exist a positive integer $\tilde{a} \in \mathbf{N}$, a torsion element $\tilde{\zeta} \in H_2(X;\mathbf{Z})$ and a primitive element $\tilde{x} \in H_2(X;\mathbf{Z})$ such that $[F] = \tilde{a}\tilde{x} + \tilde{\zeta}$. Note that $\tilde{a}$ is independent of the choice of such a representation. We denote $\tilde{a}$ by $a_F$. When $[F]$ is of finite order, define $a_F = 0 \in \mathbf{Z}$. Note that if $a_F$ is even, then there exists a unique element $e \in H_2(X;\mathbf{Z})$ such that $2e = [F]$, since $H_1(X;\mathbf{Z}_2) = 0$. We denote $e$ by $[F]/2$.

REMARK 2.7. In the notation of Definition 2.6, whether $\tilde{x}$ is characteristic or not depends only on the homology class $[F]$.

### 2.2. Computation of $H_1(W;\mathbf{Z})$

**Proposition 2.8.** *Let $X$ be an oriented connected closed smooth 4-manifold which satisfies $H_1(X;\mathbf{Z}_2) = 0$ and $F$ a connected closed surface smoothly embedded in $X$.*

(1) *If $F$ is oriented and $a_F \neq 0$, then $H_1(W;\mathbf{Z})$ contains a cyclic group $\widetilde{H}$ of order $a_F$ which is generated by the homology class represented by an oriented meridian such that the index $|H_1(W;\mathbf{Z}) : \widetilde{H}|$ is odd. If $F$ is oriented and $a_F = 0$, then $H_1(W;\mathbf{Z})$ contains an infinite cyclic group $\widetilde{H}$ which is generated by the homology class represented by an oriented meridian such that the index $|H_1(W;\mathbf{Z}) : \widetilde{H}|$ is odd.*

(2) *Suppose that $F$ is non-orientable. If $[F]_2 = 0$, then $H_1(W;\mathbf{Z})$ contains a cyclic group $\widetilde{H}$ of order 2 which is generated by the homology class represented by an oriented meridian such that the index $|H_1(W;\mathbf{Z}) : \widetilde{H}|$ is odd. If $[F]_2 \neq 0$, then the order of $H_1(W;\mathbf{Z})$ is odd.*

(3) *If $F$ is oriented and $a_F = 0$, then $H^1(W;\mathbf{Z}) \cong \mathbf{Z}$. Otherwise, $H^1(W;\mathbf{Z}) = 0$ holds. In each case, $H^1(W;\mathbf{Z}_2) \cong \mathbf{Z}_2$ if and only if $[F]_2 = 0$.*

Proof. Consider the following exact sequence for the pair $(X, W)$, where $k_*$ denotes the homomorphism induced by the inclusion $k : X \to (X, W)$ and $\partial$ the connecting homomorphism:

$$H_2(X;\mathbf{Z}) \xrightarrow{k_*} H_2(X, W;\mathbf{Z}) \xrightarrow{\partial} H_1(W;\mathbf{Z}) \longrightarrow H_1(X;\mathbf{Z}).$$

Set $\widetilde{H} = \partial(H_2(X, W; \mathbf{Z}))$. Recall that $L$ has a $D^2$-bundle structure over $F$. Let $u : (D^2, \partial D^2) \to (L, \partial L) \subset (X, W)$ be an embedding (orientation preserving when $F$ is oriented) onto one of its fibers (see Definition 2.1). We denote by $[D^2, \partial D^2] \in H_2(X, W; \mathbf{Z})$ the homology class represented by $u$. Since $H_2(X, W; \mathbf{Z}) \cong H_2(L, \partial L; \mathbf{Z}) \cong \mathbf{Z}$ (resp. $\mathbf{Z}_2$) by excision when $F$ is oriented (resp. non-orientable) and $H_2(X, W; \mathbf{Z})$ is generated by $[D^2, \partial D^2]$, $\widetilde{H}$ is a cyclic group which is generated by the homology class represented by an oriented meridian. By the exactness of the above sequence, we have only to determine the order of $\widetilde{H}$ to complete the proof of (1) and (2), since $H_1(X; \mathbf{Z})$ is a finite module without 2-torsion by our assumption that $H_1(X; \mathbf{Z}_2) = 0$.

(1) Suppose $a_F \neq 0$. Let a primitive element $x_F \in H_2(X; \mathbf{Z})$ and a torsion element $\zeta_F \in H_2(X; \mathbf{Z})$ be such that $[F] = a_F x_F + \zeta_F$. Let $y \in H_2(X; \mathbf{Z})$ be such that $y \cdot x_F = 1$. Let a positive integer $n$ denote the order of $\widetilde{H}$. Then we have $y \cdot [F] = a_F$ and $k_*(y) = a_F[D^2, \partial D^2]$. Hence $a_F$ is divisible by $n$. Since we can construct a cycle representing $z \in H_2(X; \mathbf{Z})$ such that $z \cdot [F] = n$, $n$ must be divisible by $a_F$. Therefore $n = a_F$ and $\widetilde{H} \cong \mathbf{Z}_{a_F}$. Next suppose $a_F = 0$. If $k_*$ is not the zero map, then there exists a homology class $z \in H_2(X; \mathbf{Z})$ such that $z \cdot [F] \neq 0$. This is a contradiction. Hence $k_*$ is the zero map and $\widetilde{H} \cong \mathbf{Z}$.

(2) We have only to show that $\widetilde{H} \cong \mathbf{Z}_2$ if and only if $[F]_2 = 0$, since $H_2(X, W; \mathbf{Z}) \cong \mathbf{Z}_2$. For this, it suffices to show that $[F]_2 = 0$ if and only if $H_1(W; \mathbf{Z}_2) \cong \mathbf{Z}_2$, since $H_1(W; \mathbf{Z}) \otimes \mathbf{Z}_2 \cong H_1(W; \mathbf{Z}_2)$. Consider the following exact sequence for the pair with coefficients in $\mathbf{Z}_2$, where $(k_*)_2$ denotes the homomorphism induced by $k$ and $(\partial)_2$ the connecting homomorphism:

$$H_2(X; \mathbf{Z}_2) \xrightarrow{(k_*)_2} H_2(X, W; \mathbf{Z}_2) \xrightarrow{(\partial)_2} H_1(W; \mathbf{Z}_2) \longrightarrow H_1(X; \mathbf{Z}_2) = 0.$$

Suppose $[F]_2 = 0$. If $(k_*)_2$ is not the zero map, then there exists a homology class $y_2 \in H_2(X; \mathbf{Z}_2)$ such that $y_2 \cdot [F]_2 = 1$. This is a contradiction and $(k_*)_2$ is the zero map. Hence $\mathbf{Z}_2 \cong H_2(X, W; \mathbf{Z}_2) \cong H_1(W; \mathbf{Z}_2)$. Next suppose $[F]_2 \neq 0$. Then there exists a homology class $y_2 \in H_2(X; \mathbf{Z}_2)$ such that $y_2 \cdot [F]_2 = 1$. Then we have $(k_*)_2(y_2) = 1 \in H_2(X, W; \mathbf{Z}_2) \cong \mathbf{Z}_2$. Hence $(k_*)_2$ is a surjection and $(\partial)_2$ is the zero map. Hence $H_1(W; \mathbf{Z}_2) = 0$.

