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ON DISORDERED PHASE IN THE FERROMAGNETIC POTTS MODEL ON THE BETHE LATTICE

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1. Introduction

The Bethe lattice Γ^k of degree $k \geq 1$ is a lattice in which each lattice point has $k + 1$ nearest neighbors and for every two points there is only one way connecting them.

In the Potts model spin variables $\sigma(x)$ which take values on a discrete set $\Phi = \{1, 2, ..., q\}$ are associated with each site x of the lattice.

The ferromagnetic Potts model on the Bethe lattice is defined by the Hamiltonian

$$H(\sigma) = -J \sum_{\langle x,y \rangle} \delta_{\sigma(x)\sigma(y)} \tag{1}$$

where the sum is taken over all pairs of the nearest neighbors $\langle x, y \rangle, \delta$ is the Kroneker's symbol and $J > 0$.

P. M. Bleher [1] proved that the disordered Phase in the ferromagnetic Ising model on the Bethe lattice is extreme for $T \geq T_c$, where T_c is the critical temperature of the spin glass model on the Bethe lattice, and it is not extreme for $T < T_c$. Denote $\theta = \exp(J/T)$.

The main result of this paper is the following theorem

Theorem. *For* $\theta < 1 + 1/\{(q - 1)k^{1/2} - 1\}$ *the disordered phase is extreme.*

The content of the paper is the following. In Sect. 2, following [1-6] we construct the disordered phase for all values of the temperature. In the main part of this work, in Sect. 3, we describe conditions under which the phase is extreme.

Some other extreme phases of the Potts model were studied in [5-7].

2. Construction of the Disordered Phase.

Let V and L be respectively the sets of vertices and edges of the graph Γ^k and $x^0 \in V$ be an arbitrary vertex. Denote

$$W_n = \{x \in V | d(x, x^0) = n\}$$

where the distance $d(x, y)$ on V is introduced as the length (the number of edges) of the shortest path connecting x with y. Let

$$V_n = \bigcup_{m=1}^{n} W_n = \{x \in V | d(x, x^0) \leq n\}$$

$$L_n = \{l = \langle x, y \rangle \in L | x, y \in V_n\}.$$

We say that $x < y$ if the path from x^0 to y goes through x. Moreover, y is called a direct successor of x if $y > x$ and x, y are the nearest neighbours. Denote $S(x)$ the set of direct successors of x. Observe that any vertex $x \neq x^0$ has k direct successors and x^0 has $k+1$ ones.

Let $\Phi = \{\sigma_1, \sigma_2, ..., \sigma_q\} \subset R^{q-1}$, where $\sigma_i \sigma_j = 1$ if $i = j$ and $\sigma_i \sigma_j = (-1)/(q-1)$ if $i \neq j$. It is clear that

$$(*) \qquad \sum_{i=1}^{q} \sigma_i = 0.$$

Then for any $x, y \in V$

$$\frac{q-1}{q}\left(\sigma(x)\sigma(y) + \frac{1}{q-1}\right) = \delta_{\sigma(x)\sigma(y)}$$

hence

$$H(\sigma) = -J' \sum_{\langle x,y \rangle} \sigma(x)\sigma(y)$$

where $J' = J(q-1)/q$.

For $A \subset V$ denote $\Omega_A = \Phi^A$, the configurational space of the set A. Let $h_x \in R^{q-1}$ be a vector-valued function of $x \in V$. Consider for each n the probability distribution on Ω_{V_n} defined by the formula

$$(2) \qquad \mu_n(\sigma_n) = Z_n{}^{-1} \exp\left\{\frac{J'}{T} \sum_{\langle x,y \rangle \in L_n} \sigma(x)\sigma(y) + \sum_{x \in W_n} h_x \sigma(x)\right\}$$

where

$$\sigma_n = \{\sigma(x), x \in V_n\} \in \Omega_{V_n}$$

and $Z_n{}^{-1}$ is a normalizing factor. We say that the probability distributions $\mu_n(\sigma_n)$ are compatible if for all $n \geq 1$

$$\sum_{\sigma^{(n)}} \mu_n \left(\sigma_{n-1}, \sigma^{(n)}\right) = \mu_{n-1}(\sigma_{n-1}) \tag{3}$$

where $\sigma^{(n)} = \{\sigma(x), x \in W_n\}$. In such case there exists a Gibbs distribution μ on Ω_V such that $\mu(\sigma_n) = \mu_n(\sigma_n)$. The following proposition describes the conditions on the h_x which ensures the compatibility of the probability distributions $\mu_n(\sigma_n)$.

