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PERIODIC SOLUTIONS OF A TWO-POINT BOUNDARY VALUE PROBLEM AT RESONANCE

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1. Introduction

We consider the periodic boundary value problem

$$
\begin{aligned}
&u'' + f(u)u' + g(x, u) = h \text{ in } (0, 2\pi), \\
&u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0,
\end{aligned}
\tag{1.1}
$$

where $h \in L^1(0, 2\pi)$ is given, $f : \mathbf{R} \to \mathbf{R}$ is a continuous function and $g : (0, 2\pi) \times \mathbf{R} \to \mathbf{R}$ is a Caratheodory function. That is, $g(x, u)$ is continuous in $u \in \mathbf{R}$ for a.e. $x \in (0, 2\pi)$, is measurable in $x \in (0, 2\pi)$ for all $u \in \mathbf{R}$ and satisfies for each $r > 0$, there exists $a_r \in L^1(0, 2\pi)$ such that

$$
|g(x, u)| \le a_r(x) \tag{1.2}
$$

for a.e. $x \in (0, 2\pi)$ and all $|u| \le r$. Concerning the growth condition of the nonlinear term g, we assume that either

(H) There exist a constant $r_0 > 0$, and $a, b, c, d \in L^1(0, 2\pi)$, $a, b \ge 0$ and $a(x) \le 1$ for a.e. $x \in (0, 2\pi)$ with strict inequality on a positive measurable subset of $(0, 2\pi)$, such that for a.e. $x \in (0, 2\pi)$ and all $u \ge r_0$

$$
c(x) \le g(x, u) \le a(x)|u| + b(x),
$$

and for a.e. $x \in (0, 2\pi)$ and all $u \le -r_0$

$$
-a(x)|u| - b(x) \le g(x, u) \le d(x);
$$

or

(G) There exist a constant $r_0 \ge 0$, and $a, b, c, d \in L^1(0, 2\pi)$, $a, b \ge 0$ and $a(x) \le 1/4$ for a.e. $x \in (0, 2\pi)$ with strict inequality on a positive measurable subset of $(0, 2\pi)$, such that for a.e. $x \in (0, 2\pi)$ and all $u \ge r_0$

$$
c(x) \le g(x, u),
$$

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and for a.e. $x \in (0, 2\pi)$ and all $u \leq -r_0$

$$-a(x)|u| - b(x) \leq g(x, u) \leq d(x);$$

holds. The solvability of the problem (1.1) has been extensively studied if the nonlinearity g is assumed to have either linear growth in u as $|u| \to \infty$ (see [2,3,4,5,7,8]); or superlinear growth in u in one of directions $u \to \infty$ and $u \to -\infty$, and may be bounded in the other (see [6,9]). The purpose of this paper is to give solvability theorems to (1.1) when g is allowed to grow superlinearly in u in one of directions $u \to \infty$ and $u \to -\infty$, and may grow linearly in the other. An example is given in [7] shows that our results are almost sharp and still are new. Based on the well-known Leray-Schauder continuation method (see [1]), we obtain solvability results under assumptions either with or without a Landesman-Lazer condition (see (2.2) below).

In the following we shall make use of real Banach spaces $L^p(0, 2\pi)$, $C[0, 2\pi]$, and Sobolev spaces $W^{2,1}(0, 2\pi)$ and $H^1(0, 2\pi)$. Norms of $L^p(0, 2\pi)$, $C[0, 2\pi]$ and $H^1(0, 2\pi)$ are denoted by $\|u\|_{L^p}$, $\|u\|_C$ and $\|u\|_{H^1}$, respectively. By a solution of (1.1), we mean a function $u \in W^{2,1}(0, 2\pi)$ satisfies the differential equation in (1.1) a.e. $x \in (0, 2\pi)$.

2. Existence Theorems

For $v \in W^{2,1}(0, 2\pi)$, we write $\overline{v} = (2\pi)^{-1} \int_0^{2\pi} v(x)dx$, $\tilde{v} = v - \overline{v}$ and so $\int_0^{2\pi} \tilde{v}(x)dx = 0$. We now state in the following lemmas, their proofs can be obtained in [7], Lemmas 2,3 and 4, and so are omitted.

Lemma 1. *Let m be a nonnegative function in $L^1(0, 2\pi)$ and for a.e. $x \in (0, 2\pi)$, $m(x) \leq 1$ with strict inequality on a positive measurable subset of $(0, 2\pi)$, then there exists a constant $K_1(m) > 0$ such that*

$$\int_0^{2\pi} (\overline{u} - \tilde{u}(x))(u''(x) + f(u(x))u'(x) + p(x)u(x))dx$$
$$\geq K_1 \|\tilde{u}\|_{H^1}^2$$

whenever $p \in L^1(0, 2\pi)$ with $p(x) \leq m(x)$ for a.e. $x \in (0, 2\pi)$, $u \in W^{2,1}(0, 2\pi)$ with $u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0$ and $f : \mathbf{R} \to \mathbf{R}$ is a continuous function.