(3) follows from (1) and (2) together with the following isomorphisms which are consequences of the universal coefficient theorem for cohomology:

$$H^1(W; \mathbf{Z}) \cong \mathrm{Hom}(H_1(W; \mathbf{Z}), \mathbf{Z}) \oplus \mathrm{Ext}(H_0(W; \mathbf{Z}), \mathbf{Z}),$$

$$H^1(W; \mathbf{Z}_2) \cong \mathrm{Hom}(H_1(W; \mathbf{Z}_2), \mathbf{Z}_2) \oplus \mathrm{Ext}(H_0(W; \mathbf{Z}_2), \mathbf{Z}_2).$$

This completes the proof of Proposition 2.8. □

REMARK 2.9. Let $H$ be a finitely generated $\mathbf{Z}$-module. If $H$ contains a cyclic subgroup $I$ of even order such that the index $|\mathrm{Tor}(H): I|$ is odd, then there exists a unique nonzero element $j \in H$ such that $2j = 0$. Note that when $[F]_2 = 0$, $H_1(W;\mathbf{Z})$ satisfies this condition unless $F$ is orientable and $a_F = 0$ for a fixed orientation of $F$ by the above proposition.

Applying Proposition 2.8, we have the following corollary (see [3, p. 122], [16, p. 41]).

**Corollary 2.10.** *A double branched covering space $\widetilde{X}$ over $X$ branched along $F$ exists if and only if $[F]_2 = 0$.*

REMARK 2.11. For a given surface $F$ embedded in $X$, $\widetilde{X}$ is determined uniquely up to equivalence, since we have assumed that $H_1(X;\mathbf{Z}_2) = 0$.

We conclude this section with a fact, which will be used in Sections 3 and 5.

**Fact 2.12.** For a topological space $Y$, let $Sq^1 : H^1(Y;\mathbf{Z}_2) \to H^2(Y;\mathbf{Z}_2)$ denote the Steenrod squaring operation, $\rho_Y : H^2(Y;\mathbf{Z}) \to H^2(Y;\mathbf{Z}_2)$ the reduction modulo 2, and $\beta_Y : H^1(Y;\mathbf{Z}_2) \to H^2(Y;\mathbf{Z})$ the Bockstein homomorphism. Then we have $Sq^1 = \rho_Y \circ \beta_Y$.

See [17, p. 280, G 2, 5] for a proof of this fact. See also [12, p. 148].

## 3. Some Lemmas

In this section, we give several lemmas which we need for the proof of Theorem 1.1. From now on, we assume that the homology class represented by the branch locus $F$ with coefficients in $\mathbf{Z}_2$ vanishes, i.e., $[F]_2 = 0 \in H_2(X;\mathbf{Z}_2)$. Then by Corollary 2.10, we have the double branched covering space $\widetilde{X}$ over $X$ branched along $F$ (see Remark 2.11).

First we compute the first (co)homology group of $\widetilde{W}$ with coefficients in $\mathbf{Z}_2$. For the computation of $H_1(\widetilde{W};\mathbf{Z}_2) \cong H^1(\widetilde{W};\mathbf{Z}_2)$, we prepare two lemmas. The first lemma is shown by using Proposition 2.8 and the universal coefficient theorem.

**Lemma 3.1.** (1) *Suppose $F$ is oriented. If $a_F \neq 0$, then $H^2(W;\mathbf{Z})$ contains $\mathbf{Z}_{a_F}$ as a submodule such that the index $|\mathrm{Tor}(H^2(W;\mathbf{Z})) : \mathbf{Z}_{a_F}|$ is odd. If $a_F = 0$, then $H^2(W;\mathbf{Z})$ has no 2-torsion.*
(2) *If $F$ is non-orientable, then $H^2(W;\mathbf{Z})$ contains $\mathbf{Z}_2$ as a submodule such that the index $|\mathrm{Tor}(H^2(W;\mathbf{Z})) : \mathbf{Z}_2|$ is odd.*

*In particular, if $F$ is oriented and $a_F \neq 0$, or if $F$ is non-orientable, then there exists a unique nonzero element $j_W \in H^2(W;\mathbf{Z})$ such that $2j_W = 0$ by* Remark 2.9.

Before giving the second lemma, consider the following Gysin exact sequence for the 2-fold covering $p : \widetilde{W} \to W$, where $w_1 \in H^1(W;\mathbf{Z}_2) \cong \mathbf{Z}_2$ denotes the generator and $(p^*)_2$ the homomorphisms induced by the covering map $p : \widetilde{W} \to W$:

$$\begin{aligned} 0 &\longrightarrow H^0(W;\mathbf{Z}_2) \xrightarrow{(p^*)_2} H^0(\widetilde{W};\mathbf{Z}_2) \longrightarrow H^0(W;\mathbf{Z}_2) \\ &\xrightarrow{\cup w_1} H^1(W;\mathbf{Z}_2) \xrightarrow{(p^*)_2} H^1(\widetilde{W};\mathbf{Z}_2) \longrightarrow H^1(W;\mathbf{Z}_2) \\ &\xrightarrow{\cup w_1} H^2(W;\mathbf{Z}_2) \xrightarrow{(p^*)_2} H^2(\widetilde{W};\mathbf{Z}_2). \end{aligned}$$

Since $H^0(W;\mathbf{Z}_2) \cong H^0(\widetilde{W};\mathbf{Z}_2) \cong H^1(W;\mathbf{Z}_2) \cong \mathbf{Z}_2$ by the connectedness of $W$ and Proposition 2.8 (3), we obtain the following exact sequence:

$$0 \longrightarrow H^1(\widetilde{W};\mathbf{Z}_2) \longrightarrow H^1(W;\mathbf{Z}_2) \xrightarrow{\cup w_1} H^2(W;\mathbf{Z}_2) \xrightarrow{(p*)_2} H^2(\widetilde{W};\mathbf{Z}_2).$$

The following lemma is an immediate consequence of the above sequence.

