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QUANTUM DEFORMATIONS OF CERTAIN PREHOMOGENEOUS VECTOR SPACES. II

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Introduction

Let G be a reductive algebraic group over the complex number field $\mathbb{C}$ and let $\mathfrak{g}$ be its Lie algebra. The quantized coordinate algebra $A_q(G)$ of G is constructed as a certain dual Hopf algebra of the quantized enveloping algebra $U_q(\mathfrak{g})$ of $\mathfrak{g}$. The Hopf algebras $U_q(\mathfrak{g})$ and $A_q(G)$ over $\mathbb{C}(q)$ tend to the ordinary enveloping algebra $U(\mathfrak{g})$ and the coordinate algebra $A(G)$ respectively when the parameter q tends to 1 in a certain sense (Drinfeld [1], Jimbo [3]).

Let us consider what object we should regard as a quantum deformation of an affine variety X with G-action.

An affine variety X is endowed with an action of G if and only if its coordinate algebra $A(X)$ is equipped with a right $A(G)$-comodule structure

$$\tau : A(X) \to A(X) \otimes A(G)$$

which is simultaneously an algebra homomorphism. By the duality between $U(\mathfrak{g})$ and $A(G)$ we obtain a locally finite left $U(\mathfrak{g})$-module structure

$$(*) \qquad \gamma : U(\mathfrak{g}) \otimes A(X) \to A(X)$$

given by

$$(**) \qquad \tau(n) = \sum_i n_i \otimes f_i \Rightarrow \gamma(u \otimes n) = \sum_i \langle u, f_i \rangle n_i,$$

where $\langle\ ,\ \rangle : U(\mathfrak{g}) \times A(G) \to \mathbb{C}$ is the dual pairing. Since τ is an algebra homomorphism, we have

$$(***) \qquad u \in U(\mathfrak{g}),\ m, n \in A(X),\ \Delta(u) = \sum_i u_i \otimes v_i \Rightarrow u(mn) = \sum_i (u_i m)(v_i n),$$

where $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is the coproduct. Then the action of G on X is uniquely determined by the infinitesimal action γ. Moreover, for a locally finite left

$U(\mathfrak{g})$-module structure $(*)$ on $A(X)$ satisfying $(***)$ and a certain condition on irreducible $U(\mathfrak{g})$-modules appearing as submodules of $A(X)$, there exists a unique action of G on X whose infinitesimal action is given by γ.

Now we define the notion of a quantum deformation of an affine variety X with G-action as follows. A (not necessarily commutative) $\mathbb{C}(q)$-algebra $A_q(X)$ endowed with a locally finite left $U_q(\mathfrak{g})$-module structure

$$\gamma_q : U_q(\mathfrak{g}) \otimes A_q(X) \to A_q(X)$$

is called a quantum deformation of X if $A_q(X)$ and γ_q tend to $A(X)$ and $\gamma : U(\mathfrak{g}) \otimes A(X) \to A(X)$ respectively when q tends to 1 and if it satisfies

$$u \in U_q(\mathfrak{g}), \quad m, n \in A_q(X), \quad \Delta(u) = \sum_i u_i \otimes v_i \Rightarrow u(mn) = \sum_i (u_i m)(v_i n).$$

It seems to be an interesting problem to determine in which case X admits a quantum deformation. In this paper we consider the problem when X is a prehomogeneous vector space, that is, when X is a vector space with a linear G-action containing an open G-orbit. Such a quantum deformation was intensively studied in the case where $G = GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$ and $X = M_{mn}(\mathbb{C})$ (see Taft-Towber [10], Hashimoto-Hayashi [2] and Noumi-Yamada-Mimachi [7]), and also in the case where $G = GL_n(\mathbb{C})$ and X is the set of skew symmetric matrices of degree n (see Strickland [8]).

In our previous paper [4] we gave a general method to construct quantum deformations of prehomogeneous vector spaces of parabolic type. Moreover, for each non-open G-orbit C on X, we have shown that the defining ideal of the closure $\overline{C}$ and its canonical generators admit quantum deformations inside $A_q(X)$. It includes the existence of the quantum deformation of the irreducible relative invariant when X is a regular prehomogeneous vector space. Indeed, the canonical generator of the defining ideal of the closure of the one-codimensional orbit is nothing but the irreducible relative invariant.

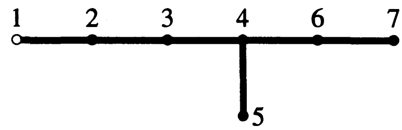
Quantum deformations of prehomogeneous vector spaces of commutative parabolic type associated to classical simple Lie algebras are intensively studied in Kamita [5]. In this paper we shall deal with the remaining two cases

(I) $G = \mathbb{C}^\times \times \mathrm{Spin}(10, \mathbb{C})$, $X = \mathbb{C}^{16}$, the scalar multiplication and the half-spin representation,

(II) $G = \mathbb{C}^\times \times E_6$, $X = \mathbb{C}^{27}$, the scalar multiplication and the 27-dimensional irreducible representation of E_6,

which naturally arise from the exceptional simple Lie algebras of type E_6 and E_7 respectively using the method in our previous paper [4]. In Introduction we shall only state the results in case (II).

Let $\mathfrak{g}_{E_7}$ be a simple Lie algebra of type E_7 over $\mathbb{C}$ and let $\mathfrak{h}$ be its Cartan subalgebra. We shall use the labelling of the vertices of the Dynkin diagram 1.



Dynkin diagram 1.

Set $I_0 = \{1, 2, \dots, 7\}$, $I = I_0 \setminus \{1\}$. Let $\Delta \subset \mathfrak{h}^*$ be the root system of type E_7. We denote the set of simple roots by $\{\alpha_i\}_{i \in I_0}$ and the set of positive roots by Δ^+. Let $(\ ,\ ) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ be a standard symmetric bilinear form. Set $D = \Delta^+ \setminus \sum_{i \in I} \mathbb{Z}\alpha_i$. Then we have $\sharp D = 27$. Set $\Lambda = \{1, 2, \dots, 27\}$, and fix a bijection $\Lambda \ni j \mapsto \beta_j \in D$ such that $\beta_k - \beta_j \in \sum_{i \in I_0} \mathbb{Z}_{\geq 0}\alpha_i$ implies $j \leq k$, where $\mathbb{Z}_{\geq 0} = \{n \in \mathbb{Z} \mid n \geq 0\}$. Set $\delta = 3\alpha_1 + 4\alpha_2 + 5\alpha_3 + 6\alpha_4 + 3\alpha_5 + 4\alpha_6 + 2\alpha_7$. For each $n \in \Lambda$ there exist exactly five pairs $(i, j) \in \Lambda^2$ such that $\beta_i + \beta_j = \delta - \beta_n$, $i < j$. We denote them by $(i_1^n, j_1^n), (i_2^n, j_2^n), (i_3^n, j_3^n), (i_4^n, j_4^n), (i_5^n, j_5^n) \in \Lambda^2$ where $i_5^n < i_4^n < i_3^n < i_2^n < i_1^n$. Let $K_i^{\pm 1}, E_i, F_i$ $(i \in I_0)$ be the canonical generators of $U_q(\mathfrak{g}_{E_7})$, and set $U_q(\mathfrak{g}) = \langle K_1^{\pm 3},\ K_j^{\pm 1}, E_j, F_j \mid j \in I \rangle \subset U_q(\mathfrak{g}_{E_7})$. Then $U_q(\mathfrak{g})$ is isomorphic to the tensor product of $\mathbb{C}(q)[K, K^{-1}]$ and the quantized enveloping algebra of type E_6, where $K = K_1^3 K_2^4 K_3^5 K_4^6 K_5^3 K_6^4 K_7^2$.

Theorem 0.1. *A quantum deformation of the 27-dimensional irreducible prehomogeneous vector space X of $G = \mathbb{C}^\times \times E_6$ is given by the following.*

(a) *$A_q(X)$ is an associative $\mathbb{C}(q)$-algebra defined by the following generators and fundamental relations:*
Generators: Y_i with $i = 1, \dots, 27$.
Fundamental relations: For $i < j$

$$Y_iY_j = \begin{cases} qY_jY_i & \text{if } \beta_i + \beta_j \text{ does not have another decomposition } \beta + \beta',\ \beta, \beta' \in D, \\ Y_jY_i + qY_bY_a - q^{-1}Y_aY_b & \\ \qquad \text{if there exist } k \in I,\ a, b \in \Lambda \text{ such that } \beta_a = \beta_i + \alpha_k,\ \beta_b = \beta_j - \alpha_k, \\ Y_jY_i & \text{otherwise.} \end{cases}$$

(b) *The action $\gamma_q : U_q(\mathfrak{g}) \otimes A_q(X) \to A_q(X)$ is given by the following. For $2 \leq k \leq 7$, $1 \leq m \leq 7$*

$$\gamma_q(F_k \otimes Y_i) = \begin{cases} Y_j & \text{if there exists } j \text{ such that } \beta_j = \beta_i + \alpha_k, \\ 0 & \text{otherwise,} \end{cases}$$

$$\gamma_q(E_k \otimes Y_i) = \begin{cases} Y_j & \text{if there exists } j \text{ such that } \beta_j = \beta_i - \alpha_k, \\ 0 & \text{otherwise,} \end{cases}$$

$$\gamma_q(K_m \otimes Y_i) = q^{-(\alpha_m, \beta_i)} Y_i.$$

(c) *The quantum deformation of the irreducible relative invariant of X is given by*

$$\varphi = \sum_{n\in\Lambda}(-q)^{|\beta_n|-1} Y_n \psi_n,$$

where $|\beta| = \sum_{i\in I_0} m_i$ $(\beta = \sum_{i\in I_0} m_i\alpha_i)$, $\psi_n = Y_{i_5^n}Y_{j_5^n} - qY_{i_4^n}Y_{j_4^n} + q^2Y_{i_3^n}Y_{j_3^n} - q^3Y_{i_2^n}Y_{j_2^n} + q^4Y_{i_1^n}Y_{j_1^n}$.