Proposition 1 (see [5, 6]). *The probability distributions* $\mu_n(\sigma_n)$, $n = 1, 2, ...,$ (2) *are compatible iff for any* $x \in V$ *the following equation holds:*

$$h_x = \sum_{y \in S(x)} F(h_y, q, \theta) \tag{4}$$

where $F : R^{q-1} \to R^{q-1}$:

$$F_i = \ln \left[\frac{(\theta - 1) \exp h_i + \sum_{j=1}^{q-1} \exp h_j + 1}{\sum_{j=1}^{q-1} \exp h_j + \theta} \right]$$

and $\theta = \exp(J/T)$, $h = (h_1, ..., h_{q-1})$, $F(h) = (F_1, ..., F_{q-1})$.

Let $h_x = h$ for any $x \in V$. For h, (4) implies the equation

$$h = kF(h, q, \theta) \tag{5}$$

For any k, q, θ this equation has a solution $h_0 = (0, 0, ..., 0)$. The distribution μ_0 corresponding to the solution h_0 is called the disordered phase or disordered Gibbs distribution.

In [5] proved that for $T < T_c = J/\ln(1 + q/(k-1))$ the equation (5) has q non-zero solutions $h_*{}^i, i = 1, 2, ..., q$, and for Potts model q pure translation invariant phases and uncountable many pure non translation invariant phases exist, the constructive description of these phases has been given.

3. Proof of Theorem

We shall prove that the disordered phase of the ferromagnetic Potts model is extreme for $\theta < 1 + 1/\{(q-1)k^{1/2} - 1\}$. We shall verify the following property.

Property E. For any $\varepsilon > 0$, $n > 0$ and any configuration

$$\sigma_n = \{\sigma(x), x \in V_n\} \in \Omega_{V_n}$$

there exist $N > n$ and $A_N \subset \Omega_{W_N}$ such that

1. $\mu_0(A_N) > 1 - \varepsilon$
2. $|\mu_0\left(\sigma_n \mid \sigma^{(N)}\right) - \mu_0(\sigma_n)| < \varepsilon, \forall \sigma^{(N)} \in A_N$.

Property E means that for typical boundary conditions $\sigma^{(N)}$ the conditional distributions $\mu_0(\sigma_n \mid \sigma^{(N)})$ converge to the unconditional ones $\mu_0(\sigma_n)$ as $N \to \infty$. For the sake of brevity here and later we denote for $A \subset \Omega_A$,

$$\mu_0(A) = \mu_0\left(A \times \Omega_{V\setminus A}\right)$$

and for $\sigma_n \in \Omega_{V_n}$,

$$\mu_0(\sigma_n) = \mu_0\left(\{\sigma_n\} \times \Omega_{V\setminus V_n}\right).$$

Moreover,

$$\mu_0\left(\sigma_n \mid \sigma^{(N)}\right) = \frac{\mu_0\left(\sigma_n, \sigma^{(N)}\right)}{\mu_0\left(\sigma^{(N)}\right)}$$

where

$$\mu_0\left(\sigma_n, \sigma^{(N)}\right) = \mu_0\left(\{\sigma_n\} \times \{\sigma^{(N)}\} \times \Omega_{V\setminus(V_n \cup W_N)}\right).$$

From the Property E it follows that μ_0 is extreme (see [2]). Let us verify Property E. Substituting $h_x = (0, 0, ..., 0) \in R^{q-1}, x \in W_N$, in (2), we have

$$\mu_0(\sigma_N) = {Z_N}^{-1} \exp\left\{\frac{-1}{T} H_N(\sigma_N)\right\} \tag{6}$$

where

$$H_N(\sigma_N) = -J' \sum_{\langle x,y\rangle \in L_N} \sigma(x)\sigma(y), \quad \sigma(x) \in \Phi,$$

for any

$$x \in V_N, \sigma_N = \{\sigma(x), x \in V_N\}.$$

This formula can be interpreted in the following way: if

$$h_x^{(N)} = \frac{J'}{T} \sum_{y \in S(x)} \sigma(y), \quad x \in W_{N-1}, \tag{7}$$

then

$$H_N(\sigma_N) = H_{N-1}(\sigma_{N-1}) - T \sum_{x \in W_{N-1}} h_x{}^{(N)}\sigma(x)$$

so

$$\mu_0(\sigma_N) = Z_N^{-1} \exp\left\{\frac{-1}{T} H_{N-1}(\sigma_{N-1}) + \sum_{x \in W_{N-1}} h_x^{(N)}\sigma(x)\right\}$$

