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RATIONAL CUSPIDAL CURVES OF TYPE $(d, d-2)$ WITH ONE OR TWO CUSPS

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1. Introduction

A plane curve $C \subset \mathbf{P}^2 = \mathbf{P}^2(\mathbf{C})$ is said to be of *type* (d, ν) if the degree of C is d and the maximal multiplicity of C is ν. In case C is rational and cuspidal, we have proved the inequality: $d < 3\nu$ ([4]). A *cusp* means a unibranched (i.e., locally irreducible) singular point. Rational cuspidal curves of type $(d, d-2)$ are classified in [2] under the assumption that C has at least three cusps. The type $(d, d-3)$ case with at least three cusps is also discussed in [3]. In this note, we classify rational cuspidal curves of type $(d, d-2)$ having at most two cusps. For a cusp $P \in C$, let $\underline{m}_P = (m_{0,P}, \dots, m_{n+1,P})$ denote the *multiplicity sequence* ([2, 3, 4]), where $m_{0,P}$ is the multiplicity $m_P = m_P(C)$ of C at P. The notation (2_4) means the sequence $(2, 2, 2, 2, 1, 1)$. We prove the following

Theorem. *Let C be a rational cuspidal curve of type $(d, d-2)$. Let Q be the cusp with multiplicity $d-2$.*

(i) *If C has the unique cusp Q, then d is even and $\underline{m}_Q = (d-2, 2_{d-2})$. Furthermore, the equation of C has the form (up to projective equivalence):*

$$\left(y^k z + \sum_{i=1}^{k+1} a_i x^i y^{k+1-i}\right)^2 - xy^{2k+1} = 0 \quad (a_{k+1} \neq 0),$$

where $d = 2k+2$.

(ii) *If C has two cusps Q and P, then one of the following occurs.*

(a) $\underline{m}_Q = (d-2)$, $\underline{m}_P = (2_{d-2})$. *In this case, C is unique up to projective equivalence. For small d, the equations can be written as follows:*

$$(yz + x^2)^2 - x^3 z = 0,$$
$$(yz + x^2)^2 y + 2(yz + x^2)x^3 - x^4 z = 0,$$
$$(yz + 2x^2)^2 y^2 + 2(yz + 2x^2)yx^3 + (2yz + 5x^2)x^4 - x^5 z = 0.$$

(b) *d is odd, $\underline{m}_Q = (d-2, 2_{(d-3)/2})$, $\underline{m}_P = (2_{(d-1)/2})$. The equation of C has* *the form (up to projective equivalence)*:

$$\left(y^{k-1}z+\sum_{i=2}^{k}a_ix^iy^{k-i}\right)^2 y - x^{2k+1}=0,$$

where $d=2k+1$.

(c) d *is even*, $\underline{m}_Q=(d-2,2_{d-2-j})$, $\underline{m}_P=(2_j)$, $1\le j\le (d-2)/2$. *The equation of* C *has the form (up to projective equivalence)*:

$$\left(y^{k+j}z+\sum_{i=2}^{k+j+1}a_ix^iy^{k+j+1-i}\right)^2 - x^{2j+1}y^{2k+1}=0 \quad (a_{k+j+1}\neq 0),$$

where $d=2k+2j+2$ *and* $k\ge 0$, $j\ge 1$.

By construction, any of the above rational cuspidal curves of type $(d,d-2)$ with one or two cusps can be transformed into a line by a Cremona transformation. Combining this with the result of [2], we obtain

Corollary. *Any rational cuspidal curve of type* $(d,d-2)$ *is transformable into a line by a Cremona transformation.*

2. Quadratic transformation

Let C be a plane curve. Choose a Cremona transformation ψ (i.e., a birational transformation of $\mathbf{P}^2$). In certain circumstances, the strict transform C' of C via ψ is a rational cuspidal curve. We successively use the Cremona transformation $\psi(O,A,\ell)$, which was used in [3]. We discuss this process in detail.

Let O, A be two points in $\mathbf{P}^2$. Let t be the line passing through O and A, and let ℓ be another line passing through O. Let $\pi: X\to\mathbf{P}^2$ be the blowing up at O. We denote by E the exceptional curve. Note that X is the geometric ruled surface $\mathbf{F}_1$. Put $\tilde{O}=\tilde{\ell}\cap E$, where $\tilde{\ell},\tilde{t}$ are the strict transforms of ℓ, t. Let $\varphi: X \rightsquigarrow X'$ be the result of elementary transformations at $\tilde{O}$ and $\tilde{A}=\pi^{-1}(A)$. More precisely, let $\eta: Z\to X$ be the blowing up at $\tilde{O}$ and $\tilde{A}$. Let $\check{\ell}',\check{t}'$ denote the exceptional curves over $\tilde{O}$ and $\tilde{A}$, respectively. Let $\check{\ell}$, $\check{t}$ and $\check{E}$ be the strict transforms of $\tilde{\ell}$, $\tilde{t}$ and E. Let $\eta': Z\to X'$ be the contraction of $\check{\ell}$ and $\check{t}$. Let $\tilde{\ell}',\tilde{t}'$ be the images of $\check{\ell}',\check{t}'$. Since $\tilde{O}\in E$ and $\tilde{A}\notin E$, $E'=\eta'(\check{E})$ is again a (-1) curve. So X' is the surface $\mathbf{F}_1$. Set $\tilde{O}'=\tilde{t}'\cap E'$. Let $\pi': X'\to\mathbf{P}^2$ be the contraction of E'. Let ℓ', t' denote the images of $\tilde{\ell}',\tilde{t}'$. Set $O'=\eta'(\tilde{O}')$. Then ℓ' and t' are lines with $O'=\ell'\cap t'$.