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1. Preliminaries

Let $\mathfrak{g}$ be a simple Lie algebra of type E_6 or E_7 over the complex number field $\mathbb{C}$, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let $\Delta \subset \mathfrak{h}^*$ be the root system, and let $W \subset GL(\mathfrak{h})$ be the Weyl group. We denote the set of positive roots by Δ^+ and the set of simple roots by $\{\alpha_i\}_{i\in I_0}$, where I_0 is an index set. For $i \in I_0$ we denote the simple reflection corresponding to α_i by $s_i \in W$. Let $(\ ,\ ) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be the invariant symmetric bilinear form such that $(\alpha, \alpha) = 2$ for any $\alpha \in \Delta$. Set $a_{ij} = (\alpha_i, \alpha_j)$. The matrix $(a_{ij})_{i,j\in I_0}$ is called the Cartan matrix of type E_6 or E_7. For $\alpha \in \Delta$ we denote the corresponding root space by $\mathfrak{g}_\alpha$. Set $\mathfrak{n}^+ = \bigoplus_{\alpha\in\Delta^+}\mathfrak{g}_\alpha$, $\mathfrak{n}^- = \bigoplus_{\alpha\in\Delta^+}\mathfrak{g}_{-\alpha}$. For a subset $I \subset I_0$ we define

$$\Delta_I = \Delta \cap \sum_{i\in I}\mathbb{Z}\alpha_i, \qquad W_I = \langle s_i \mid i \in I\rangle.$$

We set

$$\mathfrak{l}_I = \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in\Delta_I}\mathfrak{g}_\alpha\right), \quad \mathfrak{n}_I^+ = \bigoplus_{\alpha\in\Delta^+\setminus\Delta_I}\mathfrak{g}_\alpha, \quad \mathfrak{n}_I^- = \bigoplus_{\alpha\in\Delta^+\setminus\Delta_I}\mathfrak{g}_{-\alpha}.$$

Let G be a connected algebraic group with Lie algebra $\mathfrak{g}$. We denote by L_I the subgroup of G corresponding to $\mathfrak{l}_I$. Then L_I acts on $\mathfrak{n}_I^\pm$ via the adjoint action.

The quantized enveloping algebra $U_q(\mathfrak{g})$ (Drinfel'd [1], Jimbo [3]) is an associative algebra over the rational function field $\mathbb{C}(q)$ generated by the elements E_i, F_i, K_i, K_i^{-1} $(i \in I_0)$ satisfying the following fundamental relations:

$$\begin{aligned}
&K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1,\\
&K_iE_j = q^{a_{ij}}E_jK_i, \quad K_iF_j = q^{-a_{ij}}F_jK_i,\\
&E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q - q^{-1}},\\
&E_iE_j = E_jE_i && (i \neq j,\ a_{ij} = 0),\\
&E_i^2E_j - (q + q^{-1})E_iE_jE_i + E_jE_i^2 = 0 && (i \neq j,\ a_{ij} = -1),\\
&F_iF_j = F_jF_i && (i \neq j,\ a_{ij} = 0),\\
&F_i^2F_j - (q + q^{-1})F_iF_jF_i + F_jF_i^2 = 0 && (i \neq j,\ a_{ij} = -1).
\end{aligned}$$

A Hopf algebra structure on $U_q(\mathfrak{g})$ is defined as follows. The comultiplication $\Delta : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ is the algebra homomorphism satisfying

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes K_i^{-1} + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i \otimes F_i.$$

The counit $\epsilon : U_q(\mathfrak{g}) \to \mathbb{C}(q)$ is the algebra homomorphism satisfying

$$\epsilon(K_i) = 1, \quad \epsilon(E_i) = \epsilon(F_i) = 0.$$

The antipode $S : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ is the algebra antiautomorphism satisfying

$$S(K_i) = K_i^{-1}, \quad S(E_i) = -E_i K_i, \quad S(F_i) = -K_i^{-1} F_i.$$

Using the Hopf algebra structure, we define the adjoint action of $U_q(\mathfrak{g})$ on $U_q(\mathfrak{g})$ as follows. For $x, y \in U_q(\mathfrak{g})$ write $\Delta(x) = \sum_k x_k^1 \otimes x_k^2$ and set $\mathrm{ad}(x)y = \sum_k x_k^1 y S(x_k^2)$. Then $\mathrm{ad} : U_q(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}(q)}(U_q(\mathfrak{g}))$ is an algebra homomorphism. For $x, y, z \in U_q(\mathfrak{g})$ we have $\mathrm{ad}(x)(yz) = \sum_k (\mathrm{ad}(x_k^1)y)(\mathrm{ad}(x_k^2)z)$, where $\Delta(x) = \sum_k x_k^1 \otimes x_k^2$.

We define subalgebras $U_q(\mathfrak{n}^-)$ and $U_q(\mathfrak{l}_I)$ for $I \subset I_0$ by

$$U_q(\mathfrak{n}^-) = \langle F_i \mid i \in I_0 \rangle, \quad U_q(\mathfrak{l}_I) = \langle E_i, F_i, K_j, K_j^{-1} \mid i \in I, \ j \in I_0 \rangle.$$

For $i \in I_0$ we define an algebra automorphism T_i of $U_q(\mathfrak{g})$ by

$$\begin{aligned} T_i(K_j) &= K_j K_i^{-a_{ij}}, \\ T_i(E_j) &= \begin{cases} -F_i K_i & (i = j) \\ E_j & (i \neq j, \ a_{ij} = 0) \\ E_i E_j - q^{-1} E_j E_i & (i \neq j, \ a_{ij} = -1), \end{cases} \\ T_i(F_j) &= \begin{cases} -K_i^{-1} E_i & (i = j) \\ F_j & (i \neq j, \ a_{ij} = 0) \\ F_j F_i - q F_i F_j & (i \neq j, \ a_{ij} = -1) \end{cases} \end{aligned}$$

(see Lusztig [6]). For $w \in W$ choose a reduced expression $w = s_{i_1} \cdots s_{i_r}$ and set $T_w = T_{i_1} \cdots T_{i_r}$. It is known that T_w does not depend on the choice of a reduced expression.

We shall use the following later (see Lusztig [6]).

Lemma 1.1. *If $w(\alpha_i) = \alpha_j$ for $w \in W$ and $i, j \in I_0$, then we have $T_w(F_i) = F_j$.*

For $I \subset I_0$ let w_I be the longest element of W_I and let w_0 be the longest element of W. Choose a reduced expression $w_I w_0 = s_{i_1} \cdots s_{i_r}$ of $w_I w_0$ and set

$$\beta_j = s_{i_1} s_{i_2} \cdots s_{i_{j-1}}(\alpha_{i_j}), \quad Y_j = Y_{\beta_j} = T_{i_1} \cdots T_{i_{j-1}}(F_{i_j})$$

for $1 \leq j \leq r$. Then it is known that $\{\beta_j \mid 1 \leq j \leq r\} = \Delta^+ \setminus \Delta_I$. Set

$$U_q(\mathfrak{n}_I^-) = \sum_{d_j \geq 0} \mathbb{C}(q) Y_1^{d_1} \cdots Y_r^{d_r}.$$

Then $\{Y_1^{d_1} \cdots Y_r^{d_r} \mid d_j \in \mathbb{Z}_{\geq 0}, 1 \leq j \leq r\}$ is a basis of $U_q(\mathfrak{n}_I^-)$ and $U_q(\mathfrak{n}_I^-)$ is a subalgebra of $U_q(\mathfrak{n}^-)$. we have

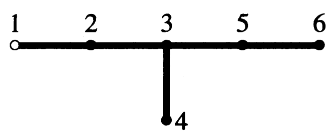
$$U_q(\mathfrak{n}_I^-) = U_q(\mathfrak{n}^-) \cap T_{w_I}^{-1} U_q(\mathfrak{n}^-)$$

and $U_q(\mathfrak{n}_I^-)$ does not depend on the choice of a reduced expression of $w_I w_0$ (see Lusztig [6]).

If $\mathfrak{n}_I^+ \neq \{0\}$, $[\mathfrak{n}_I^+, \mathfrak{n}_I^+] = \{0\}$, then Y_β for $\beta \in \Delta^+ \setminus \Delta_I$ does not depend on the choice of a reduced expression of $w_I w_0$ (see [4]). In this case we denote the $\mathbb{C}(q)$-algebra $U_q(\mathfrak{n}_I^-)$ by A_q. We can regard it as a quantum deformation of the coordinate algebra $A = \mathbb{C}[\mathfrak{n}_I^+]$ of $\mathfrak{n}_I^+$ as explained in [4].

2. Case of type E_6

Let $\mathfrak{g}$ be a simple Lie algebra of type E_6. We shall use the labelling of the vertices of the Dynkin diagram 2.



Dynkin diagram 2.

Hence we have $I_0 = \{1, 2, 3, 4, 5, 6\}$. Set $I = \{2, 3, 4, 5, 6\}$. In this case we have $\mathfrak{n}_I^+ \neq \{0\}$, $[\mathfrak{n}_I^+, \mathfrak{n}_I^+] = \{0\}$. Then $\mathfrak{l}_I$ is isomorphic to $\mathbb{C} \oplus \mathfrak{o}(10, \mathbb{C})$ and $\mathfrak{n}_I^+$ is a 16-dimensional irreducible prehomogeneous vector space. There are three L_I-orbits $\{0\}, C_0, O$ on $\mathfrak{n}_I^+$ satisfying $\{0\} \subset \overline{C_0} \subset \overline{O}$. Let $J_{C_0} \subset \mathbb{C}[\mathfrak{n}_I^+]$ be the defining ideal of the closure of C_0, and let $J_{C_0}^0$ denote the subspace of J_{C_0} consisting of the polynomials in J_{C_0} with homogeneous degree 2. Then $J_{C_0}^0$ is a ten-dimensional irreducible $\mathfrak{l}_I$-module, and it generates the ideal J_{C_0}.

We fix a reduced expression

$$w_I w_0 = s_1 s_2 s_3 s_4 s_5 s_3 s_2 s_1 s_6 s_5 s_3 s_2 s_4 s_3 s_5 s_6$$

of $w_I w_0$ and define the elements Y_i $(i \in \Lambda = \{1, 2, \ldots, 16\})$ as in Section 1.

Set $I_0' = \{1, 2, 3, 4, 5\}$, $I' = \{2, 3, 4, 5\}$, $\Lambda' = \{1, 2, \ldots, 8\}$. Then $\{\alpha_i\}_{i \in I_0'}$ is a set of simple roots of type D_5. Let $\mathfrak{g}'$ be the simple subalgebra of $\mathfrak{g}$ corresponding to I_0'. We choose a reduced expression $w_{I'} w_{I_0'} = s_1 s_2 s_3 s_4 s_5 s_3 s_2 s_1$ of $w_{I'} w_{I_0'}$. The elements Y_i $(i \in \Lambda')$ can be computed inside $U_q(\mathfrak{g}')$.