**Lemma 3.2.**
(1) $H^1(\widetilde{W};\mathbf{Z}_2) = 0$ *if and only if* $H^1(W;\mathbf{Z}_2) \xrightarrow{\cup w_1} H^2(W;\mathbf{Z}_2)$ *is an injection.*
(2) $H^1(\widetilde{W};\mathbf{Z}_2) \cong \mathbf{Z}_2$ *if and only if* $H^1(W;\mathbf{Z}_2) \xrightarrow{\cup w_1} H^2(W;\mathbf{Z}_2)$ *is the zero map.*

By applying Fact 2.12, Lemmas 3.1 and 3.2, we now compute $H_1(\widetilde{W};\mathbf{Z}_2) \cong H^1(\widetilde{W};\mathbf{Z}_2)$.

**Lemma 3.3.**
(1) *Suppose $F$ is oriented.*
   (a) *If $a_F \equiv 0 \mod 4$, then $H_1(\widetilde{W};\mathbf{Z}_2) \cong H^1(\widetilde{W};\mathbf{Z}_2) \cong \mathbf{Z}_2$. The generator of $H_1(\widetilde{W};\mathbf{Z}_2)$ is the homology class represented by a meridian.*
   (b) *If $a_F \equiv 2 \mod 4$, then $H_1(\widetilde{W};\mathbf{Z}_2) \cong H^1(\widetilde{W};\mathbf{Z}_2) = 0$.*
(2) *If $F$ is non-orientable, then $H_1(\widetilde{W};\mathbf{Z}_2) \cong H^1(\widetilde{W};\mathbf{Z}_2) = 0$.*

Proof. (1)(a) Since $H^1(W;\mathbf{Z}_2) \cong \mathbf{Z}_2$ by Proposition 2.8 (3), the map $H^1(W;\mathbf{Z}_2) \xrightarrow{\cup w_1} H^2(W;\mathbf{Z}_2)$ is either an injection or the zero map. Then by Lemma 3.2, either $H^1(\widetilde{W};\mathbf{Z}_2) = 0$ or $H^1(\widetilde{W};\mathbf{Z}_2) \cong \mathbf{Z}_2$ holds. Hence, since $H^1(\widetilde{W};\mathbf{Z}_2) \cong H_1(\widetilde{W};\mathbf{Z}_2)$, we have only to show that the homology class represented by a meridian with coefficients in $\mathbf{Z}_2$ does not vanish. First note the following. For a connected manifold $Q$, let $s$ be an embedded 1-sphere in $Q$. Then $[s]_2 \neq 0 \in H_1(Q;\mathbf{Z}_2)$ if and only if there exists a regular 2-fold covering $q : \widetilde{Q} \to Q$ such that $q^{-1}(s)$ is connected. Since $a_F \equiv 0 \mod 4$, $H_1(W;\mathbf{Z}_4) \cong \mathbf{Z}_4$ by virtue of Proposition 2.8 (1). Then we have a connected cyclic regular 4-fold covering $p' : W' \to W$ such that $W'/\tau'^2 = \widetilde{W}$, where $\tau' : W' \to W'$ denotes the covering transformation map and $W'/\tau'^2$ is the quotient manifold. Let $p'' : W' \to \widetilde{W}$ denote the quotient map. Then $p''$ is a regular 2-fold covering map. Let $S^1_W$ be an oriented meridian in $W$ and set $S^1_{\widetilde{W}} = p^{-1}(S^1_W)$. Then $S^1_{\widetilde{W}}$ is a meridian in $\widetilde{W}$ and $p^{''-1}(S^1_{\widetilde{W}}) = p^{'-1}(S^1_W)$. Hence the inverse image of $S^1_{\widetilde{W}}$ under $p^{''}$ is connected, and the homology class represented by $S^1_{\widetilde{W}}$ with coefficients in $\mathbf{Z}_2$ does not vanish. This completes the proof of (1)(a).

(1)(b) First note that $H^2(W;\mathbf{Z})$ contains a unique nonzero element $j_W \in H^2(W;\mathbf{Z})$ such that $2j_W = 0$ by Lemma 3.1. Assume $a_F = 2a$ and $a \equiv 1 \mod 2$. Since $H^1(W;\mathbf{Z}_2) \cong \mathbf{Z}_2$ by Proposition 2.8 (3), the map $H^1(W;\mathbf{Z}_2) \xrightarrow{\cup w_1} H^2(W;\mathbf{Z}_2)$ coincides with the map $Sq^1 : H^1(W;\mathbf{Z}_2) \to H^2(W;\mathbf{Z}_2)$. By Fact 2.12, $Sq^1$ coincides with the composite $H^1(W;\mathbf{Z}_2) \xrightarrow{\beta_W} H^2(W;\mathbf{Z}) \xrightarrow{\rho_W} H^2(W;\mathbf{Z}_2)$, where $\beta_W$ denotes the Bockstein homomorphism and $\rho_W$ the reduction modulo 2. Since $a_F \not\equiv 0$, $H^1(W;\mathbf{Z}) = 0$ holds by Proposition 2.8 (3). Hence we see that $\beta_W$ is an injection by the following exact sequence associated with the coefficient sequence $0 \to \mathbf{Z} \xrightarrow{\times 2} \mathbf{Z} \to \mathbf{Z}_2 \to 0$:

$$0 = H^1(W;\mathbf{Z}) \longrightarrow H^1(W;\mathbf{Z}_2) \xrightarrow{\beta_W} H^2(W;\mathbf{Z}).$$

Therefore, we have $\beta_W(w_1) = j_W \in H^2(W;\mathbf{Z})$. If $\rho_W(j_W) = 0$, then there exists an element $a'$ in $H^2(W;\mathbf{Z})$ such that $2a' = j_W$. Then we see that $a'$ and $j_W$ are contained in the submodule $\mathbf{Z}_{2a}$ of $H^2(W;\mathbf{Z})$ which is described in Lemma 3.1 (1). Since $a$ is odd, there is no element $b \in \mathbf{Z}_{2a}$ such that $2b = a$. Therefore $\rho_W(j_W) \not\equiv 0$ holds. This implies $H^1(\widetilde{W};\mathbf{Z}_2) = 0$ by Lemma 3.2 (1). This completes the proof of (1).

(2) Since, by Lemma 3.1 (2), $H^2(W;\mathbf{Z})$ contains $\mathbf{Z}_2$ as a submodule such that the index $|\mathrm{Tor}(H^2(W;\mathbf{Z})) : \mathbf{Z}_2|$ is odd, the proof is similar to that of (1)(b). This completes the proof of Lemma 3.3. $\square$

Next we give a formula for the second Stiefel-Whitney class $w_2(\widetilde{X})$ of $\widetilde{X}$.