Definition. The Cremona transformation $\psi=\pi'\circ\varphi\circ\pi^{-1}$ is denoted by $\psi(O,A,\ell)$. Note that $\psi:\mathbf{P}^2\setminus\{\ell\cup t\}\to\mathbf{P}^2\setminus\{\ell'\cup t'\}$ is an isomorphism and that $\psi:\mathbf{P}^2\setminus\{O,A\}\to\mathbf{P}^2$ is a morphism. Let C be an irreducible plane curve $(\neq \ell, t)$. Then there exists an irreducible plane curve C' such that ψ gives an isomorphism $C \setminus \{\ell \cup t\} \cong C' \setminus \{\ell' \cup t'\}$. The curve C' is called the *strict transform* of C via ψ. Symbolically, we write as $C' = \psi[C]$.

Set $\check{B}' = \check{\ell} \cap \check{\ell}'$, $\tilde{B}' = \eta'(\check{B}')$, $B' = \pi'(\tilde{B}')$. The inverse Cremona transformation $\pi \circ \varphi^{-1} \circ \pi'^{-1}$ can be written as $\psi(O', B', t')$.

NOTATION. As a convention, we use the following notation:

$$O' = \psi\langle O\rangle,\ B' = \psi\langle A\rangle,\ \ell' = \psi\langle \ell\rangle,\ t' = \psi\langle t\rangle.$$

It is straightforward to prove the following

Lemma 1. *Let (x, y, z) be the homogeneous coordinates of $\boldsymbol{P}^2$ so that the line ℓ is defined by $x = 0$, the line t is defined by $y = 0$ and the point A has the coordinates $(1, 0, a)$. Suppose that the line ℓ' is defined by $x = 0$, the line t' is defined by $y = 0$ and the point B' has the coordinates $(0, 1, 0)$. Then the Cremona transformation $\psi(O, A, \ell)$ is given by the quadratic transformation $(xy, y^2, x(z - ax))$. The inverse quadratic transformation has the form $(x^2, xy, yz + ax^2)$.*

So if a plane curve C has the defining equation $f(x, y, z) = 0$, then the equation of the strict transform of C via ψ is obtained from the irreducible factor of the polynomial $f(x^2, xy, yz + ax^2)$.

Lemma 2. *Let C be an irreducible plane curve. We choose O, A, ℓ, t as $C \neq \ell, t$. Set $C' = \psi(O, A, \ell)[C]$. Then*

$$\deg(C') - m_{O'}(C') = \deg(C) - m_O(C).$$

Proof. We have the relations:

$$\pi^*(\ell) = \tilde{\ell} + E, \quad \pi'^*(\ell') = \tilde{\ell}' + E', \quad \eta^*(\tilde{\ell}) = \check{\ell} + \check{\ell}' = \eta'^*(\tilde{\ell}').$$

We denote by $\tilde{C}$, $\check{C}$ and $\tilde{C}'$ the strict transforms of C in X, Z and X', respectively. Then we have

$$\pi^*(C) = \tilde{C} + m_O(C)E, \quad \pi'^*(C') = \tilde{C}' + m_{O'}(C')E'.$$

We also have

$$\begin{aligned} \eta^*(\tilde{C}) &= \check{C} + m_{\tilde{O}}(\tilde{C})\check{\ell}' + m_{\tilde{A}}(\tilde{C})\check{t}', \\ \eta'^*(\tilde{C}') &= \check{C} + m_{\tilde{O}'}(\tilde{C}')\check{t} + m_{\tilde{B}'}(\tilde{C}')\check{\ell}. \end{aligned}$$

It follows that $\deg(C) = \ell \cdot C = \tilde{\ell} \cdot \tilde{C} + m_O(C)$ and $\deg(C') = \ell' \cdot C' = \tilde{\ell}' \cdot \tilde{C}' + m_{O'}(C')$. Since $\tilde{\ell} \cdot \tilde{C} = (\check{\ell} + \check{\ell}')\check{C} = \tilde{\ell}' \cdot \tilde{C}'$, we obtain the desired formula. □

Let (γ, P) be an irreducible curve germ in $\mathbf{P}^2$. For such a germ (γ, P), the multiplicity sequence $\underline{m}_P = (m_0, m_1, m_2, \dots)$ is also defined. We employ the convention: $\underline{m}_P^* = (m_1, m_2, \dots)$, $\underline{m}_P^{**} = (m_2, m_3, \dots)$. Sometimes, we simply write as $P = (\gamma, P)$.