Let $\beta_j = \sum_{i \in I_0} m_i^j \alpha_i$ and set $\mathbf{m}^j = (m_1^j, \ldots, m_6^j)$ for $j \in \Lambda$. Then we have

$\mathbf{m}^1 = (1,0,0,0,0,0)$, $\mathbf{m}^2 = (1,1,0,0,0,0)$, $\mathbf{m}^3 = (1,1,1,0,0,0)$,
$\mathbf{m}^4 = (1,1,1,1,0,0)$, $\mathbf{m}^5 = (1,1,1,0,1,0)$, $\mathbf{m}^6 = (1,1,1,1,1,0)$,
$\mathbf{m}^7 = (1,1,2,1,1,0)$, $\mathbf{m}^8 = (1,2,2,1,1,0)$, $\mathbf{m}^9 = (1,1,1,0,1,1)$,
$\mathbf{m}^{10} = (1,1,1,1,1,1)$, $\mathbf{m}^{11} = (1,1,2,1,1,1)$, $\mathbf{m}^{12} = (1,2,2,1,1,1)$,
$\mathbf{m}^{13} = (1,1,2,1,2,1)$, $\mathbf{m}^{14} = (1,2,2,1,2,1)$, $\mathbf{m}^{15} = (1,2,3,1,2,1)$,
$\mathbf{m}^{16} = (1,2,3,2,2,1)$.

If $(\beta_j, \alpha_k) = -1$ for $j \in \Lambda$ and $k \in I$, then $s_k(\beta_j) = \beta_j + \alpha_k \in \Delta^+$. Since $k \neq 1$ and $m_1^j = 1$, we have $\beta_j + \alpha_k \notin \Delta_I$. Therefore there exists $l \in \Lambda$ satisfying $\beta_j + \alpha_k = \beta_l$. Conversely if $\beta_j + \alpha_k = \beta_l$ $(j, l \in \Lambda, k \in I)$, then we have $(\beta_j, \alpha_k) = -1$, $s_k(\beta_j) = \beta_l$.

There exist 20 triplets $(k, j, l) \in I \times \Lambda \times \Lambda$ satisfying $\beta_j + \alpha_k = \beta_l$. The triplets are the following: $(2,1,2)$, $(3,2,3)$, $(4,3,4)$, $(5,3,5)$, $(5,4,6)$, $(4,5,6)$, $(3,6,7)$, $(2,7,8)$, $(6,5,9)$, $(4,9,10)$, $(3,10,11)$, $(2,11,12)$, $(5,11,13)$, $(5,12,14)$, $(2,13,14)$, $(3,14,15)$, $(4,15,16)$, $(6,6,10)$, $(6,7,11)$, $(6,8,12)$.

For $k \in I$, $j \in \Lambda$, we have $\beta_j - 2\alpha_k, \beta_j + 2\alpha_k \notin \Delta^+ \setminus \Delta_I$.

Lemma 2.1. *Let $\beta, \beta' \in \Delta^+ \setminus \Delta_I$ satisfying $\beta + \alpha_k = \beta'$ $(k \in I)$. Then we can choose a reduced expression $w_I w_0 = s_{i_1} s_{i_2} \cdots s_{i_{16}}$ and $p \in \Lambda$ satisfying*

$$\beta = s_{i_1} s_{i_2} \cdots s_{i_{p-1}}(\alpha_{i_p}),\ \beta' = s_{i_1} s_{i_2} \cdots s_{i_{p-1}} s_{i_p}(\alpha_{i_{p+1}}),\ (\alpha_{i_p}, \alpha_{i_{p+1}}) = -1,$$
$$\alpha_k = s_{i_1} s_{i_2} \cdots s_{i_{p-1}}(\alpha_{i_{p+1}}).$$

Proof. Among the 20 triplets (k, j, l) satisfying $\beta_j + \alpha_k = \beta_l$ $(k \in I,\ j, k \in \Lambda)$, the 12 triplets satisfy $l = j + 1$, $(\alpha_{i_j}, \alpha_{i_{j+1}}) = -1$. Therefore it is sufficient to deal with the remaining 8 cases. In the cases $(k, j, l) = (5,3,5), (5,4,6), (5,11,13), (5,12,14)$, the reduced expression

$$w_I w_0 = s_1 s_2 s_3 s_5 s_4 s_3 s_2 s_1 s_6 s_5 s_3 s_4 s_2 s_3 s_5 s_6$$

of $w_I w_0$ with $p = 3, 5, 11, 13$ respectively satisfies the required properties. In the cases $(k, j, l) = (6,5,9), (6,6,10), (6,7,11), (6,8,12)$, the reduced expression

$$w_I w_0 = s_1 s_2 s_3 s_4 s_5 s_6 s_3 s_5 s_2 s_3 s_1 s_2 s_4 s_3 s_5 s_6$$

of $w_I w_0$ with $p = 5, 7, 9, 11$ respectively satisfies the required properties. □

It is known that $U_q(\mathfrak{n}_I^+)^1 = \bigoplus_{\beta \in \Delta^+ \setminus \Delta_I} \mathbb{C}(q) Y_\beta$ is an irreducible $U_q(\mathfrak{l}_I)$-module. (see [4])

Lemma 2.2. *For $k \in I$, $j \in \Lambda$, we have*

$$\mathrm{ad}(F_k) Y_j = \begin{cases} Y_l & \text{if there exists } l \in \Lambda \text{ such that } \beta_l = \beta_j + \alpha_k, \\ 0 & \text{otherwise,} \end{cases}$$

$$\operatorname{ad}(E_k)Y_j = \begin{cases} Y_l \ \textit{if there exists } l \in \Lambda \textit{ such that } \beta_l = \beta_j - \alpha_k, \\ 0 \ \textit{otherwise.} \end{cases}$$

Proof. Since $\bigoplus_{j\in\Lambda} \mathbb{C}(q)Y_j$ is a $U_q(\mathfrak{l}_I)$-module, we have $\operatorname{ad}(F_k)Y_j = 0$ if $\beta_j+\alpha_k \notin \Delta^+ \setminus \Delta_I$, and we have $\operatorname{ad}(E_k)Y_j = 0$ if $\beta_j - \alpha_k \notin \Delta^+ \setminus \Delta_I$.

We shall show $\operatorname{ad}(F_k)Y_\beta = Y_{\beta'}$ for $\beta, \beta' \in \Delta^+ \setminus \Delta_I$ and $k \in I$ satisfying $\beta' = \beta + \alpha_k$. By Lemma 2.1 we can choose a reduced expression of $w_I w_0 = s_{i_1}s_{i_2}\cdots s_{i_{16}}$ satisfying $\beta = s_{i_1}s_{i_2}\cdots s_{i_{p-1}}(\alpha_{i_p})$, $\beta' = s_{i_1}s_{i_2}\cdots s_{i_{p-1}}s_{i_p}(\alpha_{i_{p+1}})$, $(\alpha_{i_p}, \alpha_{i_{p+1}}) = -1$. Then we can write $Y_\beta = T_{i_1}T_{i_2}\cdots T_{i_{p-1}}(F_{i_p})$, $Y_{\beta'} = T_{i_1}T_{i_2}\cdots T_{i_{p-1}}T_{i_p}(F_{i_{p+1}})$. Since $(\alpha_{i_p}, \alpha_{i_{p+1}}) = -1$, we have $T_{i_p}(F_{i_{p+1}}) = F_{i_{p+1}}F_{i_p} - qF_{i_p}F_{i_{p+1}}$. Moreover, since $\alpha_k = s_{i_1}s_{i_2}\cdots s_{i_{p-1}}(\alpha_{i_{p+1}})$, we have $T_{i_1}T_{i_2}\cdots T_{i_{p-1}}(F_{i_{p+1}}) = F_k$ by Lemma 1.1, and hence

$$\begin{aligned} Y_{\beta'} &= T_{i_1}T_{i_2}\cdots T_{i_{p-1}}T_{i_p}(F_{i_{p+1}}) \\ &= T_{i_1}T_{i_2}\cdots T_{i_{p-1}}(F_{i_{p+1}}F_{i_p} - qF_{i_p}F_{i_{p+1}}) = F_kY_\beta - qY_\beta F_k. \end{aligned}$$

Since $(\beta, \alpha_k) = -1$, we have $\operatorname{ad}(F_k)Y_\beta = F_kY_\beta - qY_\beta F_k$. Hence we have $\operatorname{ad}(F_k)Y_\beta = Y_{\beta'}$.

Let us show $\operatorname{ad}(E_k)Y_\beta = Y_{\beta'}$ for $\beta, \beta' \in \Delta^+ \setminus \Delta_I$ and $k \in I$ satisfying $\beta' = \beta - \alpha_k$. By the above argument we have $Y_\beta = \operatorname{ad}(F_k)Y_{\beta'} = F_kY_{\beta'} - qY_{\beta'}F_k$. Since $\beta' - \alpha_k = \beta - 2\alpha_k \notin \Delta^+ \setminus \Delta_I$, we have $\operatorname{ad}(E_k)Y_{\beta'} = 0$, and hence $E_kY_{\beta'} = Y_{\beta'}E_k$. Since $(\beta', \alpha_k) = -1$, we have $K_kY_{\beta'} = qY_{\beta'}K_k$. Hence we have

$$\begin{aligned} \operatorname{ad}(E_k)Y_\beta &= (E_kY_\beta - Y_\beta E_k)K_k = \big(E_k(F_kY_{\beta'} - qY_{\beta'}F_k) - (F_kY_{\beta'} - qY_{\beta'}F_k)E_k\big)K_k \\ &= \left(\frac{K_k - K_k^{-1}}{q - q^{-1}}Y_{\beta'} - qY_{\beta'}\frac{K_k - K_k^{-1}}{q - q^{-1}}\right)K_k = (Y_{\beta'}K_k^{-1})K_k = Y_{\beta'}. \qquad \square \end{aligned}$$