**Lemma 3.4.** *We have $w_2(\widetilde{X}) = (\pi^*)_2(w_2(X)) + [\widetilde{F}]_2^*$, where $[\widetilde{F}]_2^*$ denotes the Poincaré dual of the homology class $[\widetilde{F}]_2 \in H_2(\widetilde{X};\mathbf{Z}_2)$ and $(\pi^*)_2 : H^2(X;\mathbf{Z}_2) \to H^2(\widetilde{X};\mathbf{Z}_2)$ the homomorphism induced by $\pi : \widetilde{X} \to X$.*

Proof. First recall that a characteristic surface of $X$ exists (see Remark 2.4 (2)). Let $S$ be a (possibly non-orientable) characteristic surface of $X$. Since $[F]_2 = 0$ and $F$ is connected, we may assume that $S \cap F = \emptyset$ by applying a surgery to $S$ if necessary. Set $\widetilde{S} = \pi^{-1}(S)$. We shall show that $\widetilde{S} \cup \widetilde{F}$ is a characteristic surface of $\widetilde{X}$. By virtue of the universal coefficient theorem together with Remark 2.4 (1), we have only to show that $[\widetilde{S} \cup \widetilde{F}]_2 \cdot x_2 = x_2 \cdot x_2$ for all $x_2 \in H_2(\widetilde{X};\mathbf{Z}_2)$.

To prove (A), we prepare a sublemma.

**Sublemma 3.5.** *We have $(\pi_*)_2(x_2) \cdot (\pi_*)_2(x_2) = x_2 \cdot x_2 + x_2 \cdot [\widetilde{F}]_2$ for all $x_2 \in H_2(\widetilde{X};\mathbf{Z}_2)$, where $(\pi_*)_2 : H_2(\widetilde{X};\mathbf{Z}_2) \to H_2(X;\mathbf{Z}_2)$ denotes the homomorphism induced by $\pi : \widetilde{X} \to X$.*

Proof. Let $H$ be a closed surface in $\widetilde{X}$ which represents $x$. We may assume that $H$ is transverse to both $\widetilde{F}$ and $\tau(H)$, where $\tau(H)$ denotes the image of $H$ under the covering transformation $\tau$ (see §2 (II)). Then perturb $H$ slightly to $H'$ so that $H'$ is transverse to both $H$ and $\tau(H)$, and $H' \cap (H \cup \tau(H)) \subset \widetilde{W}$. Then we have

$$\begin{aligned}(\pi_*)_2([H]_2) \cdot (\pi_*)_2([H]_2) &= (\pi_*)_2([H]_2) \cdot (\pi_*)_2([H']_2) \\ &= [H]_2 \cdot [H']_2 + [\tau(H)]_2 \cdot [H']_2 \\ &= [H]_2 \cdot [H']_2 + [\widetilde{F}]_2 \cdot [H]_2.\end{aligned}$$

For a proof of the last equality, see [3, §5]. This completes the proof of Sublemma 3.5. □

Let us show (A). For every $x_2 \in H_2(\widetilde{X}; \mathbf{Z}_2)$, we have

$$\begin{aligned}x_2 \cdot x_2 &= x_2 \cdot [\widetilde{F}]_2 + (\pi_*)_2(x_2) \cdot (\pi_*)_2(x_2) \text{(by Sublemma 3.5)} \\ &= x_2 \cdot [\widetilde{F}]_2 + (\pi_*)_2(x_2) \cdot [S]_2 \\ &= x_2 \cdot [\widetilde{F}]_2 + x_2 \cdot [\widetilde{S}]_2 \\ &= x_2 \cdot [\widetilde{S} \cup \widetilde{F}]_2,\end{aligned}$$

where the second equality holds, since $S$ is a characteristic surface of $X$. Hence, $\widetilde{S} \cup \widetilde{F}$ is characteristic. Then for every $\zeta_2 \in H_2(\widetilde{X}; \mathbf{Z}_2)$, we have

$$\begin{aligned}\langle w_2(\widetilde{X}), \zeta_2\rangle &= [\widetilde{S}]_2 \cdot \zeta_2 + [\widetilde{F}]_2 \cdot \zeta_2 \\ &= [S]_2 \cdot (\pi_*)_2(\zeta_2) + [\widetilde{F}]_2 \cdot \zeta_2 \\ &= \langle (\pi^*)_2(w_2(X)), \zeta_2\rangle + \langle [\widetilde{F}]_2^*, \zeta_2\rangle \\ &= \langle (\pi^*)_2(w_2(X)) + [\widetilde{F}]_2^*, \zeta_2\rangle.\end{aligned}$$

Therefore, we have $w_2(\widetilde{X}) = (\pi^*)_2(w_2(X)) + [\widetilde{F}]_2^*$ by the universal coefficient theorem. This completes the proof of Lemma 3.4. □

REMARK 3.6. Corollary 2.10 and Lemma 3.4 hold even if the branch locus is not connected. Let $F'$ be a (possibly disconnected) closed surface smoothly embedded in $X$. Then a double branched covering space $\widetilde{X}'$ over $X$ branched precisely along $F'$ exists if and only if $[F']_2 = 0 \in H_2(X; \mathbf{Z}_2)$. Moreover, we have $w_2(\widetilde{X}') = (\pi'^*)_2(w_2(X)) + [\widetilde{F}']_2^*$, where $\pi' : \widetilde{X}' \to X$ denotes the covering projection, $\widetilde{F}' = \pi'^{-1}(F')$, and $[\widetilde{F}']_2^*$ denotes the Poincaré dual of $[\widetilde{F}']_2 \in H_2(\widetilde{X}'; \mathbf{Z}_2)$. This equation can be shown as follows. Let $S$ be a characteristic surface of $X$. We may assume that $S$ is transverse to $F'$. Then the covering projection $\pi'$ is transverse to $S$. Hence $\pi'^{-1}(S)$ is a closed surface smoothly embedded in $\widetilde{X}'$. We can show that the homology class represented by $\pi'^{-1}(S) \cup \widetilde{F}'$ is characteristic and its Poincaré dual is equal to $(\pi'^*)_2(w_2(X)) + [\widetilde{F}']_2^*$ by a method similar to that in the proof of Lemma 3.4.

Lemmas 3.3 and 3.4 are essential for the proof of Theorem 1.1. The following lemma will be used for the proof in the spin case.

**Lemma 3.7.** *We have that* $[\widetilde{F}]_2 = 0 \in H_2(\widetilde{X};\mathbf{Z}_2)$ *if and only if* $H_1(\widetilde{W};\mathbf{Z}_2) \cong \mathbf{Z}_2$.