If $(\gamma, P) \neq (\ell, P), (t, P)$, then we can define the strict transform (γ', P') of (γ, P) via the Cremona transformation $\psi = \psi(O, A, \ell)$. Symbolically, we write as $(\gamma', P') = \psi[(\gamma, P)]$, or simply $P' = \psi[P]$. We denote by $i(\ell)$ (resp. $i(t)$) the contact order $(\ell \cdot \gamma)_P$ (resp. $(t \cdot \gamma)_P$). Note that $i(\ell) = m_0$ or $i(\ell) = m_0 + m_1 + \dots + m_s$ for some $s > 0$ with $m_0 = \dots = m_{s-1}$. The same thing holds for $i(t)$. Cf. [2], Lemma 1.3. According to the position of P and the contact orders with ℓ and t, we divide the germs (γ, P) into the following types. If (γ, P) is of type T (resp. T^*) with respect to $\{O, A, \ell\}$, then (γ', P') is of type T^* (resp. T) with respect to $\{O', B', t'\}$. Type I_a is self dual.

type	P	$i(\ell)$	$i(t)$
I_a	O	m_0	m_0
I_b	O	$m_1 + m_0$ $(m_0 > m_1)$	m_0
I_b^*	O	m_0	$> m_0$
II	O	$> 2m_0$ $(m_0 = m_1)$	m_0
II*	A	0	$> m_0$
III	O	$2m_0$ $(m_0 = m_1)$	m_0
III*	$\in t \setminus \{O, A\}$	0	$\geq m_0$
IV	A	0	m_0
IV*	$\in \ell \setminus \{O\}$	$\geq m_0$	0

Lemma 3. *The strict transform (γ', P') has the following invariants.*

type	$\underline{m}_{P'}$	P'	$i(t')$	$i(\ell')$
I_a	$\underline{m}_P$	O'	m_0	m_0
I_b	$(m_0 - m_1, \underline{m}_P^{**})$	O'	$m_0 - m_1$	m_0
I_b^*	$(i(t), i(t) - m_0, \underline{m}_P^*)$	O'	$2i(t) - m_0$	$i(t)$
II	$(i(\ell) - 2m_0, \underline{m}_P^{**})$	B'	0	$i(\ell) - m_0$
II*	$(i(t) - m_0, i(t) - m_0, \underline{m}_P^*)$	O'	$2i(t) - m_0$	$i(t) - m_0$
III	$\underline{m}_P^{**}$	$\in \ell' \setminus \{O', B'\}$	0	m_0
III*	$(i(t), i(t), \underline{m}_P)$	O'	$2i(t)$	$i(t)$
IV	$\underline{m}_P^*$	$\in t' \setminus \{O'\}$	m_0	0
IV*	$(i(\ell), \underline{m}_P)$	B'	0	$i(\ell)$

Proof. We here prove the case in which $P = O$. We can similarly deal with the other cases. Write $\underline{m}_{P'} = (m'_0, m'_1, m'_2, \dots)$. Let $(\tilde{\gamma}, \tilde{P})$, $(\check{\gamma}, \check{P})$, $(\tilde{\gamma}', \tilde{P}')$ be the strict transforms of (γ, P) in X, Z, X', respectively. Note that $\tilde{P} = \tilde{O}$ if and only if $i(\ell) > m_0$. If $\tilde{P} = \tilde{O}$, then

$$m_0 = (E\tilde{\gamma})_{\tilde{P}} = (\check{E}\check{\gamma})_{\check{P}} + m_1, \quad m_1 = (\check{\gamma}\check{\ell}')_{\check{P}}, \quad i(\ell) = m_0 + m_1 + (\check{\gamma}\check{\ell})_{\check{P}}.$$

If $\tilde{P}' = \tilde{O}'$, or equivalently if $i(t) > m_0$, then

$$m'_0 = (E'\tilde{\gamma}')_{\tilde{P}'} = (\check{E}\check{\gamma})_{\check{P}} + m'_1, \quad m'_1 = (\check{\gamma}\check{t})_{\check{P}}, \quad i(t') = m'_0 + m'_1 + (\check{\gamma}\check{t}')_{\check{P}}.$$

Type I_a. Neither ℓ nor t is tangent to (γ, P). We infer that $\tilde{P} \in E \setminus \{\tilde{\ell}, \tilde{t}\}$. It follows that neither t' nor ℓ' is tangent to (γ', P').

Type I_b. Since $m_0 > m_1$, the exceptional curve E is tangent to $\tilde{\gamma}$ at $\tilde{P}$, hence $\check{P} \in \check{E}$. We infer that $\check{P}$ is the infinitely near point of P'. We have $\underline{m}_{P'} = (m'_0, \underline{m}_P^{**})$. Since $\tilde{P}' \neq \tilde{O}'$, $m'_0 = (E'\tilde{\gamma}')_{\tilde{P}'} = (\check{E}\check{\gamma})_{\check{P}}$, hence $m_0 = m'_0 + m_1$. Thus $m'_0 = m_0 - m_1$. We have

$$i(\ell') = (\gamma'\ell')_{P'} = (\tilde{\gamma}'\tilde{\ell}')_{\tilde{P}'} + m'_0 = m_1 + m'_0 = m_0.$$