This implies that the joint distribution of the random vectors $\{\sigma(x), x \in V_{N-1}\}$ and $\{h_x^{(N)}, x \in W_{N-1}\} = h^{(N,N-1)}$ with respect to μ_0 has the form

$$(8) \qquad \mu_0\left(\sigma_{N-1}, h^{(N,N-1)}\right) = \sum_{(J'/T) \sum_{y \in S(x)} \sigma(y) = h_x^{(N)},\ x \in W_{N-1}} \mu_0(\sigma_N)$$

$$= Z_N^{-1} \exp\left\{\frac{-1}{T} H_{N-1}(\sigma_{N-1}) + \sum_{x \in W_{N-1}} h_x^{(N)}\sigma(x)\right\} \prod_{x \in W_{N-1}} \nu\left(h_x^{(N)}\right)$$

where $\nu(h_x{}^{(N)})$ is the distribution of the random vector (7) under the condition that $\sigma(y)$ are independent, $\sigma(y) = \sigma_i, \sigma_i \in \Phi$ with probability $1/q$. Formula (8) resembles (2) but now the vectors $h_x{}^{(N)}$ are random. Using the recurrent equations

$$(9) \qquad h_x{}^{(N)} = \sum_{y \in S(x)} F\left(h_y{}^{(N)}, q, \theta\right)$$

where $\theta = \exp(J/T)$, define the set of random vectors

$$\{h_x{}^{(N)} = h_x{}^{(N)}(\sigma^{(N)}), x \in V_{N-1}\}.$$

Since the random vectors $h_x{}^{(N)}$ satisfy the compatibility conditions (9). Proposition 1 implies that the joint distribution of the random vectors

$$\{\sigma(x), x \in V_n\}$$

and $\{h_x{}^{(N)}, x \in W_n\} = h^{(N,n)}$ with respect to μ_0 has the form

$$(10) \qquad \mu_0(\sigma_n, h^{(N,n)}) = Z_{N,n}^{-1} \exp\left[\frac{-1}{T} H_n(\sigma_n) + \sum_{x \in W_n} h_x^{(N)}\sigma(x)\right] \prod_{x \in W_n} \nu_{N-n}(h_x^{(N)})$$

where the probability distribution $\nu_{N-n}(h_x{}^{(N)})$ is defined in the following way. Consider the set of independent random vectors $\{\sigma(x), x \in W_N\}$ taking values $\sigma_i \in \Phi$ with probability $1/q$ and the corrresponding probability space (Ω_{W_N}, B, μ), where μ is the multinomial distribution with $p_i = 1/q, i = 1, ..., q$. Consider on this probability space the random vectors $h_x{}^{(N)}$ which are defined recurrently by Eqs. (7), (9).

Then for any fixed $n < N$ the random vectors $\{h_x{}^{(N)}, x \in W_n\}$ are independent, identically distributed. By $\nu_{N-1}(h_x{}^{(N)})$ we denote the distribution of $h_x{}^{(N)}$ for $x \in W_n$. Let $h = (h_1, ..., h_{q-1}) \in R^{q-1}$. Denote

$$\|h\| = \max_{1 \le i \le q-1} |h_i|.$$

Lemma 1. *For any $h \in R^{q-1}$ the following inequalites holds*:

a) $|(\partial F_i)/(\partial h_j)| \le (\theta - 1)/\theta, \quad j = 1, 2, ..., q-1;$

b) $\|F(h)\| \le (q-1)(\theta - 1)/\theta \, \|h\|.$

Proof.

a) For $j \neq i$ we get

$$\frac{\partial F_i}{\partial h_j} = \frac{\exp h_j (1 - \exp h_i)(\theta - 1)}{\left[(\theta - 1)\exp h_i + \sum_{j=1}^{q-1} \exp h_j + 1\right]\left[\sum_{j=1}^{q-1} \exp h_j + \theta\right]}.$$

Consider several cases:

CASE 1.1. $h_i > 0$ then

$$\frac{-(1 - \exp h_i)}{(\theta - 1)\exp h_i + \sum_{j=1}^{q-1} \exp h_j + 1} \le \frac{1}{\theta}, \quad \frac{\exp h_j}{\sum_{j=1}^{q-1} \exp h_j + \theta} < 1.$$