Next we shall consider quadratic fundamental relations among the elements Y_i. Since we have

$$\sum_{i,j\in\Lambda} \mathbb{C}(q)Y_iY_j = \bigoplus_{s\le t} \mathbb{C}(q)Y_sY_t,$$

we can write

$$Y_iY_j = \sum_{\substack{s\le t \\ \beta_i+\beta_j=\beta_s+\beta_t}} a_{s,t}^{i,j}Y_sY_t \quad (a_{s,t}^{i,j} \in \mathbb{C}(q))$$

for $i > j$ (see [4]). Hence if $\beta_i+\beta_j$ does not have another decomposition $\beta+\beta'$ ($\beta, \beta' \in \Delta^+ \setminus \Delta_I$, $\beta_i + \beta_j = \beta + \beta'$) then we have $Y_iY_j = a_{i,j}Y_jY_i$ for some $a_{i,j} \in \mathbb{C}(q)$. We denote the set of weights of the ten-dimensional irreducible highest weight $\mathfrak{l}_I$-module $J_{C_0}^0$ with highest weight $-\beta_1 - \beta_8$ by Γ. For $\beta, \beta' \in \Delta^+ \setminus \Delta_I$ a weight $\beta + \beta'$ has another decomposition if and only if we have $-(\beta + \beta') \in \Gamma$. We fix a bijection

$\{1, 2, \dots, 10\} \ni n \mapsto -\delta_n \in \Gamma$ such that if $\delta_m - \delta_n \in \sum_{i\in I_0} \mathbb{Z}_{\geq 0}\alpha_i$, then $n \leq m$. For each n there exist exactly four pairs $(i, j) \in \Lambda^2$ such that $i < j$, $\beta_i + \beta_j = \delta_n$. We denote them by $(i_1^n, j_1^n), (i_2^n, j_2^n), (i_3^n, j_3^n), (i_4^n, j_4^n) \in \Lambda^2$ where $i_4^n < i_3^n < i_2^n < i_1^n$. Set $\mathbf{A}(n) = (i_4^n, i_3^n, i_2^n, i_1^n, j_1^n, j_2^n, j_3^n, j_4^n) \in \Lambda^8$ $(1 \leq n \leq 10)$. Then we have

$$\begin{aligned}
&\mathbf{A}(1) = (1, 2, 3, 4, 5, 6, 7, 8), && \mathbf{A}(2) = (1, 2, 3, 4, 9, 10, 11, 12),\\
&\mathbf{A}(3) = (1, 2, 5, 6, 9, 10, 13, 14), && \mathbf{A}(4) = (1, 3, 5, 7, 9, 11, 13, 15),\\
&\mathbf{A}(5) = (2, 3, 5, 8, 9, 12, 14, 15), && \mathbf{A}(6) = (1, 4, 6, 7, 10, 11, 13, 16),\\
&\mathbf{A}(7) = (2, 4, 6, 8, 10, 12, 14, 16), && \mathbf{A}(8) = (3, 4, 7, 8, 11, 12, 15, 16),\\
&\mathbf{A}(9) = (5, 6, 7, 8, 13, 14, 15, 16), && \mathbf{A}(10) = (9, 10, 11, 12, 13, 14, 15, 16).
\end{aligned}$$

We denote the set $\{i_4^n, i_3^n, i_2^n, i_1^n, j_1^n, j_2^n, j_3^n, j_4^n\}$ by $|\mathbf{A}(n)|$ for $1 \leq n \leq 10$. For any $i, j \in \Lambda$ there exists n satisfying $i, j \in |\mathbf{A}(n)|$.

Set

$$\mathcal{A} = \{(k, n, n') \in I \times \Lambda \times \Lambda \mid \delta_n + \alpha_k = \delta_{n'}\}.$$

Then

$$\begin{aligned}\mathcal{A} = \{&(6, 1, 2), (5, 2, 3), (3, 3, 4), (2, 4, 5), (4, 4, 6),\\ &(2, 6, 7), (4, 5, 7), (3, 7, 8), (5, 8, 9), (6, 9, 10)\}.\end{aligned}$$

For any $n \in \{2, 3, \dots, 10\}$ we can take a sequence $((k_1, n_1, n_1'), \dots, (k_s, n_s, n_s'))$ of $\mathcal{A}$ satisfying $n_1 = 1$, $n_s' = n$, $n_j' = n_{j+1}$ $(1 \leq j \leq s-1)$.

For $(k, n, n') \in \mathcal{A}$ and $m \in \{1, 2, 3, 4\}$, we have either

(P_m^+) $(\beta_{i_m^n}, \alpha_k) = 0, i_m^{n'} = i_m^n, (\beta_{j_m^n}, \alpha_k) = -1, \beta_{j_m^{n'}} = \beta_{j_m^n} + \alpha_k$

or

(P_m^-) $(\beta_{i_m^n}, \alpha_k) = -1, \beta_{i_m^{n'}} = \beta_{i_m^n} + \alpha_k, (\beta_{j_m^n}, \alpha_k) = 0, j_m^{n'} = j_m^n$.

Proposition 2.3. *For any $i, j \in \Lambda$ satisfying $i < j$, we have*

$$\text{(Q6)}\qquad Y_iY_j = \begin{cases} Y_jY_i & \text{if there exists } n \text{ such that } i = i_1^n, j = j_1^n,\\ Y_{j_2^n}Y_{i_2^n} + (q - q^{-1})Y_{i_1^n}Y_{j_1^n} & \text{if there exists } n \text{ such that } i = i_2^n, j = j_2^n,\\ Y_{j_m^n}Y_{i_m^n} + qY_{j_{m-1}^n}Y_{i_{m-1}^n} - q^{-1}Y_{i_{m-1}^n}Y_{j_{m-1}^n} & \\ \qquad \text{if there exist } n,\ m \in \{3, 4\} \text{ such that } i = i_m^n, j = j_m^n, & \\ qY_jY_i & \text{otherwise}.\end{cases}$$

Proof. Since there exists some n satisfying $i, j \in |\mathbf{A}(n)|$ for any $i, j \in \Lambda$, it is sufficient to show that for any $1 \leq n \leq 10$ the elements $Y_{i_m^n}, Y_{j_m^n}$ $(1 \leq m \leq 4)$ satisfy the following relations.

$$(\mathrm{R}n)\quad \begin{cases} Y_{i_1^n}Y_{j_1^n} = Y_{j_1^n}Y_{i_1^n} & (\mathrm{R}n,1)\\ Y_{i_m^n}Y_{j_m^n} = Y_{j_m^n}Y_{i_m^n} + qY_{j_{m-1}^n}Y_{i_{m-1}^n} - q^{-1}Y_{i_{m-1}^n}Y_{j_{m-1}^n}\ (2\le m\le 4) & (\mathrm{R}n,2)\\ Y_{l_1}Y_{l_2} = qY_{l_2}Y_{l_1} & \\ \qquad (l_1,l_2\in|\mathbf{A}(n)|, l_1<l_2, (l_1,l_2)\neq(i_m^n, j_m^n)\ (1\le m\le 4)) & (\mathrm{R}n,3)\end{cases}$$

When $n = 1$, the elements Y_i $(1 \le i \le 8)$ satisfy the same relations as those for type D_5, hence the relations (R1) hold.

For any $m > 1$ there exists a sequence $((k_1, n_1, n_1'), \dots, (k_s, n_s, n_s'))$ of $\mathcal{A}$ satisfying $n_1 = 1, n_s' = m, n_j' = n_{j+1}$ $(1 \le j \le s-1)$, and hence it is sufficient to show the relations $(\mathrm{R}n')$ for $(k, n, n') \in \mathcal{A}$ assuming the relations $(\mathrm{R}n)$.

Let $(k, n, n') \in \mathcal{A}$. Assume that the relations $(\mathrm{R}n)$ hold.

We first show that the relation $(\mathrm{R}n',1)$ holds. If the condition (P_1^+) is satisfied, then we have $Y_{i_1^{n'}} = Y_{i_1^n}$, $F_kY_{i_1^n} = Y_{i_1^n}F_k$, $Y_{j_1^{n'}} = \mathrm{ad}(F_k)Y_{j_1^n} = F_kY_{j_1^n} - qY_{j_1^n}F_k$. Since $Y_{i_1^n}Y_{j_1^n} = Y_{j_1^n}Y_{i_1^n}$, we have

$$\begin{aligned} Y_{i_1^{n'}}Y_{j_1^{n'}} &= Y_{i_1^n}\,\mathrm{ad}(F_k)Y_{j_1^n} = Y_{i_1^n}(F_kY_{j_1^n} - qY_{j_1^n}F_k)\\ &= (F_kY_{j_1^n} - qY_{j_1^n}F_k)Y_{i_1^n} = Y_{j_1^{n'}}Y_{i_1^{n'}}. \end{aligned}$$

If the condition (P_1^-) is satisfied, then we can prove the formula $(\mathrm{R}n',1)$ similarly.

Next we prove the formula $(\mathrm{R}n',2)$. Assume the condition (P_m^+) is satisfied, then we have

$$\begin{aligned} Y_{i_m^{n'}}Y_{j_m^{n'}} &= Y_{i_m^n}(F_kY_{j_m^n} - qY_{j_m^n}F_k)\\ &= F_kY_{j_m^n}Y_{i_m^n} - qY_{j_m^n}F_kY_{i_m^n}\\ &\quad + q(F_kY_{j_{m-1}^n}Y_{i_{m-1}^n} - qY_{j_{m-1}^n}Y_{i_{m-1}^n}F_k)\\ &\quad - q^{-1}(F_kY_{i_{m-1}^n}Y_{j_{m-1}^n} - qY_{i_{m-1}^n}Y_{j_{m-1}^n}F_k). \end{aligned}$$

If the condition (P_{m-1}^+) is satisfied, then we have

$$\begin{aligned} F_kY_{j_{m-1}^n}Y_{i_{m-1}^n} - qY_{j_{m-1}^n}Y_{i_{m-1}^n}F_k &= Y_{j_{m-1}^n}(F_kY_{i_{m-1}^n} - qY_{i_{m-1}^n}F_k) = Y_{j_{m-1}^{n'}}Y_{i_{m-1}^{n'}},\\ F_kY_{i_{m-1}^n}Y_{j_{m-1}^n} - qY_{i_{m-1}^n}Y_{j_{m-1}^n}F_k &= (F_kY_{i_{m-1}^n} - qY_{i_{m-1}^n}F_k)Y_{j_{m-1}^n} = Y_{i_{m-1}^{n'}}Y_{j_{m-1}^{n'}}, \end{aligned}$$

and if the condition (P_{m-1}^-) is satisfied, then we have

$$\begin{aligned} F_kY_{j_{m-1}^n}Y_{i_{m-1}^n} - qY_{j_{m-1}^n}Y_{i_{m-1}^n}F_k &= (F_kY_{j_{m-1}^n} - qY_{j_{m-1}^n}F_k)Y_{i_{m-1}^n} = Y_{j_{m-1}^{n'}}Y_{i_{m-1}^{n'}},\\ F_kY_{i_{m-1}^n}Y_{j_{m-1}^n} - qY_{i_{m-1}^n}Y_{j_{m-1}^n}F_k &= Y_{i_{m-1}^n}(F_kY_{j_{m-1}^n} - qY_{j_{m-1}^n}F_k) = Y_{i_{m-1}^{n'}}Y_{j_{m-1}^{n'}}. \end{aligned}$$

Hence we have $Y_{i_m^{n'}}Y_{j_m^{n'}} = Y_{j_m^{n'}}Y_{i_m^{n'}} + qY_{j_{m-1}^{n'}}Y_{i_{m-1}^{n'}} - q^{-1}Y_{i_{m-1}^{n'}}Y_{j_{m-1}^{n'}}$. The formula $(\mathrm{R}n',2)$ is proved. When the condition (P_m^-) is satisfied, we can prove it similarly.