Proof. By Lemma 3.3, it suffices to show that $[\widetilde{F}]_2 \neq 0 \in H_2(\widetilde{X};\mathbf{Z}_2)$ if and only if $H_1(\widetilde{W};\mathbf{Z}_2) = 0$. Suppose that $[\widetilde{F}]_2 \neq 0$. Then there exists a $\mathbf{Z}_2$ 2-cycle representing $y_2 \in H_2(\widetilde{X};\mathbf{Z}_2)$ such that $y_2 \cdot [\widetilde{F}]_2 = 1$. Therefore, the $\mathbf{Z}_2$-homology class in $H_1(\widetilde{W};\mathbf{Z}_2)$ represented by a meridian must vanish. Hence we obtain $H_1(\widetilde{W};\mathbf{Z}_2) = 0$ by virtue of Lemma 3.3. Conversely, suppose $H_1(\widetilde{W};\mathbf{Z}_2) = 0$. Then we can construct a $\mathbf{Z}_2$ 2-cycle representing $z_2 \in H_2(\widetilde{X};\mathbf{Z}_2)$ such that $z_2 \cdot [\widetilde{F}]_2 = 1$. Therefore, $[\widetilde{F}]_2 \neq 0$. This completes the proof of Lemma 3.7. □

The following three lemmas will be used for the proof in the nonspin case.

**Lemma 3.8.** *We always have* $H_1(\widetilde{X};\mathbf{Z}_2) = 0$.

Proof. Consider the following Mayer-Vietoris exact sequence with coefficients in $\mathbf{Z}_2$, where $(e_{1*})_2$ and $(e_{2*})_2$ denote the homomorphisms induced by the inclusions $e_1 : \partial\widetilde{L} \to \widetilde{W}$ and $e_2 : \partial\widetilde{L} \to \widetilde{L}$ respectively:

$$H_1(\partial\widetilde{L};\mathbf{Z}_2) \xrightarrow{(e_{1*})_2 \oplus (e_{2*})_2} H_1(\widetilde{W};\mathbf{Z}_2) \oplus H_1(\widetilde{L};\mathbf{Z}_2) \longrightarrow H_1(\widetilde{X};\mathbf{Z}_2) \longrightarrow 0.$$

Since either $H_1(\widetilde{W};\mathbf{Z}_2) = 0$ or the generator of $H_1(\widetilde{W};\mathbf{Z}_2)$ is the homology class represented by a meridian by Lemma 3.3, $(e_{1*})_2 \oplus (e_{2*})_2$ is a surjection. Hence, $H_1(\widetilde{X};\mathbf{Z}_2) = 0$. □

**Lemma 3.9.** *Whenever* $X$ *is not spin,* $W$ *is not spin.*

Proof. Since $H^1(X;\mathbf{Z}_2) \cong H_1(X;\mathbf{Z}_2) = 0$ by our assumption, we have the cohomology exact sequence for the pair $(X, W)$ with coefficients in $\mathbf{Z}_2$, where $(i^*)_2$ is the homomorphism induced by the inclusion $i : W \to X$:

$$0 \longrightarrow H^1(W;\mathbf{Z}_2) \longrightarrow H^2(X, W;\mathbf{Z}_2) \longrightarrow H^2(X;\mathbf{Z}_2) \xrightarrow{(i^*)_2} H^2(W;\mathbf{Z}_2).$$

Note that $H^2(X, W;\mathbf{Z}_2) \cong H^2(L, \partial L;\mathbf{Z}_2) \cong H_2(L;\mathbf{Z}_2) \cong H_2(F;\mathbf{Z}_2) \cong \mathbf{Z}_2$ holds by excision and Poincaré-Lefschetz duality. Hence $H^1(W;\mathbf{Z}_2) \cong H^2(X, W;\mathbf{Z}_2) \cong \mathbf{Z}_2$, and $(i^*)_2$ is an injection. Thus we have $w_2(W) = (i^*)_2(w_2(X)) \neq 0$. This completes the proof. □

**Lemma 3.10.** *Assume that* $X$ *is not spin and* $H^1(W;\mathbf{Z}) = 0$. *Thus* $H^2(W;\mathbf{Z})$ *contains a unique nonzero element* $j_W \in H^2(W;\mathbf{Z})$ *such that* $2j_W = 0$ *by* Proposition 2.8 (3) *and* Lemma 3.1. *Then,* $\widetilde{W}$ *is spin if and only if* $\rho_W(j_W) = w_2(W)$, *where* $\rho_W : H^2(W;\mathbf{Z}) \to H^2(W;\mathbf{Z}_2)$ *denotes the reduction modulo 2.*

Proof. Consider the following two exact sequences, where $w_1 \in H^1(W;\mathbf{Z}_2) \cong \mathbf{Z}_2$ denotes the generator, $(p^*)_2$ the homomorphism induced by $p : \widetilde{W} \to W$, and $\beta_W$ the Bockstein homomorphism:

$$H^1(W;\mathbf{Z}_2) \xrightarrow{\cup w_1} H^2(W;\mathbf{Z}_2) \xrightarrow{(p^*)_2} H^2(\widetilde{W};\mathbf{Z}_2),$$

$$0 \longrightarrow H^1(W;\mathbf{Z}_2) \xrightarrow{\beta_W} H^2(W;\mathbf{Z}) \longrightarrow H^2(W;\mathbf{Z}) \xrightarrow{\rho_W} H^2(W;\mathbf{Z}_2).$$

The first one is the Gysin exact sequence for the regular 2-fold covering $p : \widetilde{W} \to W$, and the second one is the exact sequence associated with the coefficient sequence $0 \to \mathbf{Z} \overset{\times 2}{\to} \mathbf{Z} \to \mathbf{Z}_2 \to 0$. Since we have assumed that $X$ is not spin, $w_2(W) \neq 0$ by Lemma 3.9. Hence, using the Gysin exact sequence above, we see that $w_2(\widetilde{W}) = (p^*)_2(w_2(W)) = 0$ if and only if $w_1 \cup w_1 = Sq^1(w_1) = w_2(W)$. Since $\beta_W$ is an injection and $2\beta_W(w_1) = \beta_W(2w_1) = 0$, we have $\beta_W(w_1) = j_W$. Thus, by Fact 2.12, $Sq^1(w_1) = \rho_W \circ \beta_W(w_1) = w_2(W)$ if and only if $\rho_W(j_W) = w_2(W)$. This completes the proof of Lemma 3.10. □