Type II. In this case, $\tilde{\ell}$ is tangent to $\tilde{\gamma}$ at $\tilde{P}$. It follows that $\check{P} = \check{B}'$, which is the infinitely near point of $P' = B'$, hence $\underline{m}_{P'} = (m'_0, \underline{m}_P^{**})$. We have $m'_0 = (\check{\gamma}\check{\ell})_{\check{P}} = i(\ell) - m_0 - m_1 = i(\ell) - 2m_0$. Now we have

$$i(\ell') = (\gamma'\ell')_{P'} = (\tilde{\gamma}'\tilde{\ell}')_{\tilde{P}'} = (\check{\gamma}\check{\ell}')_{\check{P}} + m'_0 = m_1 + m'_0 = i(\ell) - m_0.$$

Type III. In this case, $\check{P} \neq \check{B}'$, $\check{P} \notin \check{E}$. We conclude that $P' \in \ell' \setminus \{O', B'\}$. We also have $P' \cong \check{P}$. It follows that $\underline{m}_{P'} = \underline{m}_P^{**}$, $i(\ell') = (\check{\gamma}\check{\ell}')_{\check{P}} = m_1 = m_0$.

Type I_b^*. In this case, $\tilde{P} = \check{P}$ and $\tilde{P}' = \tilde{O}'$. So $\underline{m}_{P'} = (m'_0, m'_1, \underline{m}_P^{**})$. Since $\tilde{P} \neq \tilde{O}$, we have $m_0 = (E\tilde{\gamma})_{\tilde{P}} = (\check{E}\check{\gamma})_{\check{P}}$. Thus $m'_0 = m_0 + m'_1$. On the other hand, $i(t) = (\tilde{\gamma}\tilde{t})_{\tilde{P}} + m_0$, $m'_1 = (\check{\gamma}\check{t})_{\check{P}}$. Combining these, we get $m'_1 = i(t) - m_0$, $m'_0 = i(t)$. Since $(\check{\gamma}\check{t}')_{\check{P}} = 0$, we find that $i(t') = m'_0 + m'_1 = 2i(t) - m_0$. □

3. Proof of Theorem

Let C be a rational cuspidal curve of type $(d, d-2)$. Let Q be the cusp with $m_Q = d - 2$. The possible multiple sequences of cusps are given by the following

Lemma 4.

(i) *If C has the unique cusp Q, then d is even and $\underline{m}_Q = (d-2, 2_{d-2})$.*

(ii) *If C has two cusps Q and P, then one of the following occurs*

  (a) $\underline{m}_Q = (d-2)$, $\underline{m}_P = (2_{d-2})$,

  (b) *d is odd,* $\underline{m}_Q = (d-2, 2_{(d-3)/2})$, $\underline{m}_P = (2_{(d-1)/2})$,

  (c) *d is even,* $\underline{m}_Q = (d-2, 2_{d-2-j})$, $\underline{m}_P = (2_j)$, $1 \le j \le (d-2)/2$.

Proof. Since $m_{0,Q}+m_{1,Q} \leq d$, we find that the multiplicity sequence $\underline{m}_Q$ can take one of the following types: (i) $(d-2)$, (ii) $(d-2, 2_k)$. In the second case, if d is odd, then we must have $d-2 = 2k+1$, hence $k = (d-3)/2$, and if d is even, then $d-2 \leq 2k$ (Cf. [2], Proposition 1.2). The genus formula implies that if C is unicuspidal, then d is even and $\underline{m}_Q = (d-2, 2_{d-2})$. Suppose C is bicuspidal. Since $m_Q+m_P \leq d$, we have $\underline{m}_P = (2_j)$ for some j. If $\underline{m}_Q = (d-2)$, then we have $\underline{m}_P = (2_{d-2})$. Consider the case $\underline{m}_Q = (d-2, 2_k)$. If d is odd, then $k = (d-3)/2$, hence we have $\underline{m}_P = (2_{(d-1)/2})$. If d is even, then we have $j = d-2-k$. Since $d-2 \leq 2k$, we infer that $j \leq (d-2)/2$. □

REMARK. The characteristic sequence $\mathrm{Ch}(P)$ (Cf. [4]) of the cusp P with $\underline{m}_P = (d-2)$ is $(d-2, d-1)$. In case $\underline{m}_P = (d-2, 2_k)$, we have

$$\mathrm{Ch}(P) = \begin{cases} (d-2, d) & \text{(if } d \text{ is odd)}, \\ (d-2, d, 2k+3) & \text{(if } d \text{ is even)}. \end{cases}$$

3.1. The bicuspidal case (a) We first construct rational bicuspidal plane curves in the class (a). For $k \geq 1$, we inductively construct a rational plane curve C_k of degree $k+2$ having two cusps Q_k and P_k with $\underline{m}_{Q_k} = (k)$, $\underline{m}_{P_k} = (2_k)$. If such a curve C_k exists, then the tangent line t_k to C_k at Q_k and the line ℓ_k passing through Q_k and P_k satisfies the intersection property:

$$\begin{aligned} \ell_k \cdot C_k &= kQ_k + 2P_k, \\ t_k \cdot C_k &= (k+1)Q_k + R_k. \end{aligned}$$