Finally we prove the formula (Rn',3). Let $l'_1, l'_2 \in |\mathbf{A}(n')|$ satisfying $l'_1 < l'_2$ and $(l'_1, l'_2) \neq (i^{n'}_m, j^{n'}_m)$ for $1 \leq m \leq 4$. When $l'_p = i^{n'}_m \in |\mathbf{A}(n')|$ (resp. $l'_p = j^{n'}_m$), we denote $i^n_m \in |\mathbf{A}(n)|$ (resp. j^n_m) by l_p for $p = 1, 2$. Since $l_1 < l_2$ and $(l_1, l_2) \neq (i^n_m, j^n_m)$ for $1 \leq m \leq 4$, we have $Y_{l_1} Y_{l_2} = q Y_{l_2} Y_{l_1}$. We have the following possibilities:

(1) $l'_1 = l_1$, $l'_2 = l_2$, $(\beta_{l_1}, \alpha_k) = (\beta_{l_2}, \alpha_k) = 0$,

(2) $l'_1 = l_1$, $(\beta_{l_1}, \alpha_k) = 0$, $\beta_{l'_2} = \beta_{l_2} + \alpha_k$, $(\beta_{l_2}, \alpha_k) = -1$,

(3) $\beta_{l'_1} = \beta_{l_1} + \alpha_k$, $(\beta_{l_1}, \alpha_k) = -1$, $l'_2 = l_2$, $(\beta_{l_2}, \alpha_k) = 0$,

(4) $\beta_{l'_1} = \beta_{l_1} + \alpha_k$, $\beta_{l'_2} = \beta_{l_2} + \alpha_k$, $(\beta_{l_1}, \alpha_k) = (\beta_{l_2}, \alpha_k) = -1$.

In the case (1) the formula (Rn',3) is obvious.

In the case (2) we have $F_k Y_{l_1} = Y_{l_1} F_k$, $Y_{l'_2} = \mathrm{ad}(F_k) Y_{l_2} = F_k Y_{l_2} - q Y_{l_2} F_k$. Hence we have

$$Y_{l'_1} Y_{l'_2} = Y_{l_1}(F_k Y_{l_2} - q Y_{l_2} F_k) = q(F_k Y_{l_2} - q Y_{l_2} F_k) Y_{l_1} = q Y_{l'_2} Y_{l'_1}.$$

In the case (3) we can prove it similarly to the case (2).

In the case (4) we have $Y_{l'_p} = F_k Y_{l_p} - q Y_{l_p} F_k$ for $p = 1, 2$. Since $\beta_{l'_p} + \alpha_k = \beta_{l_p} + 2\alpha_k \notin \Delta^+ \setminus \Delta_I$ and $(\beta_{l'_p}, \alpha_k) = 1$, we have $\mathrm{ad}(F_k) Y_{l'_p} = F_k Y_{l'_p} - q^{-1} Y_{l'_p} F_k = 0$ for $p = 1, 2$. Hence we have $F_k F_k Y_{l_p} - (q + q^{-1}) F_k Y_{l_p} F_k + Y_{l_p} F_k F_k = 0$, $F_k Y_{l_p} F_k = (q + q^{-1})^{-1}(F_k F_k Y_{l_p} + Y_{l_p} F_k F_k)$ for $p = 1, 2$. By these formulas we have

$$\begin{aligned}
Y_{l'_1} Y_{l'_2} &= (F_k Y_{l_1} - q Y_{l_1} F_k)(F_k Y_{l_2} - q Y_{l_2} F_k) \\
&= F_k Y_{l_1} F_k Y_{l_2} - q F_k Y_{l_1} Y_{l_2} F_k - q Y_{l_1} F_k F_k Y_{l_2} + q^2 Y_{l_1} F_k Y_{l_2} F_k \\
&= \frac{1}{q + q^{-1}} F_k F_k Y_{l_1} Y_{l_2} + \frac{1}{q + q^{-1}} Y_{l_1} F_k F_k Y_{l_2} - q F_k Y_{l_1} Y_{l_2} F_k - q Y_{l_1} F_k F_k Y_{l_2} \\
&\quad + \frac{q^2}{q + q^{-1}} Y_{l_1} F_k F_k Y_{l_2} + \frac{q^2}{q + q^{-1}} Y_{l_1} Y_{l_2} F_k F_k \\
&= \frac{1}{q + q^{-1}} F_k F_k Y_{l_1} Y_{l_2} - q F_k Y_{l_1} Y_{l_2} F_k + \frac{q^2}{q + q^{-1}} Y_{l_1} Y_{l_2} F_k F_k.
\end{aligned}$$

Similarly we have

$$Y_{l'_2} Y_{l'_1} = \frac{1}{q + q^{-1}} F_k F_k Y_{l_2} Y_{l_1} - q F_k Y_{l_2} Y_{l_1} F_k + \frac{q^2}{q + q^{-1}} Y_{l_2} Y_{l_1} F_k F_k.$$

Since $Y_{l_1} Y_{l_2} = q Y_{l_2} Y_{l_1}$, we have $Y_{l'_1} Y_{l'_2} = q Y_{l'_2} Y_{l'_1}$. □

By [4] and Proposition 2.3 we obtain the following:

Theorem 2.4. *The formulas* (Q6) *give fundamental relations for the generator system* $\{Y_i\}_{i \in \Lambda}$ *of the algebra* $A_q = U_q(\mathfrak{n}_I^-)$.

We shall construct a quantum deformation of the lowest degree part $J^0_{C_0}$ of the defining ideal J_{C_0} and we shall give canonical generators of a quantum analogue of

J_{C_0}.

Set

$$\psi_n = Y_{i_4^n}Y_{j_4^n} - qY_{i_3^n}Y_{j_3^n} + q^2Y_{i_2^n}Y_{j_2^n} - q^3Y_{i_1^n}Y_{j_1^n},$$

for $1 \leq n \leq 10$. Recall that $\mathrm{A}(n) = (i_4^n, i_3^n, i_2^n, i_1^n, j_1^n, j_2^n, j_3^n, j_4^n)$. Using the formulas (R$n$,1), (R$n$,2), we can write $\psi_n = Y_{j_4^n}Y_{i_4^n} - q^{-1}Y_{j_3^n}Y_{i_3^n} + q^{-2}Y_{j_2^n}Y_{i_2^n} - q^{-3}Y_{j_1^n}Y_{i_1^n}$.

Lemma 2.5. *We have*

$$\mathrm{ad}(F_k)\psi_n = \begin{cases} \psi_{n'} & \text{if there exists } n' \text{ such that } \delta_n + \alpha_k = \delta_{n'}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathrm{ad}(E_k)\psi_n = \begin{cases} \psi_{n'} & \text{if there exists } n' \text{ such that } \delta_n - \alpha_k = \delta_{n'}, \\ 0 & \text{otherwise} \end{cases}$$

for $k \in I$, and

$$\mathrm{ad}(K_k)\psi_n = q^{-(\delta_n,\alpha_k)}\psi_n$$

for $k \in I_0$.

Proof. Let $(k, n, n') \in \mathcal{A}$. We shall show $\mathrm{ad}(F_k)\psi_n = \psi_{n'}$. If the condition ($\mathrm{P}_m^+$) is satisfied, then we have $\mathrm{ad}(F_k)Y_{i_m^n} = 0$, $Y_{i_m^{n'}} = Y_{i_m^n}$, $\mathrm{ad}(K_k)Y_{i_m^n} = Y_{i_m^n}$ $\mathrm{ad}(F_k)Y_{j_m^n} = Y_{j_m^{n'}}$. Hence

$$\mathrm{ad}(F_k)(Y_{i_m^n}Y_{j_m^n}) = (\mathrm{ad}(F_k)Y_{i_m^n})Y_{j_m^n} + (\mathrm{ad}(K_k)Y_{i_m^n})(\mathrm{ad}(F_k)Y_{j_m^n}) = Y_{i_m^{n'}}Y_{j_m^{n'}}.$$

If the condition (P_m^-) is satisfied, then we have $\mathrm{ad}(F_k)Y_{i_m^n} = Y_{i_m^{n'}}$, $\mathrm{ad}(F_k)Y_{j_m^n} = 0$. Hence $\mathrm{ad}(F_k)(Y_{i_m^n}Y_{j_m^n}) = Y_{i_m^{n'}}Y_{j_m^{n'}}$ similarly. Therefore we have $\mathrm{ad}(F_k)\psi_n = \psi_{n'}$.

Next we prove $\mathrm{ad}(E_k)\psi_{n'} = \psi_n$. We have $\mathrm{ad}(E_k)Y_{i_m^{n'}} = 0$, $\mathrm{ad}(E_k)Y_{j_m^{n'}} = Y_{j_m^n}$ if the condition (P_m^+) is satisfied, and we have $\mathrm{ad}(E_k)Y_{i_m^{n'}} = Y_{i_m^n}$, $\mathrm{ad}(K_k^{-1})Y_{j_m^{n'}} = Y_{j_m^{n'}}$, $j_m^{n'} = j_m^n$, $\mathrm{ad}(E_k)Y_{j_m^{n'}} = 0$ if the condition (P_m^-) is satisfied. Hence we have

$$\mathrm{ad}(E_k)(Y_{i_m^{n'}}Y_{j_m^{n'}}) = (\mathrm{ad}(E_k)Y_{i_m^{n'}})(\mathrm{ad}(K_k^{-1})Y_{j_m^{n'}}) + Y_{i_m^{n'}}(\mathrm{ad}(E_k)Y_{j_m^{n'}}) = Y_{i_m^n}Y_{j_m^n}$$

for $1 \leq m \leq 4$. Therefore we have $\mathrm{ad}(E_k)\psi_{n'} = \psi_n$.