## 4. Proof of Theorem 1.1 — spin case

Suppose that $X$ is spin. Then by Lemma 3.4 together with our assumption that $X$ is spin, the second Stiefel-Whitney class $w_2(\widetilde{X}) \in H^2(\widetilde{X};\mathbf{Z}_2)$ of $\widetilde{X}$ coincides with the Poincaré dual $[\widetilde{F}]_2^* \in H^2(\widetilde{X};\mathbf{Z}_2)$ of the homology class $[\widetilde{F}]_2 \in H_2(\widetilde{X};\mathbf{Z}_2)$. Hence $\widetilde{X}$ is spin if and only if $[\widetilde{F}]_2 = 0$. Recall that $[\widetilde{F}]_2 = 0$ if and only if $H_1(\widetilde{W};\mathbf{Z}_2) \cong \mathbf{Z}_2$ by virtue of Lemma 3.7. Furthermore, by Lemma 3.3, $H_1(\widetilde{W};\mathbf{Z}_2) \cong \mathbf{Z}_2$ if and only if $F$ is oriented and $a_F \equiv 0 \mod 4$. This completes the proof in the spin case. □

## 5. Proof of Theorem 1.1 — nonspin case

Suppose that $X$ is not spin. The proof will be divided into the following three cases.

CASE 1. When $F$ is non-orientable.

CASE 2. When $F$ is oriented and $[F]/2 \in H_2(X;\mathbf{Z})$ is characteristic.

CASE 3. When $F$ is oriented and $[F]/2 \in H_2(X;\mathbf{Z})$ is not characteristic.

Case 1. Consider the following commutative diagram:

$$\begin{tikzcd} 0 = H^2(X, W;\mathbf{Z}) \arrow[r] & H^2(X;\mathbf{Z}) \arrow[r, "i^*"] \arrow[d, "\rho_X"] & H^2(W;\mathbf{Z}) \arrow[r, "\delta_W"] \arrow[d, "\rho_W"] & H^3(X, W;\mathbf{Z}) \arrow[d, "\rho_{(X,W)}"] \\ & H^2(X;\mathbf{Z}_2) \arrow[r, "(i^*)_2"'] & H^2(W;\mathbf{Z}_2) \arrow[r, "(\delta_W)_2"'] & H^3(X, W;\mathbf{Z}_2), \end{tikzcd}$$

where the rows are the exact sequences for the pair $(X, W)$. The vertical maps denote the reductions modulo 2, $i^*$ and $(i^*)_2$ the homomorphisms induced by the inclusion $i : W \to X$, and $\delta_W : H^2(W;\mathbf{Z}) \to H^3(X;W;\mathbf{Z})$ and $(\delta_W)_2 : H^2(W;\mathbf{Z}_2) \to H^3(X, W;\mathbf{Z}_2)$ the connecting homomorphisms. Then we have the following.

(i) $H^2(X;\mathbf{Z})$ has no 2-torsion by our assumption that $H_1(X;\mathbf{Z}_2) = 0$ together with the universal coefficient theorem.

(ii) $H^2(W;\mathbf{Z})$ contains a unique nonzero element $j_W \in H^2(W;\mathbf{Z})$ such that $2j_W = 0$ by Lemma 3.1.

(iii) We have $H^2(X, W;\mathbf{Z}) \cong H^2(L, \partial L;\mathbf{Z}) \cong H_2(L;\mathbf{Z}) \cong H_2(F;\mathbf{Z}) = 0$ by excision, Poincaré-Lefschetz duality, and our assumption that $F$ is non-orientable. Hence, by the above diagram, $i^* : H^2(X;\mathbf{Z}) \to H^2(W;\mathbf{Z})$ is an injection.

(iv) $H^3(X, W;\mathbf{Z}) \cong H^3(L, \partial L;\mathbf{Z}) \cong H_1(L;\mathbf{Z}) \cong \mathbf{Z}^{g-1} \oplus \mathbf{Z}_2$ by excision and Poincaré-Lefschetz duality, and hence there exists a unique nonzero element $j_{(X,W)} \in H^3(X, W;\mathbf{Z})$ such that $2j_{(X,W)} = 0$, where $g \geq 1$ denotes the non-orientable genus of $F$.

(v) $\rho_{(X,W)}(j_{(X,W)}) \neq 0$.

The last statement follows from the following observation. Consider the exact sequence associated with the coefficient sequence $0 \to \mathbf{Z} \overset{\times 2}{\to} \mathbf{Z} \to \mathbf{Z}_2 \to 0$:

$$H^3(X, W;\mathbf{Z}) \longrightarrow H^3(X, W;\mathbf{Z}) \xrightarrow{\rho_{(X,W)}} H^3(X, W;\mathbf{Z}_2).$$

By (iv) above, there is no element $t \in H^3(X, W;\mathbf{Z})$ such that $2t = j_{(X,W)}$. Hence we have $\rho_{(X,W)}(j_{(X,W)}) \neq 0$ by the exactness of the above sequence.

Suppose that $w_2(\widetilde{W}) = 0$. Then by Proposition 2.8 (3), Lemma 3.10, and our assumption that $X$ is not spin, we have $\rho_W(j_W) = w_2(W)$. Since $(\delta_W)_2(w_2(W)) = (\delta_W)_2 \circ (i^*)_2(w_2(X)) = 0$, we have $(\delta_W)_2 \circ \rho_W(j_W) = 0$. On the other hand, we see that $\delta_W(j_W) = j_{(X,W)}$. Otherwise, $\delta_W(j_W) = 0$ and there exists a nonzero element $\tilde{j_W} \in H^2(X;\mathbf{Z})$ such that $i^*(\tilde{j_W}) = j_W$. Since $i^*(2\tilde{j_W}) = 2j_W = 0$ and $i^*$ is an injection by (iii) above, we have $2\tilde{j_W} = 0$. This contradicts the fact that $H^2(X;\mathbf{Z})$ has no 2-torsion. Thus we have $(\delta_W)_2 \circ \rho_W(j_W) = \rho_{(X,W)} \circ \delta_W(j_W) = \rho_{(X,W)}(j_{(X,W)}) \neq 0$ by (v) above. This is a contradiction. Thus $\widetilde{W}$ is not spin. Hence $F$ must be orientable for $\widetilde{X}$ to be spin. This completes the proof for Case 1.

From now on, suppose that $F$ is oriented. Fix a primitive element $x_F \in H_2(X;\mathbf{Z})$ and a torsion element $\zeta_F \in H_2(X;\mathbf{Z})$ such that $[F] = a_F x_F + \zeta_F$ (recall, by Remark 2.7, that whether $x_F$ is characteristic or not depends only on $[F]$).