where R_k is a non-singular point of C_k. We begin with a cuspidal cubic C_1 with a cusp P_1 and a non-singular point Q_1, which is not the flex of C_1. Let ℓ_1 be the line passing through Q_1 and P_1. Let t_1 be the tangent line to C_1 at Q_1. Then t_1 intersects C_1 at another point R_1. Using $\psi_k = \psi(Q_k, R_k, \ell_k)$, we construct C_{k+1} from C_k. Set $C_{k+1} = \psi_k[C_k]$, $\ell_{k+1} = \psi_k\langle\ell_k\rangle$, $t_{k+1} = \psi_k\langle t_k\rangle$, $Q_{k+1} = \psi_k\langle Q_k\rangle$, $P_{k+1} = \psi(P_k)$. By Lemma 3, we see that $\underline{m}_{Q_{k+1}} = (k+1)$, $\underline{m}_{P_{k+1}} = (2_{k+1})$. In this case, Q_k is of type I_b^* and P_k is of type IV^* with respect to $\{Q_k, R_k, \ell_k\}$. By Lemma 2, we have $\deg(C_{k+1}) = k+2$. The line t_{k+1} is tangent to C_{k+1} at Q_{k+1} and $\ell_{k+1} \cap \Gamma_{k+1} = \{Q_{k+1}, P_{k+1}\}$.

In order to find the equation of C_k, we begin with the cuspidal cubic:

$$\begin{aligned} &C_1 : \ f_1(x, y, z) = yz^2 - x^2z + x^3 = 0, \\ &\ell_1 : \ x = 0, \ \ t_1 : \ y = 0, \\ &Q_1 : \ (0, 0, 1), \ \ P_1 : \ (0, 1, 0), \ \ R_1 : \ (1, 0, 1). \end{aligned}$$

By Lemma 1, ψ_1 is given by the quadratic transformation $(xy, y^2, x(z-x))$. Substituting $(x^2, xy, yz+x^2)$, we obtain

$$f_1(x^2, xy, yz+x^2) = xy\{(yz+x^2)^2 - x^3 z\}.$$

We infer from this that the defining equation of C_2 is $(yz+x^2)^2 - x^3 z = 0$. Continuing in this way, we arrive at the equation of C_k.

3.2. The bicuspidal case (b) We now construct rational bicuspidal plane curves of class (b). We inductively construct a plane curve Γ_k of odd degree $2k+1$ having a cusp Q_k of type $(2k-1, 2_{k-1})$ and a cusp P_k of type (2_k). If such a curve exists, then the tangent line t_k to Γ_k at Q_k and the line ℓ_k passing through Q_k and P_k satisfies the intersection property:

$$\begin{aligned} \ell_k \cdot \Gamma_k &= (2k-1)Q_k + 2P_k, \\ t_k \cdot \Gamma_k &= (2k+1)Q_k. \end{aligned}$$

We first take a cuspidal cubic Γ_1 with a cusp P_1 and a flex Q_1. Let ℓ_1 be the line passing through P_1 and Q_1 and let t_1 be the flex tangent line to Γ_1 at Q_1. We construct Γ_{k+1} from Γ_k by using $\psi_k = \psi(Q_k, A_k, \ell_k)$, where the point $A_k \in t_k \setminus \{Q_k\}$ can be arbitrarily chosen. As before, set $\Gamma_{k+1} = \psi_k[\Gamma_k]$, $\ell_{k+1} = \psi_k\langle \ell_k \rangle$, $t_{k+1} = \psi_k\langle t_k \rangle$, $Q_{k+1} = \psi_k\langle Q_k \rangle$, $P_{k+1} = \psi(P_k)$. The curve Γ_{k+1} is a rational bicuspidal curve. By Lemma 3, we see that $\underline{m}_{Q_{k+1}} = (2k+1, 2_k)$ and $\underline{m}_{P_{k+1}} = (2_{k+1})$. By Lemma 2, we have $\deg(\Gamma_{k+1}) = 2k+3$. We also know that t_{k+1} is the tangent line to Γ_{k+1} at Q_{k+1} and $\ell_{k+1} \cap \Gamma_{k+1} = \{Q_{k+1}, P_{k+1}\}$. So Γ_{k+1} is the desired rational bicuspidal curve.

Choose the equation of Γ_1 as follows:

$$\begin{aligned} &\Gamma_1 : \ f_1(x, y, z) = yz^2 - x^3 = 0, \\ &\ell_1 : \ x = 0, \ \ t_1 : \ y = 0, \\ &Q_1 : \ (0,0,1), \ \ P_1 : \ (0,1,0), \ \ A_1 : \ (1,0,a). \end{aligned}$$

In this case, substituting the inverse quadratic transformation of ψ_1, we obtain

$$f_1(x^2, xy, yz+ax^2) = x\{(yz+ax^2)^2 y - x^5\}.$$

Thus, the equation of Γ_2 is given by $(yz+ax^2)^2 y - x^5 = 0$. Continuing this process $k-1$ times, we get the equation of Γ_k given in Theorem.

REMARK. For the quintic Γ_2, we can further put $a = 0$ or $a = 1$. Cf. [9, 10].