Lemma 2. *Let $p : \mathbf{R} \to \mathbf{R}$ be a periodic function of period 2π in $L^1(0, 2\pi)$ such that $p(x) \leq 1/4$ for a.e. $x \in (0, 2\pi)$ with strict inequality on a positive measurable subset of $(0, 2\pi)$. Then there exists a constant $K_2(p) > 0$ such that*

$$\int_{x_0}^{x_0+2\pi} [(v'(x))^2 - p(x)v^2(x)]dx \geq K_2 \|v\|_{H^1(x_0, x_0+2\pi)}^2$$

for all $v \in H^1(x_0, x_0 + 2\pi)$ with $v(x_0) = v(x_0 + 2\pi) = 0$ and $x_0 \in \mathbf{R}$.

Lemma 3. *There exists a constant $K_3 > 0$ such that*

$$\|\tilde{u}\|_{H^1} \le K_3 \|u'' + f(u)u'\|_{L^1}$$

for each continuous function $f : \mathbf{R} \to \mathbf{R}$ and $u \in W^{2,1}(0, 2\pi)$ with

$$u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0.$$

Theorem 1. *Let $g : (0, 2\pi) \times \mathbf{R} \to \mathbf{R}$ be a Caratheodory function satisfying* (H), *then for each $h \in L^1(0, 2\pi)$ the problem* (1.1) *has a solution $u \in W^{2,1}(0, 2\pi)$, provided that one of the following conditions holds*:

$$\int_0^{2\pi} d(x)dx \le \int_0^{2\pi} h(x)dx \le \int_0^{2\pi} c(x)dx; \tag{2.1}$$

$$\int_0^{2\pi} g_-^0(x)dx < \int_0^{2\pi} h(x)dx < \int_0^{2\pi} g_+^0(x)dx; \tag{2.2}$$

$$\int_0^{2\pi} d(x)dx \le \int_0^{2\pi} h(x)dx < \int_0^{2\pi} g_+^0(x)dx; \tag{2.3}$$

$$\int_0^{2\pi} g_-^0(x)dx < \int_0^{2\pi} h(x)dx \le \int_0^{2\pi} c(x)dx; \tag{2.4}$$

where $g_+^0(x) = \liminf_{u\to\infty} g(x, u)$ and $g_-^0(x) = \limsup_{u\to-\infty} g(x, u)$.

Proof. Let $\alpha \in \mathbf{R}$ be fixed, $0 < \alpha < 1$. We consider the boundary value problems

$$\begin{aligned} &u'' + tf(u)u' + (1-t)\alpha u + tg(x, u) = th \text{ in } (0, 2\pi), \\ &u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0 \end{aligned} \tag{2.5}$$

for $0 \le t \le 1$, which becomes the original problem when $t = 1$. Since $0 < \alpha < 1$, (2.5) has only a trivial solution when $t = 0$. To apply the Leray-Schauder continuation method, it suffices to show that solutions to (2.5) for $0 < t < 1$ have an a priori bound in $H^1(0, 2\pi)$. To this end, let $\theta : \mathbf{R} \to \mathbf{R}$ be a continuous function such that for $u \in \mathbf{R}$, $0 \le \theta(u) \le 1$, $\theta(u) = 0$ for $|u| \le r_0$, and $\theta(u) = 1$ for $|u| \ge 2r_0$. We define $e(x) = \max\{a_{r_0}(x), b(x), |c(x)|, |d(x)|\}$,

$$g_1(x, u) = \begin{cases} \min\{g(x, u) + e(x), a(x)u\}\theta(u) & \text{if } u \ge 0 \\ \max\{g(x, u) - e(x), a(x)u\}\theta(u) & \text{if } u \le 0 \end{cases}$$

and $g_2(x, u) = g(x, u) - g_1(x, u)$. Then $g_1, g_2 : (0, 2\pi) \times \mathbf{R} \to \mathbf{R}$ are Caratheodory functions such that for a.e. $x \in (0, 2\pi)$ and $u \in \mathbf{R}$, $u \ne 0$

$$0 \le \frac{g_1(x, u)}{u} \le a(x), \tag{2.6}$$

and for a.e. $x \in (0, 2\pi)$ and $u \in \mathbf{R}$

$$|g_2(x, u)| \leq e(x). \tag{2.7}$$

If u is a possible solution to (2.5) for some $0 < t < 1$, then using (2.6), (2.7) and Lemma 1, we have