Case 2. Note that $[F]/2$ is characteristic if and only if $a_F \equiv 2 \mod 4$ and $x_F \in H_2(X;\mathbf{Z})$ is characteristic. By Lemma 3.4, it suffices to show that $[\widetilde{F}]_2^* = (\pi^*)_2(w_2(X))$. First consider the following homology exact sequence associated with the coefficient sequence $0 \to \mathbf{Z} \xrightarrow{\times 2} \mathbf{Z} \to \mathbf{Z}_2 \to 0$:

$$H_2(\widetilde{X};\mathbf{Z}) \to H_2(\widetilde{X};\mathbf{Z}_2) \to H_1(\widetilde{X};\mathbf{Z}).$$

By the above sequence together with Lemma 3.8, each $y_2 \in H_2(\widetilde{X};\mathbf{Z}_2)$ has an integral lift $y \in H_2(\widetilde{X};\mathbf{Z})$. Hence, by virtue of the universal coefficient theorem, we have only to show that $x \cdot [\widetilde{F}] \equiv \langle (\pi^*)_2(w_2(X)), \rho_{\widetilde{X}}(x)\rangle \mod 2$ for all $x \in H_2(\widetilde{X};\mathbf{Z})$, where $\rho_{\widetilde{X}}$ denotes the reduction modulo 2.

Let $x \in H_2(\widetilde{X};\mathbf{Z})$ be a homology class such that $x \cdot [\widetilde{F}] = 2m+1$, where $m$ is an integer. Then, by Remark 2.2, we have $2(2m+1) = \pi_*(x)\cdot[F] = 2(2a+1)\pi_*(x)\cdot x_F$, where $a_F = 2(2a+1)$ with $a \in \mathbf{Z}$. Hence, $\pi_*(x)\cdot x_F \equiv 1 \mod 2$. Then, since $x_F$ is characteristic, we have $x\cdot[\widetilde{F}] \equiv 1 \equiv \pi_*(x)\cdot x_F \equiv \rho_X(\pi_*(x))\cdot w_2(X)^* = \langle (\pi^*)_2(w_2(X)), \rho_{\widetilde{X}}(x)\rangle$ mod 2, where $w_2(X)^*$ denotes the Poincaré dual of $w_2(X) \in H^2(X;\mathbf{Z}_2)$ and $\rho_X$ the reduction modulo 2.

Let $x \in H_2(\widetilde{X};\mathbf{Z})$ be a homology class such that $x \cdot [\widetilde{F}] = 2m$, where $m$ is an integer. Then we have $2(2m) = \pi_*(x)\cdot[F] = 2(2a+1)\pi_*(x)\cdot x_F$. Hence, $\pi_*(x)\cdot x_F \equiv 0 \mod 2$. Then we have $x\cdot[\widetilde{F}] \equiv 0 \equiv \pi_*(x)\cdot x_F \equiv \rho_X(\pi_*(x))\cdot w_2(X)^* = \langle (\pi^*)_2(w_2(X)), \rho_{\widetilde{X}}(x)\rangle \mod 2$. Hence $\widetilde{X}$ is spin. This completes the proof for Case 2.

Case 3. Note that $[F]/2$ is not characteristic if and only if

(Case 3.1) $a_F \equiv 0 \mod 4$ or

(Case 3.2) $a_F \equiv 2 \mod 4$ and $x_F \in H_2(X;\mathbf{Z})$ is not characteristic.

(Case 3.1) Consider the following Gysin exact sequence (see the exact sequences following Lemma 3.1):

$$0 \longrightarrow H^1(\widetilde{W};\mathbf{Z}_2) \longrightarrow H^1(W;\mathbf{Z}_2) \longrightarrow H^2(W;\mathbf{Z}_2) \xrightarrow{(p*)_2} H^2(\widetilde{W};\mathbf{Z}_2).$$

By Proposition 2.8 (3) and Lemma 3.3 (1)(a), we have $H^1(\widetilde{W};\mathbf{Z}_2) \cong H^1(W;\mathbf{Z}_2) \cong \mathbf{Z}_2$. Hence $(p^*)_2$ is an injection. Since we have assumed that $X$ is not spin, $w_2(W) \neq 0$ holds by Lemma 3.9. Hence $w_2(\widetilde{W}) = (p^*)_2(w_2(W)) \neq 0$ and $\widetilde{X}$ is not spin. This completes the proof for Case 3.1.

(Case 3.2) First note the following.

(i) The homomorphism $(i^*)_2 : H^2(X;\mathbf{Z}_2) \to H^2(W;\mathbf{Z}_2)$ induced by the inclusion $i : W \to X$ is an injection. This fact has been shown in the proof of Lemma 3.9.

(ii) Since $a_F \neq 0$, we have $H^1(W;\mathbf{Z}) = 0$ by Proposition 2.8 (3). Moreover, $H^2(W;\mathbf{Z})$ contains a unique nonzero element $j_W \in H^2(W;\mathbf{Z})$ such that $2j_W = 0$ by Lemma 3.1.

(iii) We have $H^n(X, W;\mathbf{Z}) \cong H^n(L, \partial L;\mathbf{Z}) \cong H_{4-n}(L;\mathbf{Z}) \cong H_{4-n}(F;\mathbf{Z})$ for every integer $n$ by excision and Poincaré-Lefschetz duality. We denote this isomorphism

for the case $n = 2$ by $\phi : H^2(X, W; \mathbf{Z}) \overset{\cong}{\to} H_2(F; \mathbf{Z})$.
By (i) above, we have the following commutative diagram:

$$\begin{tikzcd} H^2(X, W; \mathbf{Z}) \arrow[r, "k^*"] & H^2(X; \mathbf{Z}) \arrow[r, "i^*"] \arrow[d, "\rho_X"] & H^2(W; \mathbf{Z}) \arrow[r, "\delta_W"] \arrow[d, "\rho_W"] & H^3(X, W; \mathbf{Z}) \\ 0 \arrow[r] & H^2(X; \mathbf{Z}_2) \arrow[r, "(i^*)_2"] & H^2(W; \mathbf{Z}_2), & \end{tikzcd}$$

where the rows are exact, the vertical maps denote the reductions modulo 2, $k^*$ the homomorphism induced by the inclusion $k : X \to (X, W)$, $i^*$ and $(i^*)_2$ the homomorphisms induced by the inclusion $i : W \to X$, and $\delta_W : H^2(W; \mathbf{Z}) \to H^3(X, W; \mathbf{Z})$ the connecting homomorphism. The upper row is the exact sequence for the pair $(X, W)$.