3.3. The unicuspidal case We pass to the unicuspidal case. We construct a rational unicuspidal curve D_k of degree $2k+2$ with a cusp Q_k of type $(2k, 2_{2k})$. Let Γ_k be the rational bicuspidal curve in the class (b) constructed as above. Namely, Γ_k has a cusp Q_k of type $(2k-1, 2_{k-1})$ and a cusp P_k of type (2_k). Let ℓ_k be the line joining Q_k and P_k and let t_k be the tangent line to Γ_k at Q_k. We construct D_k from Γ_k by performing quadratic transformations k-times.

Put $\Gamma_{k,0} = \Gamma_k$, $O_{k,0} = Q_{k,0} = Q_k$, $P_{k,0} = P_k$. Here, by $Q_{k,0}$ and $P_{k,0}$ we also mean the germ (Γ_k, Q_k) and (Γ_k, P_k). Interchanging the roles of ℓ_k and t_k, we set $\ell_{k,0} = t_k$, $t_{k,0} = \ell_k$. For $0 \le j \le k$, we inductively construct plane curves $\Gamma_{k,j}$. Set $\psi_j = \psi(O_{k,j}, A_{k,j}, \ell_{k,j})$, where $A_{k,0} \in \ell_{k,0} \setminus \{O_{k,0}, P_{k,0}\}$ and $A_{k,j} \in \ell_{k,j} \setminus \{O_{k,j}\}$ for $0 < j \le k$. We define $\Gamma_{k,j+1} = \psi_j[\Gamma_{k,j}]$, $O_{k,j+1} = \psi_j\langle O_{k,j}\rangle$, $\ell_{k,j+1} = \psi_j\langle \ell_{k,j}\rangle$, $t_{k,j+1} = \psi_j\langle t_{k,j}\rangle$, $Q_{k,j+1} = \psi_j[Q_{k,j}]$, $P_{k,j+1} = \psi_j[P_{k,j}]$. In case $k = 1$, we easily see that $D_1 = \Gamma_{1,1}$ is a quartic curve having a cusp $Q_1 = P_{1,1}$ with $\underline{m}_{Q_1} = (2_3)$. Now we assume that $k \ge 2$. For $1 \le j < k$, we see that both $Q_{k,j}$ and $P_{k,j}$ are locally irreducible branches centered at $O_{k,j}$ with $\underline{m}_{Q_{k,j}} = (2k-2j-1, 2_{k-j-1})$ and $\underline{m}_{P_{k,j}} = (2j, 2_{k+j})$. Indeed, $Q_{k,0}$ is of type I_b and $P_{k,0}$ is of type III*. By Lemma 3, we find that as points $Q_{k,1} = P_{k,1} = O_{k,1}$ and $\underline{m}_{Q_{k,1}} = (2k-3, 2_{k-2})$ and $\underline{m}_{P_{k,1}} = (2_{k+2})$. Furthermore, $t_{k,1}$ (resp. $\ell_{k,1}$) is the tangent line to $Q_{k,1}$ (resp. $P_{k,1}$) with $\ell_{k,1} \cdot Q_{k,1} = 2k-1$ and $t_{k,1} \cdot P_{k,1} = 4$. Thus $Q_{k,1}$ is of type I_b and $P_{k,1}$ is of type I_b^* with respect to $\{O_{k,1}, A_{k,1}, \ell_{k,1}\}$. Again by Lemma 3, we see that $\underline{m}_{Q_{k,2}} = (2k-5, 2_{k-3})$ and $\underline{m}_{P_{k,2}} = (4, 2_{k+2})$. By induction, we can prove the claim for $Q_{k,j}$ and $P_{k,j}$. Since $m_{O_{k,j}}(\Gamma_{k,j}) = 2k-1$, it follows from Lemma 2 that $\deg(\Gamma_{k,j}) = 2k+1$. Finally, the curve $\Gamma_{k,k-1}$ has two irreducible branches $Q_{k,k-1}$, $P_{k,k-1}$ at $O_{k,k-1}$ with $\underline{m}_{Q_{k,k-1}} = (1)$, $\underline{m}_{P_{k,k-1}} = (2k-2, 2_{2k-1})$. Namely, $Q_{k,k-1}$ is a non-singular branch at $O_{k,k-1}$ with $\ell_{k,k-1} \cdot Q_{k,k-1} = 3$. So $Q_{k,k-1}$ is of type II with respect to $\{Q_{k,k-1}, A_{k,k-1}, \ell_{k,k-1}\}$. It follows that $Q_{k,k} = \psi_k[Q_{k,k-1}]$ is a non-singular branch whose center $\neq O_{k,k} = \psi_k\langle O_{k,k-1}\rangle$. We conclude that the rational curve $D_k = \Gamma_{k,k}$ has only one cusp $Q_k = P_{k,k} = \psi_k[P_{k,k-1}]$ with $\underline{m}_{Q_k} = (2k, 2_{2k})$ as desired. By Lemma 2, we see that $\deg(D_k) = 2k+2$.