CASE 1.2. $h_i < 0$ then

$$\frac{\exp h_j}{(\theta - 1)\exp h_i + \sum_{j=1}^{q-1} \exp h_j + 1} < 1, \quad \frac{1 - \exp h_i}{\sum_{j=1}^{q-1} \exp h_j + \theta} \le \frac{1}{\theta}.$$

For $j = i$ we get

$$\frac{\partial F_i}{\partial h_i} = \frac{\exp h_i (1 + \sum_{j=1, j \neq i}^{q-1} \exp h_j + \theta)(\theta - 1)}{\left[(\theta - 1)\exp h_i + \sum_{j=1}^{q-1} \exp h_j + 1\right]\left[\sum_{j=1}^{q-1} \exp h_j + \theta\right]}.$$

Consider several cases:

CASE 2.1. $h_i > 0$ then

$$\frac{\exp h_i}{(\theta - 1)\exp h_i + \sum_{j=1}^{q-1} \exp h_j + 1} \le \frac{1}{\theta}, \quad \frac{1 + \sum_{j=1, j \neq i}^{q-1} \exp h_j + \theta}{\sum_{j=1}^{q-1} \exp h_j + \theta} < 1.$$

CASE 2.2. $h_i < 0$ then

$$\frac{\exp h_i(1+\sum_{j=1,j\neq i}^{q-1}\exp h_j+\theta)}{(\theta-1)\exp h_i+\sum_{j=1}^{q-1}\exp h_j+1}\leq 1, \quad \frac{1}{\sum_{j=1}^{q-1}\exp h_j+\theta}<\frac{1}{\theta}.$$

Hence, from above cases we get a).

b) (11)
$$\begin{aligned}\|F(h)-F(\widetilde{h})\| &= \max_{1\leq i\leq q-1}|F_i(h)-F_i(\widetilde{h})|\\ &\leq \max_{1\leq i\leq q-1}\sum_{j=1}^{q-1}|F_{ih_j}||h_j-\widetilde{h}_j| \leq \frac{\theta-1}{\theta}\sum_{j=1}^{q-1}\max|h_j-\widetilde{h}_j|\\ &= \frac{\theta-1}{\theta}\sum_{j=1}^{q-1}\|h-\widetilde{h}\| = \frac{\theta-1}{\theta}(q-1)\|h-\widetilde{h}\|.\end{aligned}$$

Substituting $\widetilde{h}=(0,0,...,0)$ in (11), we have:

$$\|F(h)\|\leq(q-1)\frac{\theta-1}{\theta}\|h\|.$$

The lemma is proved. □

Lemma 2. *For any $x\in W_n$, $n\leq N-1$ the following equalities holds*:

$$Eh_{xi}^{(N)}=0, \quad i=1,2,...,q-1,$$

where $h_x^{(N)}=\left(h_{x1}^{(N)},h_{x2}^{(N)},...,h_{x(q-1)}^{(N)}\right)\in R^{q-1}$.

Proof. For $x\in W_{N-1}$ we have

(**)
$$\begin{aligned}Eh_{xi}^{(N)} &= \sum_{\sigma^{(N)}}h_{xi}^{(N)}\left(\sigma^{(N)}\right)\mu\left(\sigma^{(N)}\right)\\ &= \frac{1}{q^{(k+1)k^{N-1}}}\sum_{\sigma^{(N)}}h_{xi}^{(N)}\left(\sigma^{(N)}\right) = \frac{J'}{Tq^{(k+1)k^{N-1}}}\sum_{\sigma^{(N)}}\left(\sum_{y\in S(x)}\sigma^{(i)}(y)\right),\end{aligned}$$

where $\sigma(y)=\left(\sigma^{(1)}(y),\sigma^{(2)}(y),...,\sigma^{(q-1)}(y)\right)\in\Phi$. According to (*) it follows from (**) that $Eh_{xi}^{(N)}=0$.