In other 50 cases, where $\delta_n + \alpha_k \notin \{\delta_l \mid 1 \leq l \leq 10\}$, we can check $\mathrm{ad}(F_k)\psi_n = 0$ by a case-by-case consideration as follows.

In the 10 cases where there exists n' satisfying $\mathrm{ad}(F_k)\psi_{n'} = \psi_n$, ($(k, n) = (6, 2)$, $(5, 3)$, $(3, 4)$, $(2, 5)$, $(4, 6)$, $(2, 7)$, $(4, 7)$, $(3, 8)$, $(5, 9)$, $(6, 10)$), we have $\mathrm{ad}(F_k)Y_{i_m^n} = \mathrm{ad}(F_k)Y_{j_m^n} = 0$ for $1 \leq m \leq 4$, and hence the assertion is obvious.

In the 8 cases $(k, n) = (5, 1)$, $(6, 3)$, $(6, 4)$, $(6, 5)$, $(6, 6)$, $(6, 7)$, $(6, 8)$, $(5, 10)$, we have $\mathrm{ad}(F_k)Y_{i_m^n} = \mathrm{ad}(F_k)Y_{j_m^n} = 0$ for $m = 3, 4$, $\mathrm{ad}(F_k)Y_{i_2^n} = Y_{j_1^n}$, $\mathrm{ad}(F_k)Y_{j_2^n} = 0$,

$\mathrm{ad}(F_k)Y_{i_1^n} = Y_{j_2^n}$, $\mathrm{ad}(F_k)Y_{j_1^n} = 0$, and hence $\mathrm{ad}(F_k)(Y_{i_2^n}Y_{j_2^n}) = Y_{j_1^n}Y_{j_2^n}$, $\mathrm{ad}(F_k)(Y_{i_1^n}Y_{j_1^n}) = Y_{j_2^n}Y_{j_1^n}$. Thus we have $\mathrm{ad}(F_k)\psi_n = q^2(Y_{j_1^n}Y_{j_2^n} - qY_{j_2^n}Y_{j_1^n}) = 0$ by Proposition 2.3.

In the remaining 32 cases there exists $m' \in \{2,3,4\}$ such that $\mathrm{ad}(F_k)Y_{i_m^n} = 0$ $(m \neq m')$, $\mathrm{ad}(F_k)Y_{j_m^n} = 0$ $(m \neq m'-1)$, $\mathrm{ad}(F_k)Y_{i_{m'}^n} = Y_{i_{m'-1}^n}$, $\mathrm{ad}(F_k)Y_{j_{m'-1}^n} = Y_{j_{m'}^n}$, $\mathrm{ad}(K_k)Y_{i_{m'-1}^n} = q^{-1}Y_{i_{m'-1}^n}$. Then we have $\mathrm{ad}(F_k)(Y_{i_{m'}^n}Y_{j_{m'}^n}) = Y_{i_{m'-1}^n}Y_{j_{m'}^n}$, $\mathrm{ad}(F_k)(Y_{i_{m'-1}^n}Y_{j_{m'-1}^n}) = q^{-1}Y_{i_{m'-1}^n}Y_{j_{m'}^n}$, $\mathrm{ad}(F_k)\psi_n = q^{4-m'}(1-qq^{-1})Y_{i_{m'-1}^n}Y_{j_{m'}^n} = 0$.

The weight $\beta_{i_m^n}+\beta_{j_m^n}$ does not depend on m. Hence we have $\mathrm{ad}(K_k)\psi_n = q^{-(\delta_n,\alpha_k)}\psi_n$ where $\delta_n = \beta_{i_m^n} + \beta_{j_m^n}$.

Finally we show $\mathrm{ad}(E_k)\psi_n = 0$ if $\delta_n - \alpha_k \notin \{\delta_l \mid 1 \le l \le 10\}$. We can check $\mathrm{ad}(E_k)\psi_1 = 0$ for any $k = 2, 3, \dots, 6$ directly. It follows that $\sum_{n=1}^{10}\mathbb{C}(q)\psi_n = U_q(\mathfrak{l}_I)\psi_1$ and hence $\sum_{n=1}^{10}\mathbb{C}(q)\psi_n$ is an $\mathrm{ad}\,U_q(\mathfrak{l}_I)$-stable subspace with weights in $\{-\delta_l \mid 1 \le l \le 10\}$. Therefore we have $\mathrm{ad}(E_k)\psi_n = 0$ if $\delta_n - \alpha_k \notin \{\delta_l \mid 1 \le l \le 10\}$. □

Proposition 2.6. *$\sum_{n=1}^{10}\mathbb{C}(q)\psi_n$ is an irreducible highest weight $U_q(\mathfrak{l}_I)$-module with highest weight vector ψ_1.*

Proof. By Lemma 2.5 $\sum_{n=1}^{10}\mathbb{C}(q)\psi_n$ is a finite dimensional $U_q(\mathfrak{l}_I)$-submodule generated by a highest weight vector ψ_1 with highest weight $-\delta_1$. Thus it is irreducible. □

By [4] and Proposition 2.6 we obtain the following:

Theorem 2.7. *A quantum analogue of the defining ideal J_{C_0} of the closure of the non-trivial non-open orbit C_0 is given by the two-sided ideal of A_q generated by $\{\psi_n \mid 1 \le n \le 10\}$.*

3. Case of type E_7

Let $\mathfrak{g}$ be a simple Lie algebra of type E_7. We shall use the labelling of the vertices of the Dynkin diagram 1. Hence we have $I_0 = \{1,2,3,4,5,6,7\}$. Set $I = \{2,3,4,5,6,7\}$. In this case we have $\mathfrak{n}_I^+ \neq \{0\}$, $[\mathfrak{n}_I^+, \mathfrak{n}_I^+] = \{0\}$. Then $\mathfrak{l}_I$ is isomorphic to $\mathbb{C} \oplus \mathfrak{g}_{E_6}$, where $\mathfrak{g}_{E_6}$ is a Lie algebra of type E_6 over $\mathbb{C}$, and $\mathfrak{n}_I^+$ is a 27-dimensional irreducible prehomogeneous vector space. There are four L_I-orbits $\{0\}, C_1, C_2, O$ on $\mathfrak{n}_I^+$ satisfying $\{0\} \subset \overline{C_1} \subset \overline{C_2} \subset \overline{O}$. Let $J_{C_1} \subset \mathbb{C}[\mathfrak{n}_I^+]$ be the defining ideal of the closure of C_1, and let $J_{C_1}^0$ denote the subspace of J_{C_1} consisting of the polynomials in J_{C_1} with homogeneous degree 2. Then $J_{C_1}^0$ is a 27-dimensional irreducible $\mathfrak{l}_I$-module, and it generates the ideal J_{C_1}. Let $J_{C_2} \subset \mathbb{C}[\mathfrak{n}_I^+]$ be the defining ideal of the closure of C_2, and let $J_{C_2}^0$ denote the subspace of J_{C_2} consisting of the polynomials in J_{C_2} with homogeneous degree 3. Then $J_{C_2}^0$ is a one-dimensional irreducible $\mathfrak{l}_I$-module generated by the irreducible relative invariant, and it generates the ideal J_{C_2}.

We fix a reduced expression

$$w_I w_0 = s_1 s_2 s_3 s_4 s_5 s_6 s_4 s_3 s_2 s_1 s_7 s_6 s_4 s_3 s_5 s_4 s_6 s_7 s_2 s_3 s_4 s_6 s_5 s_4 s_3 s_2 s_1$$

of $w_I w_0$ and define the elements Y_i $(i \in \Lambda = \{1, 2, \dots, 27\})$ as in Section 1.

Set $I'_0 = \{1, 2, 3, 4, 5, 6\}$, $I' = \{2, 3, 4, 5, 6\}$, $\Lambda' = \{1, 2, \dots, 10\}$. Then $\{\alpha_i\}_{i \in I'_0}$ is a set of simple roots of type D_6. Let $\mathfrak{g}'$ be the simple subalgebra of $\mathfrak{g}$ corresponding to I'_0. We choose a reduced expression $w_{I'} w_{I'_0} = s_1 s_2 s_3 s_4 s_5 s_6 s_4 s_3 s_2 s_1$ of $w_{I'} w_{I'_0}$. The elements Y_i $(i \in \Lambda')$ can be computed inside $U_q(\mathfrak{g}')$.

Let $\beta_j = \sum_{i \in I_0} m_i^j \alpha_i$ and set $\mathbf{m}^j = (m_1^j, \dots, m_7^j)$ for $j \in \Lambda$. Then we have

$\mathbf{m}^1 = (1, 0, 0, 0, 0, 0, 0)$, $\mathbf{m}^2 = (1, 1, 0, 0, 0, 0, 0)$, $\mathbf{m}^3 = (1, 1, 1, 0, 0, 0, 0)$,
$\mathbf{m}^4 = (1, 1, 1, 1, 0, 0, 0)$, $\mathbf{m}^5 = (1, 1, 1, 1, 1, 0, 0)$, $\mathbf{m}^6 = (1, 1, 1, 1, 0, 1, 0)$,
$\mathbf{m}^7 = (1, 1, 1, 1, 1, 1, 0)$, $\mathbf{m}^8 = (1, 1, 1, 2, 1, 1, 0)$, $\mathbf{m}^9 = (1, 1, 2, 2, 1, 1, 0)$,
$\mathbf{m}^{10} = (1, 2, 2, 2, 1, 1, 0)$, $\mathbf{m}^{11} = (1, 1, 1, 1, 0, 1, 1)$, $\mathbf{m}^{12} = (1, 1, 1, 1, 1, 1, 1)$,
$\mathbf{m}^{13} = (1, 1, 1, 2, 1, 1, 1)$, $\mathbf{m}^{14} = (1, 1, 2, 2, 1, 1, 1)$, $\mathbf{m}^{15} = (1, 1, 1, 2, 1, 2, 1)$,
$\mathbf{m}^{16} = (1, 1, 2, 2, 1, 2, 1)$, $\mathbf{m}^{17} = (1, 1, 2, 3, 1, 2, 1)$, $\mathbf{m}^{18} = (1, 1, 2, 3, 2, 2, 1)$,
$\mathbf{m}^{19} = (1, 2, 2, 2, 1, 1, 1)$, $\mathbf{m}^{20} = (1, 2, 2, 2, 1, 2, 1)$, $\mathbf{m}^{21} = (1, 2, 2, 3, 1, 2, 1)$,
$\mathbf{m}^{22} = (1, 2, 2, 3, 2, 2, 1)$, $\mathbf{m}^{23} = (1, 2, 3, 3, 1, 2, 1)$, $\mathbf{m}^{24} = (1, 2, 3, 3, 2, 2, 1)$,
$\mathbf{m}^{25} = (1, 2, 3, 4, 2, 2, 1)$, $\mathbf{m}^{26} = (1, 2, 3, 4, 2, 3, 1)$, $\mathbf{m}^{27} = (1, 2, 3, 4, 2, 3, 2)$.