$$\begin{aligned}
0 &= \int_0^{2\pi} [(\bar{u} - \tilde{u}(x))(u''(x) + tf(u(x))u'(x) + (1-t)\alpha u(x) \\
&\quad + tg_1(x, u(x)) + tg_2(x, u(x)) - th(x))]dx \\
&= \int_0^{2\pi} [(\bar{u} - \tilde{u}(x))(u''(x) + tf(u(x))u'(x) + p_t(u(x))u(x) \\
&\quad + tg_2(x, u(x)) - th(x))]dx \\
&\geq K_1(m)\|\tilde{u}\|_{H^1}^2 - (\|e\|_{L^1} + \|h\|_{L^1})(|\bar{u}| + \|\tilde{u}\|_C) \\
&\geq K_1(m)\|\tilde{u}\|_{H^1}^2 - C_1(|\bar{u}| + \|\tilde{u}\|_{H^1}),
\end{aligned}$$

if we set $m(x) = \max\{\alpha, a(x)\}$ and for $u(x) \neq 0$, $p_t(u(x)) = (1-t)\alpha + tg_1(x, u(x))/u(x)$, and $p_t(u(x)) = (1-t)\alpha$ for $u(x) = 0$. Consequently,

$$\|\tilde{u}\|_{H^1}^2 \leq \frac{C_1}{K_1}(|\bar{u}| + \|\tilde{u}\|_{H^1}) \tag{2.8}$$

for some constant $C_1 > 0$ independent of u. It remains to show that solutions to (2.5) for $0 < t < 1$ have an a priori bound in $H^1(0, 2\pi)$. We argue by contradiction, and suppose that there exist a sequence $\{u_n\}$ in $W^{2,1}(0, 2\pi)$ and a corresponding sequence $\{t_n\}$ in $(0, 1)$ such that u_n is a solution to (2.5) with $t = t_n$ and $\|u_n\|_{H^1} \geq n$ for all n. Let $v_n = u_n/\|u_n\|_{H^1}$, then $\|v_n\|_{H^1} = 1$ for all $n \in \mathbf{N}$, and by (2.8) we have $\|\tilde{v}_n\|_{H^1} \to 0$ as $n \to \infty$. Because $\|\bar{v}_n\|_{H^1} \leq \|v_n\|_{H^1} + \|\tilde{v}_n\|_{H^1}$ is bounded, we may assume without loss of generality that $\{\bar{v}_n\}$ converges to β in $\mathbf{R}$. In this case, we have $v_n \to \beta$ in $H^1(0, 2\pi)$ with $\beta \neq 0$ because $\|v_n\|_{H^1}$ for all $\in \mathbf{N}$. We consider only the case $\beta > 0$, for the case $\beta < 0$ can be treated similarly. Using the compact imbedding of $H^1(0, 2\pi) \to C[0, 2\pi]$, we have $v_n \to \beta$ in $C[0, 2\pi]$, and so there exists $n_0 \in \mathbf{N}$ such that $v_n > \beta/2$ on $[0, 2\pi]$ for all $n \geq n_0$. Hence, we may assume that $u_n \geq r_0$ on $[0, 2\pi]$ independent of n. Clearly, $u_n(x) \to \infty$ for each $x \in [0, \pi]$. Integrating (2.5) over $(0, 2\pi)$ when $u = u_n$ and $t = t_n$, we have

$$
\begin{aligned}
t_n \int_0^{2\pi} g(x, u_n(x))dx &< (1 - t_n)\alpha \int_0^{2\pi} u_n(x)dx + t_n \int_0^{2\pi} g(x, u_n(x))dx \\
&= t_n \int_0^{2\pi} h(x)dx.
\end{aligned} \tag{2.9}
$$

We first assume that h satisfies either (2.2) or (2.3). It follows from (H) and the fact that $u_n \geq r_0$ on $[0, 2\pi]$ independent of n, $g(x, u_n)$ is bounded from below by a function in $L^1(0, 2\pi)$ independent of n. Applying Fatou's lemma to the inequality $\int_0^{2\pi} g(x, u_n(x))dx < \int_0^{2\pi} h(x)dx$, we have $\int_0^{2\pi} g_+^0(x)dx \leq \int_0^{2\pi} h(x)dx$, which contradicts the second inequality in either (2.2) or (2.3). In order, we assume that h satisfies (2.1) or (2.4). It follows from (H), (2.9) and the fact that $u_n \geq r_0$ on $[0, \pi]$ independent of n, we have

$$
\int_0^{2\pi} c(x)dx \leq \int_0^{2\pi} g(x, u_n(x))dx < \int_0^{2\pi} h(x)dx,
$$

which contradicts the second inequality in either (2.1) or (2.4). This completes the proof of the theorem.