For $x\in W_{N-1},\sigma^{(N)}\in\Omega_{W_N}$ there exist numbers $\alpha_{xi}=\alpha_{xi}\left(\sigma^{(N)}\right)\in\{0,1,...,k\}$, $i=1,2,...,q$ such that $\sum_{i=1}^{q}\alpha_{xi}=k$ and

$$\begin{aligned}h_x^{(N)} &= \frac{J'}{T}\sum_{y\in S(x)}\sigma(y)\\ &= \frac{J'}{T}\sum_{i=1}^{q}\alpha_{xi}\sigma_i = \left(\frac{J'}{T}(\alpha_{x1}-\alpha_{xq}),\frac{J'}{T}\left(\alpha_{x2}-\alpha_{xq}\right),...,\frac{J'}{T}\left(\alpha_{x(q-1)}-\alpha_{xq}\right)\right).\end{aligned}$$

Consider $A^{(i)}_{x,n}$, $x \in W_n$; $i = 1, 2, ..., q$ which are defined by the following recurrent equations:

$$A^{(i)}_{x,n} = \prod_{z \in S(x)} \left((\theta - 1)A^{(i)}_{z,n+1} + \sum_{j=1}^{q} A^{(j)}_{z,n+1} \right), \quad x \in W_n; \quad i = 1, 2, ..., q; \quad n < N - 1;$$

$$A^{(i)}_{y,N-1} = \exp(\frac{J'}{T}\alpha_{yi}), \quad y \in W_{N-1}.$$

It is clear that for any $i = 1, 2, ..., q$, the distribution of $\{A^{(i)}_{x,n}(\sigma^{(N)})\}$ is the same as that of $\{A^{(q)}_{x,n}(\sigma^{(N)})\}$.

For $x \in W_n$, $n < N - 1$ from (4) we get

$$\begin{aligned} E h^{(N)}_{xi} &= E \sum_{z_{n+1} \in S(x)} F_i\left(h^{(N)}_{z_{n+1}}; \theta\right) \\ &= \sum_{z_{n+1} \in S(x)} E F_i \left(\sum_{z_{n+2} \in S(z_{n+1})} F \left(\cdots \left(\sum_{z_{N-1} \in S(z_{N-2})} F\left(h^{(N)}_{z_{N-1}}; \theta\right); \theta \right) \cdots ; \theta \right) \right) \\ &= \sum_{z_{n+1} \in S(x)} E \ln \frac{(\theta - 1)A^{(i)}_{z_{n+1},n+1} + \sum_{j=1}^{q} A^{(j)}_{z_{n+1},n+1}}{(\theta - 1)A^{(q)}_{z_{n+1},n+1} + \sum_{j=1}^{q} A^{(j)}_{z_{n+1},n+1}} \\ &= \frac{1}{q^{(k+1)k^{N-1}}} \sum_{z_{n+1} \in S(x)} \ln \frac{\prod_{\sigma^{(N)}} \left[(\theta - 1)A^{(i)}_{z_{n+1},n+1} + \sum_{j=1}^{q} A^{(j)}_{z_{n+1},n+1}\right]}{\prod_{\sigma^{(N)}} \left[(\theta - 1)A^{(q)}_{z_{n+1},n+1} + \sum_{j=1}^{q} A^{(j)}_{z_{n+1},n+1}\right]}. \end{aligned}$$

From this equality we get $Eh^{(N)}_{xi} = 0$, $x \in W_n$. The lemma is proved. □

Let $D{h_{xi}}^{(N)}$ denote the variance of the random variable

$$ {h_{xi}}^{(N)} = {h_{xi}}^{(N)}(\sigma^{(N)}), \quad i = 1, 2, ..., q - 1$$

with respect to the measure μ.

Lemma 3. *If* $\theta < 1 + 1/\left\{(q - 1)k^{1/2} - 1\right\}$ *then*

$$\lim_{N-n \to \infty} D{h_{xi}}^{(N)} = 0, \quad x \in W_n, \quad i = 1, 2, ..., q - 1. \tag{12}$$

Proof. From the independence of ${h_{yi}}^{(N)}$ in (9) it follows that

$$D{h_{xi}}^{(N)} = kDF\left({h_{yi}}^{(N)}, q, \theta\right) \tag{13}$$

$$x \in W_{m-1}, \qquad y \in W_m, \qquad m \leq N - 1.$$

From lemma 1, 2 and (13) we get

$$Dh_{xi}^{(N)} \le kD\left(\frac{(q-1)(\theta-1)}{\theta}h_{yi}^{(N)}\right) = \left[k\left(\frac{(q-1)(\theta-1)}{\theta}\right)^2\right] Dh_{yi}^{(N)}.$$