If $(\beta_j, \alpha_k) = -1$ for $j \in \Lambda$ and $k \in I$, then $s_k(\beta_j) = \beta_j + \alpha_k \in \Delta^+ \setminus \Delta_I$ and there exists $l \in \Lambda$ satisfying $\beta_j + \alpha_k = \beta_l$. Conversely if $\beta_j, \beta_l \in \Delta^+ \setminus \Delta_I$ satisfying $\beta_l - \beta_j = \alpha_k$ $(k \in I)$, then we have $(\beta_j, \alpha_k) = -1$, $s_k(\beta_j) = \beta_l$.

For $k \in I$, $j \in \Lambda$, we have $\beta_j - 2\alpha_k$, $\beta_j + 2\alpha_k \notin \Delta^+ \setminus \Delta_I$.

Set

$$\mathcal{B} = \{(k, j, l) \in I \times \Lambda \times \Lambda \mid \beta_j + \alpha_k = \beta_l\}.$$

We have
$\mathcal{B} = \{(2, 1, 2), (3, 2, 3), (4, 3, 4), (5, 4, 5), (6, 4, 6), (6, 5, 7), (5, 6, 7), (4, 7, 8), (3, 8, 9),$
$(2, 9, 10), (7, 6, 11), (7, 7, 12), (7, 8, 13), (7, 9, 14), (7, 10, 19), (5, 11, 12),$
$(4, 12, 13), (3, 13, 14), (6, 13, 15), (6, 14, 16), (3, 15, 16), (4, 16, 17), (5, 17, 18),$
$(2, 14, 19), (2, 16, 20), (2, 17, 21), (2, 18, 22), (6, 19, 20), (4, 20, 21), (5, 21, 22),$
$(3, 21, 23), (3, 22, 24), (5, 23, 24), (4, 24, 25), (6, 25, 26), (7, 26, 27)\}.$
In particular, we have $|\mathcal{B}| = 36$.

Lemma 3.1. *Let $\beta, \beta' \in \Delta^+ \setminus \Delta_I$ satisfying $\beta + \alpha_k = \beta'$ $(k \in I)$. Then we can choose a reduced expression $w_I w_0 = s_{i_1} s_{i_2} \cdots s_{i_{27}}$ and $p \in \Lambda$ satisfying*

$$\beta = s_{i_1} s_{i_2} \cdots s_{i_{p-1}}(\alpha_{i_p}), \quad \beta' = s_{i_1} s_{i_2} \cdots s_{i_{p-1}} s_{i_p}(\alpha_{i_{p+1}}), \quad (\alpha_{i_p}, \alpha_{i_{p+1}}) = -1,$$
$$\alpha_k = s_{i_1} s_{i_2} \cdots s_{i_{p-1}}(\alpha_{i_{p+1}}).$$

Proof. The 21 triplets (k, j, l) in $\mathcal{B}$ satisfy $l = j+1$, $(\alpha_{i_j}, \alpha_{i_{j+1}}) = -1$. Therefore it is sufficient to deal with the remaining 15 cases. In the cases $(k, j, l) = (6, 4, 6)$, $(6, 5, 7)$, $(6, 13, 15)$, $(6, 14, 16)$, $(3, 21, 23)$, $(3, 22, 24)$, we can take

$$w_I w_0 = s_1s_2s_3s_4s_6s_5s_4s_3s_2s_1s_7s_6s_4s_5s_3s_4s_6s_7s_2s_3s_4s_5s_6s_4s_3s_2s_1$$

with $p = 4, 6, 13, 15, 21, 23$, and in the cases $(k, j, l) = (7, 6, 11)$, $(7, 7, 12)$, $(7, 8, 13)$, $(7, 9, 14)$, $(7, 10, 19)$, we can take

$$w_I w_0 = s_1s_2s_3s_4s_5s_6s_7s_4s_6s_3s_4s_2s_3s_1s_2s_5s_4s_6s_7s_3s_4s_6s_5s_4s_3s_2s_1$$

with $p = 6, 8, 10, 12, 14$, and in the cases $(k, j, l) = (2, 14, 19)$, $(2, 16, 20)$, $(2, 17, 21)$, $(2, 18, 22)$, we can take

$$w_I w_0 = s_1s_2s_3s_4s_5s_6s_4s_3s_2s_1s_7s_6s_4s_5s_3s_2s_4s_3s_6s_4s_7s_6s_5s_4s_3s_2s_1$$

with $p = 15, 17, 19, 21$. □

We can show the following similarly to the case E_6. We omit the details.

Lemma 3.2. *For $k \in I$, $j \in \Lambda$, we have*

$$\mathrm{ad}(F_k)Y_j = \begin{cases} Y_l & \text{if there exists } (k, j, l) \in \mathcal{B}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathrm{ad}(E_k)Y_j = \begin{cases} Y_l & \text{if there exists } (k, l, j) \in \mathcal{B}, \\ 0 & \text{otherwise.} \end{cases}$$

The $U_q(\mathfrak{l}_I)$-module $\bigoplus_{j\in\Lambda} \mathbb{C}(q)Y_j$ is an irreducible highest weight module with highest weight vector Y_1 and lowest weight vector Y_{27}. Hence, for any $1 \le m \le 26$, there exists a sequence $((k_1, n_1', n_1), \dots, (k_s, n_s', n_s))$ of $\mathcal{B}$ satisfying $n_1 = 27$, $n_s' = m$, $n_j' = n_{j+1}$ $(1 \le j \le s-1)$.

Next we shall consider relations among the elements Y_i. We can write

$$Y_iY_j = \sum_{\substack{s\le t \\ \beta_i+\beta_j=\beta_s+\beta_t}} a_{s,t}^{i,j} Y_sY_t \quad (a_{s,t}^{i,j} \in \mathbb{C}(q))$$

for $i > j$ (see [4]). Hence if $\beta_i+\beta_j$ does not have another decomposition $\beta+\beta'$ ($\beta, \beta' \in \Delta^+ \setminus \Delta_I$, $\beta_i + \beta_j = \beta + \beta'$) then we have $Y_iY_j = a_{i,j}Y_jY_i$ for some $a_{i,j} \in \mathbb{C}(q)$. Set $\delta = 2\varpi_1 = 3\alpha_1+4\alpha_2+5\alpha_3+6\alpha_4+3\alpha_5+4\alpha_6+2\alpha_7$, where ϖ_1 is the fundamental weight corresponding to α_1. We denote a set of weights of the 27-dimensional irreducible highest weight $\mathfrak{l}_I$-module $J_{C_1}^0$ with highest weight $-\beta_1 - \beta_{10}$ by Γ. Set $\gamma_n = \delta - \beta_n$ $(n \in \Lambda)$, and we have $\Gamma = \{-\gamma_n \mid n \in \Lambda\}$. For $\beta, \beta' \in \Delta^+ \setminus \Delta_I$ a weight $\beta + \beta'$ has another decomposition if and only if we have $-(\beta + \beta') \in \Gamma$. For each $n \in \Lambda$ there exist exactly five pairs $(i, j) \in \Lambda^2$ such that $i < j$, $\beta_i + \beta_j = \gamma_n$. We denote them by (i_1^n, j_1^n), (i_2^n, j_2^n), (i_3^n, j_3^n), (i_4^n, j_4^n), $(i_5^n, j_5^n) \in \Lambda^2$ where $i_5^n < i_4^n < i_3^n < i_2^n < i_1^n$, $j_1^n < j_2^n < j_3^n < j_4^n < j_5^n$, and i_1^n, j_1^n satisfy the following condition (P_1^+) or (P_1^-). Set $\mathbf{B}(n) = (i_5^n, i_4^n, i_3^n, i_2^n, i_1^n, j_1^n, j_2^n, j_3^n, j_4^n, j_5^n) \in \Lambda^{10}$ $(n \in \Lambda)$. Then we have