By interchanging x and y, the equation of the curve Γ_k has the form:

$$\begin{aligned}
&\Gamma_k : f_0(x, y, z) = \left(x^{k-1}z + \sum_{i=2}^{k} a_i y^i x^{k-i}\right)^2 x - y^{2k+1} = 0,\\
&\ell_k : x = 0, \quad t_k : y = 0,\\
&O_{k,0} : (0, 0, 1), \quad P_{k,0} : (1, 0, 0), \quad A_{k,0} : (1, 0, b), \quad b \neq 0.
\end{aligned}$$

The inverse quadratic transformation of ψ_0 has the form $(x^2, xy, yz + bx^2)$. The inverse transformation of the successive quadratic transformations $\psi_{k-1} \circ \cdots \circ \psi_0$ has the form $(x^{k+1}, x^k y, y^k z + \sum_{j=2}^{k+1} b_j x^j y^{k+1-j})$ with $b_{k+1} \neq 0$. Substituting this, we obtain

$$\begin{aligned}
&f_0\left(x^{k+1}, y^k x, y^k z + \sum_{j=2}^{k+1} b_j x^j y^{k+1-j}\right)\\
&= x^{2k^2+k-1}\left\{\left(y^k z + \sum_{j=2}^{k+1} b_j x^j y^{k+1-j} + \sum_{i=2}^{k} a_i y^i x^{k+1-i}\right)^2 - xy^{2k+1}\right\}.
\end{aligned}$$

Putting the same terms together, we get the equation in Theorem.

3.4. The bicuspidal case (c) We construct rational bicuspidal plane curves in the class (c). For $k \geq 0$, We inductively construct a plane curve $D_{k,j}$ of degree $2k + 2j + 2$ having a cusp $Q_{k,j}$ of type $(2k+2j, 2_{2k+j})$ and a cusp $P_{k,j}$ of type (2_j). In this case, the tangent line $t_{k,j}$ to $D_{k,j}$ at $Q_{k,j}$ and the line $\ell_{k,j}$ passing through $Q_{k,j}$ and $P_{k,j}$ satisfies the following intersection property:

$$\begin{aligned} \ell_{k,j} \cdot D_{k,j} &= (2k+2j)Q_{k,j} + 2P_{k,j}, \\ t_{k,j} \cdot D_{k,j} &= (2k+2j+2)Q_{k,j}. \end{aligned}$$

If $k \geq 1$, we start with the rational unicuspidal curve D_k. Let t_k be the tangent line to D_k at Q_k. Then we have $t_k \cdot D_k = (2k+2)Q_k$. By considering the projection from Q_k, we can find a unique non-singular point $P_k \in D_k$ such that the line ℓ_k passing through Q_k and P_k satisfies the relation:

$$\ell_k \cdot D_k = (2k)Q_k + 2P_k.$$

For the case $k = 0$, take a non-singular conic D_0 and two distinct points P_0, Q_0. Let ℓ_0, t_0 be the tangent lines to C_0 at P_0 and Q_0. Put $D_{k,0} = D_k$. By induction, using $\psi_j = \psi(Q_{k,j}, A_{k,j}, \ell_{k,j})$ with $A_{k,j} \in t_{k,j} \setminus \{Q_{k,j}\}$, we construct $D_{k,j+1}$ from $D_{k,j}$. Set $D_{k,j+1} = \psi_j[D_{k,j}]$, $Q_{k,j+1} = \psi_j\langle Q_{k,j}\rangle$, $P_{k,j+1} = \psi_j(P_{k,j})$. Since $Q_{k,j}$ is of type I_b^* and $P_{k,j}$ is of type IV* with respect to $\{Q_{k,j}, A_{k,j}, \ell_{k,j}\}$, we see from Lemma 3 that $\underline{m}_{Q_{k,j+1}} = (2k+2j+2, 2_{2k+j+1})$ and $\underline{m}_{P_{k,j+1}} = (2_{j+1})$ as desired. By Lemma 2, we have $\deg(D_{k,j}) = 2k + 2j + 2$.

The equation of D_k is given by

$$\left(y^k z + \sum_{i=1}^{k+1} a_i x^i y^{k+1-i}\right)^2 - xy^{2k+1} = 0 \quad (a_{k+1} \neq 0).$$

The inverse of the successive quadratic transformation $\psi_{j-1} \circ \cdots \circ \psi_0$ has the form $(x^{j+1}, x^j y, y^j z + \sum_{h=2}^{j+1} c_h x^h y^{j+1-h})$. Substituting this in the above equation and putting the same terms together, we obtain the equation in Theorem.

3.5. Projective equivalence It remains to prove that a rational cuspidal curve of type $(d, d-2)$ is projectively equivalent to the plane curve defined by one of the equations given in Theorem. We here prove this for rational bicuspidal curves in the class (b). We essentially follow the argument in [3]. Let C be a rational bicuspidal curve of odd degree d with a cusp Q and P such that $\underline{m}_Q = (2d-2, 2_{(d-2)/2})$, $\underline{m}_P = (2_{(d-1)/2})$. Let t be the tangent line to C at Q and let ℓ be the line passing through Q and P. Then