Iterating this inequality we have for

$$x \in W_n, \qquad y \in W_{N-1}, \qquad n \le N-1,$$
$$Dh_{xi}{}^{(N)} \le \left[k\left(\frac{(q-1)(\theta-1)}{\theta}\right)^2\right]^{N-n-1} Dh_{yi}{}^{(N)}.$$

which implies (12) for $\theta < 1 + 1/\{(q-1)k^{1/2} - 1\}$. The lemma is proved. ☐

Lemma 4. *If* $\theta < 1 + 1/\{(q-1)k^{1/2} - 1\}$ *then*

$$\text{Prob}\left\{\lim_{N\to\infty} \|h_x^{(N)}\| = 0, \quad x \in W_n\right\} = 1,$$

where n *is fixed.*

Proof. From the inequality Chebishev's

$$\text{Prob}\left\{|h_{xi}^{(N)} - Eh_{xi}^{(N)}| \ge \varepsilon\right\} \le \frac{Dh_{xi}^{(N)}}{\varepsilon^2}.$$

By lemma 2 and lemma 3 we have

$$\lim_{N\to\infty} \text{Prob}\{|h_{xi}^{(N)}| < \varepsilon\} = 1.$$

The lemma is proved. ☐

Let $\theta < 1 + 1/\{(q-1)k^{1/2} - 1\}$ and $n > 0, \delta > 0$ be fixed. To verify Property E define for $N > n$ the set

$$A_{N,\delta} = \left\{\sigma^{(N)} = \{\sigma(x), x \in W_N\} : \|h_x{}^{(N)}(\sigma^{(N)})\| \le \delta, x \in W_n\right\}.$$

Let us prove that

$$\lim_{N\to\infty} \mu_0(A_{N,\delta}) = 1. \tag{14}$$

By (10)

$$\mu_0(A_{N,\delta}) = \frac{\sum_{\left\{\|h_x^{(N)}\|\le\delta,\ x\in W_n\right\}} Q_n\left(\{h_x^{(N)},\ x \in W_n\}\right)}{\sum_{\left\{h_x^{(N)},\ x\in W_n\right\}} Q_n\left(\left\{h_x^{(N)},\ x \in W_n\right\}\right)} \tag{15}$$

where

$$Q_n(\{h_x^{(N)},\ x \in W_n\}) = \sum_{\sigma_n} \exp\left\{\frac{-1}{T} H_n(\sigma_n) + \sum_{x \in W_n} h_x^{(N)} \sigma(x)\right\} \prod_{x \in W_n} \nu_{N-n}(h_x^{(N)})$$

So (14) follows by lemma 4 from (15).

Let us estimate now $|\mu_0(\sigma_n \mid \sigma^{(N)}) - \mu_0(\sigma_n)|$. By (10) we have that

$$\mu_0(\sigma_n \mid \sigma^{(N)}) = \frac{\exp\left\{\frac{-1}{T} H_n(\sigma_n) + \sum_{x \in W_n} h_x^{(N)} \sigma(x)\right\}}{\sum_{\sigma_n} \exp\left\{\frac{-1}{T} H_n(\sigma_n) + \sum_{x \in W_n} h_x^{(N)} \sigma(x)\right\}}, \tag{16}$$

where $h_x{}^{(N)} = h_x{}^{(N)}(\sigma^{(N)})$, $x \in W_n$, are expressed via $\sigma^{(N)}$ by formulas (7), (9). Moreover according to (6)

$$\mu_0(\sigma_n) = \frac{\exp\left\{\frac{-1}{T} H_n(\sigma_n)\right\}}{\sum_{\sigma_n} \exp\left\{\frac{-1}{T} H_n(\sigma_n)\right\}}. \tag{17}$$

As n is fixed, it follows obviously from (16), (17) that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\mu_0(\sigma_n \mid \sigma^{(N)}) - \mu_0(\sigma_n)| < \varepsilon \tag{18}$$

if $\|h_x^{(N)}\| \leq \delta$ for all $x \in W_n$, i.e. for $\sigma^{(N)} \in A_{N,\delta}$. Thus for a given $\varepsilon > 0$ we can choose at first such $\delta > 0$ that (18) is fulfilled for $\sigma^{(N)} \in A_{N,\delta}$, and next by (14) such N that

$$\mu_0(A_{N,\delta}) > 1 - \varepsilon.$$

Hence we have proved that for μ_0 the Property E is valid, so μ_0 is an extreme phase for $\theta < 1 + 1/\{(q-1)k^{1/2} - 1\}$. The theorem is proved.

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