$\mathbf{B}(1)=(10, 19, 20, 21, 23, 22, 24, 25, 26, 27)$, $\mathbf{B}(2)=(9, 14, 16, 17, 23, 18, 24, 25, 26, 27)$,
$\mathbf{B}(3)=(8, 13, 15, 17, 21, 18, 22, 25, 26, 27)$, $\mathbf{B}(4)=(7, 12, 15, 16, 20, 18, 22, 24, 26, 27)$,
$\mathbf{B}(5)=(6, 11, 15, 16, 20, 17, 21, 23, 26, 27)$, $\mathbf{B}(6)=(5, 12, 13, 14, 19, 18, 22, 24, 25, 27)$,
$\mathbf{B}(7)=(4, 11, 13, 14, 19, 17, 21, 23, 25, 27)$, $\mathbf{B}(8)=(3, 11, 12, 14, 19, 16, 20, 23, 24, 27)$,
$\mathbf{B}(9)=(2, 11, 12, 13, 19, 15, 20, 21, 22, 27)$, $\mathbf{B}(10)=(1, 11, 12, 13, 14, 15, 16, 17, 18, 27)$,
$\mathbf{B}(11)=(5, 7, 8, 9, 10, 18, 22, 24, 25, 26)$, $\mathbf{B}(12)=(4, 6, 8, 9, 10, 17, 21, 23, 25, 26)$,
$\mathbf{B}(13)=(3, 6, 7, 9, 10, 16, 20, 23, 24, 26)$, $\mathbf{B}(14)=(2, 6, 7, 8, 10, 15, 20, 21, 22, 26)$,
$\mathbf{B}(15)=(3, 4, 5, 9, 10, 14, 19, 23, 24, 25)$, $\mathbf{B}(16)=(2, 4, 5, 8, 10, 13, 19, 21, 22, 25)$,
$\mathbf{B}(17)=(2, 3, 5, 7, 10, 12, 19, 20, 22, 24)$, $\mathbf{B}(18)=(2, 3, 4, 6, 10, 11, 19, 20, 21, 23)$,
$\mathbf{B}(19)=(1, 6, 7, 8, 9, 15, 16, 17, 18, 26)$, $\mathbf{B}(20)=(1, 4, 5, 8, 9, 13, 14, 17, 18, 25)$,
$\mathbf{B}(21)=(1, 3, 5, 7, 9, 12, 14, 16, 18, 24)$, $\mathbf{B}(22)=(1, 3, 4, 6, 9, 11, 14, 16, 17, 23)$,
$\mathbf{B}(23)=(1, 2, 5, 7, 8, 12, 13, 15, 18, 22)$, $\mathbf{B}(24)=(1, 2, 4, 6, 8, 11, 13, 15, 17, 21)$,
$\mathbf{B}(25)=(1, 2, 3, 6, 7, 11, 12, 15, 16, 20)$, $\mathbf{B}(26)=(1, 2, 3, 4, 5, 11, 12, 13, 14, 19)$,
$\mathbf{B}(27)=(1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$.

For $n \in \Lambda$ we denote the set $\{i_5^n, i_4^n, i_3^n, i_2^n, i_1^n, j_1^n, j_2^n, j_3^n, j_4^n, j_5^n\}$ by $|\mathbf{B}(n)|$. For any $i, j \in \Lambda$ there exists $n \in \Lambda$ satisfying $i, j \in |\mathbf{B}(n)|$.

For $(k, n', n) \in \mathcal{B}$ and $m \in \{1, 2, 3, 4, 5\}$, we have either

(P_m^+) $(\beta_{i_m^n}, \alpha_k) = 0$, $i_m^{n'} = i_m^n$, $(\beta_{j_m^n}, \alpha_k) = -1$, $\beta_{j_m^{n'}} = \beta_{j_m^n} + \alpha_k$

or

(P_m^-) $(\beta_{i_m^n}, \alpha_k) = -1$, $\beta_{i_m^{n'}} = \beta_{i_m^n} + \alpha_k$, $(\beta_{j_m^n}, \alpha_k) = 0$, $j_m^{n'} = j_m^n$.

Proposition 3.3. *For any $i, j \in \Lambda$ satisfying $i < j$, we have*

$$(\mathrm{Q7}) \qquad Y_i Y_j = \begin{cases} Y_j Y_i & \text{if there exists } n \in \Lambda \text{ such that } \{i, j\} = \{i_1^n, j_1^n\}, \\ Y_{j_2^n} Y_{i_2^n} + (q - q^{-1}) Y_{i_1^n} Y_{j_1^n} & \\ & \text{if there exists } n \in \Lambda \text{ such that } i = i_2^n,\ j = j_2^n, \\ Y_{j_m^n} Y_{i_m^n} + q Y_{j_{m-1}^n} Y_{i_{m-1}^n} - q^{-1} Y_{i_{m-1}^n} Y_{j_{m-1}^n} & \\ \quad \text{if there exist } n \in \Lambda,\ m \in \{3, 4, 5\} \text{ such that } i = i_m^n,\ j = j_m^n, & \\ q Y_j Y_i & \text{otherwise.} \end{cases}$$

Proof. Since there exists $n \in \Lambda$ satisfying $i, j \in |\mathbf{B}(n)|$ for any $i, j \in \Lambda$, it is sufficient to show

$$(\mathrm{R}n)\quad \begin{cases} Y_{i_1^n}Y_{j_1^n} = Y_{j_1^n}Y_{i_1^n} & (\mathrm{R}n,1)\\ Y_{i_m^n}Y_{j_m^n} = Y_{j_m^n}Y_{i_m^n} + qY_{j_{m-1}^n}Y_{i_{m-1}^n} - q^{-1}Y_{i_{m-1}^n}Y_{j_{m-1}^n}\ (2 \le m \le 5) & (\mathrm{R}n,2)\\ Y_{l_1}Y_{l_2} = qY_{l_2}Y_{l_1} & \\ \qquad (l_1, l_2 \in |\mathbf{B}(n)|, l_1 < l_2, \{l_1, l_2\} \ne \{i_m^n, j_m^n\}\ (1 \le m \le 5)) & (\mathrm{R}n,3) \end{cases}$$

for $n \in \Lambda$ and $1 \le m \le 5$.

When $n = 27$, the elements Y_i $(1 \le i \le 10)$ satisfy the same relations as those for type D_6, and hence relations (R27) hold.

Since there exists a sequence $((k_1, n'_1, n_1), \ldots, (k_s, n'_s, n_s))$ of $\mathcal{B}$ satisfying $n_1 = 27, n'_s = m, n'_j = n_{j+1}$ $(1 \le j \le s-1)$ for any $1 \le m \le 26$, it is sufficient to show (Rn') for $(k, n', n) \in \mathcal{B}$ assuming (Rn). This is proved similarly to Proposition 2.3. Details are omitted. □

By [4] and Proposition 3.3 we obtain the following:

Theorem 3.4. *The formulas* (Q7) *give fundamental relations for the generator system* $\{Y_i\}_{i\in\Lambda}$ *of the algebra* $A_q = U_q(\mathfrak{n}_I^-)$.

We shall construct a quantum deformation of the lowest degree part $J_{C_1}^0$ of the defining ideal J_{C_1} and we shall give canonical generators of a quantum deformation of J_{C_1}.

Set

$$\psi_n = Y_{i_5^n}Y_{j_5^n} - qY_{i_4^n}Y_{j_4^n} + q^2Y_{i_3^n}Y_{j_3^n} - q^3Y_{i_2^n}Y_{j_2^n} + q^4Y_{i_1^n}Y_{j_1^n},$$

for $n \in \Lambda$, where $\mathbf{B}(n) = (i_5^n, i_4^n, i_3^n, i_2^n, i_1^n, j_1^n, j_2^n, j_3^n, j_4^n, j_5^n)$. Using the formulas (R$n$,1), (R$n$,2), we can write

$$\psi_n = Y_{j_5^n}Y_{i_5^n} - q^{-1}Y_{j_4^n}Y_{i_4^n} + q^{-2}Y_{j_3^n}Y_{i_3^n} - q^{-3}Y_{j_2^n}Y_{i_2^n} + q^{-4}Y_{j_1^n}Y_{i_1^n}.$$

Similarly to Lemma 2.5 and Proposition 2.6 we can show the following:

Lemma 3.5. *We have*

$$\mathrm{ad}(F_k)\psi_n = \begin{cases} \psi_{n'} & \textit{if there exists } (k, n', n) \in \mathcal{B},\\ 0 & \textit{otherwise,} \end{cases}$$

$$\mathrm{ad}(E_k)\psi_n = \begin{cases} \psi_{n'} & \textit{if there exists } (k, n, n') \in \mathcal{B},\\ 0 & \textit{otherwise} \end{cases}$$

for $k \in I$, *and*

$$\mathrm{ad}(K_k)\psi_n = q^{-(\gamma_n,\alpha_k)}\psi_n$$

for $k \in I_0$.

Proposition 3.6. $\sum_{n\in\Lambda}\mathbb{C}(q)\psi_n$ *is an irreducible highest weight* $U_q(\mathfrak{l}_I)$*-module with highest weight vector* ψ_{27}.

By [4] and Proposition 3.6 we obtain the following:

Theorem 3.7. *A quantum deformation of the defining ideal* J_{C_1} *of the closure of the non-open orbit* C_1 *is given by the two-sided ideal of* A_q *generated by* $\{\psi_n \mid n \in \Lambda\}$.

Set

$$\varphi = \sum_{n\in\Lambda}(-q)^{|\beta_n|-1}Y_n\psi_n,$$

where $|\beta| = \sum_{i\in I_0} m_i$ $(\beta = \sum_{i\in I_0} m_i\alpha_i)$.

Proposition 3.8. $\mathbb{C}(q)\varphi$ *is a one-dimensional* $U_q(\mathfrak{l}_I)$*-module.*

Proof. By Proposition 3.3 we can check that the coefficient $a_{1,10,27}$ of $Y_1Y_{10}Y_{27}$ in $\varphi = \sum_{i<j<k} a_{ijk}Y_iY_jY_k$ is $1+q^8+q^{16}$. Therefore we have $\varphi \neq 0$.

Let $(k,n,n') \in \mathcal{B}$. Then we have $|\beta_{n'}| = |\beta_n|+1$, $\mathrm{ad}(F_k)Y_n = Y_{n'}$, $\mathrm{ad}(F_k)Y_{n'} = 0$, $\mathrm{ad}(F_k)\psi_{n'} = \psi_n$, $\mathrm{ad}(F_k)\psi_n = 0$, $(\beta_{n'},\alpha_k) = 1$. Hence $\mathrm{ad}(F_k)(Y_n\psi_n - qY_{n'}\psi_{n'}) = Y_{n'}\psi_n - qq^{-1}Y_{n'}\psi_n = 0$. Therefore we have $\mathrm{ad}(F_k)\varphi = 0$ for any $k \in I$, and similarly we have $\mathrm{ad}(E_k)\varphi = 0$ for any $k \in I$. Since $\gamma_n + \beta_n = \delta$ for any $n \in \Lambda$, we have $\mathrm{ad}(K_k)\varphi = q^{-(\delta,\alpha_k)}\varphi$ for any $k \in I_0$. In particular, we have $\mathrm{ad}(K_k)\varphi = \varphi$ for any $k \in I$, and $\mathrm{ad}(K_1)\varphi = q^{-2}\varphi$. □

The element φ is a quantum deformation of the irreducible relative invariant on the prehomogeneous vector space.

Theorem 3.9. *A quantum deformation of the defining ideal* J_{C_2} *of the closure of the non-open orbit* C_2 *is given by the two-sided ideal of* A_q *generated by* φ.

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