Suppose that $w_2(\widetilde{W}) = 0$. By (iii) above, we have $H^3(X, W; \mathbf{Z}) \cong H_1(F; \mathbf{Z})$. Then, since $H_1(F; \mathbf{Z})$ is a free $\mathbf{Z}$-module, we see that $\delta_W(j_W) = 0$. Therefore, there exists an element $\tilde{j_W} \in H^2(X; \mathbf{Z})$ such that $i^*(\tilde{j_W}) = j_W$. Since we have assumed that $X$ is not spin, we have $\rho_W(j_W) = w_2(W)$ by Lemma 3.10. Then we see that $\rho_X(\tilde{j_W}) = w_2(X)$ by the diagram above. On the other hand, since $i^*(2\tilde{j_W}) = 2j_W = 0$, there exists an integer $l \in \mathbf{Z}$ such that $2\tilde{j_W} = k^*(lv) = l[F]^*$, where $v \in H^2(X, W; \mathbf{Z}) \cong H_2(F; \mathbf{Z}) \cong \mathbf{Z}$ denotes the generator such that $\phi(v) \in H_2(F; \mathbf{Z})$ coincides with the orientation class of $F$, and $[F]^* \in H^2(X; \mathbf{Z})$ the Poincaré dual of $[F] \in H_2(X; \mathbf{Z})$ (for a proof of the equality $k^*(v) = [F]^*$, see [13, p. 135, Problem 11-C]). Thus $\tilde{j_W} = l(a_F/2)x_F^* + l(\zeta_F/2)^*$ holds, where $\zeta_F/2$ denotes the (unique) torsion element in $H_2(X; \mathbf{Z})$ such that $2(\zeta_F/2) = \zeta_F$, and $x_F^* \in H^2(X; \mathbf{Z})$ and $(\zeta_F/2)^* \in H^2(X; \mathbf{Z})$ the Poincaré duals of $x_F \in H_2(X; \mathbf{Z})$ and $\zeta_F/2 \in H_2(X; \mathbf{Z})$ respectively. Since $\rho_X(\tilde{j_W}) = w_2(X) \neq 0$ and $\rho_X(l(\zeta_F/2)^*) = 0$, $l$ is odd and $x_F$ is characteristic. This is a contradiction, so that $\widetilde{W}$ is not spin. Hence $\widetilde{X}$ is not spin. This completes the proof in the nonspin case. □

## 6. Examples

In this section, we give some examples.

EXAMPLE 6.1. Let $D : \mathbf{C}P^2 \to \mathbf{C}P^2$ be the involution on the complex projective plane given by the complex conjugation $D([z_0 : z_1 : z_2]) = ([\overline{z_0} : \overline{z_1} : \overline{z_2}])$. Then, it is known that the fixed point set of $D$ is the real projective plane $\mathbf{R}P^2 = \{[r_0 : r_1 : r_2] \in \mathbf{C}P^2 | r_0, r_1, r_2 \in \mathbf{R}\}$, which is non-orientable, and the quotient manifold $\mathbf{C}P^2/D$ is homeomorphic to the 4-sphere $S^4$. The latter fact is known as Arnol'd-Kuiper-Massey theorem (see [1], [10] and [11]). In this case, $\mathbf{C}P^2$ is the double branched covering space over $S^4$ branched along an embedded $\mathbf{R}P^2$ and it is not spin.

EXAMPLE 6.2. Let us give an example of the theorem of Hitchin which we have stated in Section 1. Let $X$ be the complex projective plane $\mathbf{C}P^2$ and $C$ a non-singular algebraic curve of degree 2. Then the double branched covering space over

$X$ branched along $C$ is $\widetilde{X}^1_{\mathbf{C}P^2}$, which is a hypersurface of degree 2 in $\mathbf{C}P^3$ [7, p. 23]. Note that the homology class $[C] \in H_2(\mathbf{C}P^2;\mathbf{Z})$ represented by $C$ is twice a characteristic homology class. The conic $C$ is diffeomorphic to the 2-sphere and $\widetilde{X}^1_{\mathbf{C}P^2}$ is known to be homeomorphic to $S^2 \times S^2$, which is spin (see [14, p. 23], [8]). Note that the covering transformation $\tau : S^2 \times S^2 \to S^2 \times S^2$ is given by $\tau(x, y) = (y, x)$.

EXAMPLE 6.3. Let $X$ be $S^2 \times S^2$ and $F$ an oriented surface smoothly embedded in $X$ such that $[F] = 2[\Delta(S^2)]$, where $\Delta(S^2) = \{(x, x) \in S^2 \times S^2 | x \in S^2\}$, which is not characteristic. Then by the $G$-signature theorem (see [2], [6]), we have $\sigma(S^2 \times S^2) = (\sigma(\widetilde{X}) + [\widetilde{F}] \cdot [\widetilde{F}])/2$, where $\sigma(S^2 \times S^2)$ (resp. $\sigma(\widetilde{X})$) denotes the signature of $S^2 \times S^2$ (resp. $\widetilde{X}$). Since $2[\widetilde{F}]\cdot[\widetilde{F}] = [F]\cdot[F] = 8$ (see Remark 2.2), we have $\sigma(\widetilde{X}) = -4$. Hence $\widetilde{X}$ is not spin by the following theorem of Rohlin [15], [9, p. 31]: *the signature of a closed spin smooth four-manifold is divisible by* 16.

REMARK 6.4. If both $\widetilde{X}$ and $X$ are spin and $F$ is oriented, then we have $[\widetilde{F}] \cdot [\widetilde{F}] \equiv 0 \mod 16$. This fact can be shown by using the $G$-signature theorem as follows. Since $X$ is assumed to be spin, the signature $\sigma(X)$ of $X$ is a multiple of 16 by the above theorem of Rohlin. Hence we have $(\sigma(\widetilde{X})+[\widetilde{F}]\cdot[\widetilde{F}])/2 = \sigma(X) \equiv 0 \mod 16$ by virtue of the $G$-signature theorem. Thus $\sigma(\widetilde{X}) + [\widetilde{F}] \cdot [\widetilde{F}] \equiv 0 \mod 32$. Since $\widetilde{X}$ is also assumed to be spin, $\sigma(\widetilde{X}) \equiv 0 \mod 16$. Thus we obtain $[\widetilde{F}] \cdot [\widetilde{F}] \equiv 0 \mod 16$. Also note that $[F] \cdot [F] \equiv 0 \mod 32$ holds, since $[F] \cdot [F] = 2[\widetilde{F}] \cdot [\widetilde{F}]$.

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Department of Mathematics
Osaka University
Toyonaka Osaka 560–0043
Japan
e-mail: nagami@math.sci.osaka-u.ac.jp