$$\ell \cdot C = (2d-2)Q + 2P,$$

$$t \cdot C = dQ.$$

Set $d = 2k+1$. Inductively, we construct rational bicuspidal curves $C^{(s)}$ of degree $2k+1-2(s-1)$ having cusps $Q^{(s)}$, $P^{(s)}$ with $\underline{m}_P^{(s)} = (2k-2s+1, 2_{k-s})$ and $\underline{m}_{P^{(s)}} = (2_{k+1-s})$, for $1 \leq s \leq k$. Set $C^{(1)} = C$. Assumming that $C^{(s)}$ is defined, let $t^{(s)}$ be the tangent line to $C^{(s)}$ at $Q^{(s)}$ and let $\ell^{(s)}$ be the line passing through $Q^{(s)}$ and $P^{(s)}$. Set $\psi^{(s)} = \psi(Q^{(s)}, P^{(s)}, t^{(s)})$, $Q^{(s+1)} = \psi^{(s)}\langle Q^{(s)}\rangle$, $P^{(s+1)} = \psi^{(s)}[P^{(s)}]$. By definition, $P^{(s+1)}$ is the strict transform of the germ $(C^{(s)}, P^{(s)})$. As a point, we easily see that $P^{(s+1)} \in \ell^{(s+1)} \setminus \{O^{(s+1)}\}$. We also set $B^{(s+1)} = \psi^{(s)}\langle P^{(s)}\rangle \in t^{(s+1)} \setminus \{O^{(s+1)}\}$. Since $Q^{(s)}$ (resp. $P^{(s)}$) is of type I_b (resp. IV) with respect to $\{Q^{(s)}, P^{(s)}, t^{(s)}\}$, we see that $\underline{m}_Q^{(s+1)} = (2k-2s-1, 2_{k-s-1})$ (resp. $\underline{m}_{P^{(s+1)}} = (2_{k-s})$). It turns out that $C^{(k)}$ is a cuspidal cubic and $Q^{(k)}$ is a flex and $P^{(k)}$ is the cusp. Let Γ_1 be the cuspidal cubic defined by $yz^2 - x^3 = 0$. Set $O_1 = (0,0,1)$, $\ell_1 = \{x = 0\}$, $t_1 = \{y = 0\}$. Then there is a projective transformation φ_1 with the property:

$$\varphi_1(C^{(k)}) = \Gamma_1, \quad \varphi_1(Q^{(k)}) = O_1, \quad \varphi_1(\ell^{(k)}) = \ell_1, \quad \varphi_1(t^{(k)}) = t_1.$$

Put $A_1 = \varphi_1(B^{(k)})$. Letting $\psi_1 = \psi(O_1, A_1, \ell_1)$, we set $\Gamma_2 = \psi_1[\Gamma_1]$. We easily see that $\varphi_2 = \psi_1^{-1} \circ \varphi_1 \circ \psi^{(k-1)}$ is a projective transformation which transforms $C^{(s-1)}$ to Γ_2. Inductively, we can construct a rational bicuspidal curve Γ_k and a projective transformation φ_k, which transforms C to Γ_k. Namely, if Γ_i is defined, then we set $\Gamma_{i+1} = \psi_i[\Gamma_i]$ and $\varphi_{i+1} = \psi_i^{-1} \circ \varphi_i \circ \psi^{(k-i)}$, where $\psi_i = \psi(O_i, A_i, \ell_i)$ with $A_i = \varphi_i(B^{(k+1-i)})$. As we have seen before, Γ_k is defined by the equation in Theorem.

REMARK. We can similarly deal with the other cases. In the bicuspidal case (a), the quadratic transformation $\psi_i = \psi(O_i, R_i, \ell_i)$ is uniquely determined by C_i, so the equation of C_k has no parameter.

REMARK. The techniques of constructing new curves by Cremona transformations are also discussed in [1, 5, 8].

REMARK. Prof. M. Zaidenberg informed us that T. Fenske independently classified rational cuspidal curves of type $(d, d-2)$ with at most two cusps, as well as those of type $(d, d-3)$ with at most two cusps. The second author discussed the defining equations of rational cuspidal curves of type $(d, d-3)$ with at most two cusps [7].

References

[1] A. Degtyarev: *Isotopy classification of complex plane curves of degree 5*, Leningrad Math. J. **1** (1990), 881–904.

[2] H. Flenner and M. Zaidenberg: *On a class of rational cuspidal plane curves*, Manuscripta Math. **89** (1996), 439–460.

[3] H. Flenner and M. Zaidenberg: *Rational cuspidal plane curves of type* $(d, d-3)$, Math. Nach. **210** (2000), 93–110.

[4] T. Matsuoka and F. Sakai: *The degree of rational cuspidal curves*, Math. Ann. **285** (1989), 233–247.

[5] U. Persson: *Configurations of Kodaira fibers on rational elliptic surfaces*, Math. Z. **205** (1990), 1–47.

[6] F. Sakai: *Singularities of plane curves*, Geometry of Complex Projective Varieties, Seminars and Conferences **9**, Mediterranean Press, Rende, 1993, 257–273.

[7] K. Tono: *Defining equations of rational cuspidal curves of type* $(d, d-3)$ *with one or two cusps*, preprint.

[8] C.T.C. Wall: *Is every quartic a conic of conic?*, Math. Proc. Camb. Phil. Soc. **109** (1991), 419–424.

[9] C.T.C. Wall: *Highly singular quintic curves*, Math. Proc. Camb. Phil. Soc. **119** (1996), 257–277.

[10] H. Yoshihara: *On plane rational curves*, Proc. Japan Acad. **55** (1979), 152–155.

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