In order, to obtain an existence theorem to (1.1) in which the condition (H) is replaced by (G). An example is given in [7] shows that the condition (G) in the following theorem is almost sharp, that is, when the number 1/4 in (G) is replaced by any number larger than 1/4, the assertion is false. □

Theorem 2. *Let $g : (0, 2\pi) \times \mathbf{R} \to \mathbf{R}$ be a Caratheodory function satisfying* (G). *Then for each $h \in L^1(0, 2\pi)$ the problem* (1.1) *has a solution $u \in W^{2,1}(0, 2\pi)$, provided that one of conditions* (2.1), (2.2), (2.3) *and* (2.4) *holds.*

Proof. Let $\alpha \in \mathbf{R}$ be fixed, $0 < \alpha < 1/4$. We consider the boundary value problems (2.5) for $0 \leq t \leq 1$, and define

$$
g(x, u) = \begin{cases} \{g(x, u) + e(x)\}\theta(u) & \text{if } u \geq 0 \\ \max\{g(x, u) - e(x), a(x)u\}\theta(u) & \text{if } u \leq 0 \end{cases}
$$

and $g_2 = g - g_1$, where $\theta : \mathbf{R} \to \mathbf{R}$ is a continuous function and $e \in L^1(0, 2\pi)$ which are defined as in the proof of Theorem 1. It follows that for a.e. $x \in (0, 2\pi)$ and all $u \in \mathbf{R}$

$$
0 \leq g_1(x, u)u \text{ and } |g_2(x, u)| \leq e(x), \tag{2.10}
$$

and for a.e. $x \in (0, 2\pi)$ and all $u \leq 0$

$$
-a(x)|u| \leq g_1(x, u) \leq 0. \tag{2.11}
$$

As in the proof of Theorem 1, it suffices to show that solutions to (2.5) for $0 < t < 1$ have an a priori bound in $H^1(0, 2\pi)$. Indeed, if u is a possible solution to (2.5), then we can write u as $u = u^+ - u^-$, where $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$. We will first show that u^- is bounded in $H^1(0, \pi)$. To this end, let us extend $u(x), a(x), g_1(x, u), g_2(x, u)$ and $h(x)$ 2π-periodically in x to all of $\mathbf{R}$, and then using the same notations for the periodic extensions as for the original functions. Then $u : \mathbf{R} \to \mathbf{R}$ is a periodic solution of period 2π to (2.5). Clearly, $\|u^-\|_{H^1} = \|u^-\|_{H^1(x,x+2\pi)}$ for all $x \in \mathbf{R}$. If first suppose that u^- has a zero in $[0, 2\pi]$ and set $v(x) = -u^-(x)$ for all $x \in \mathbf{R}$. Let $x_0 = \max\{x | u^-(x) = 0, 0 \le x \le 2\pi\}$, and let $[c_0, d_0]$ be a component of the support of u^- in $[x_0, x_0 + 2\pi]$. Then

$$\begin{aligned} &v'' + tf(v)v' + (1-t)\alpha v + tg_1(x, v) + tg_2(x, v) = th \quad \text{in } (c_0, d_0), \\ &v(c_0) = v(d_0) = 0. \end{aligned} \tag{2.12}$$

Multiplying each side of (2.12) by $-v$ and integrating over $[c_0, d_0]$, and then using $v(c_0) = v(d_0) = 0$ and the decomposition of g, $g = g_1 + g_2$, we obtain

$$\begin{aligned} &\int_{c_0}^{d_0} \{(v'(x))^2 - [(1-t)\alpha + ta(x)]v^2(x)\}dx \\ &\le \int_{c_0}^{d_0} \left\{(v'(x))^2 - \left[(1-t)\alpha + t\frac{g_1(x, v(x))}{v(x)}\right] v^2(x)\right\} dx \\ &= \int_{c_0}^{d_0} tg_2(x, v(x))v(x)dx - t\int_{c_0}^{d_0} h(x)v(x)dx. \end{aligned} \tag{2.13}$$

Thus (2.13) holds for all components of the support of v in $[x_0, x_0 + 2\pi]$, and also holds for the complement in $[x_0, x_0 + 2\pi]$ of the support of v because v is identically zero on that set. In this case, we obtain

$$\begin{aligned} &\int_{x_0}^{x_0+2\pi} \{(v'(x))^2 - p(x)v^2(x)\}dx \\ &\le \int_{x_0}^{x_0+2\pi} \{(v'(x))^2 - [(1-t)\alpha + ta(x)]v^2(x)\}dx \\ &\le C_2\|v\|_{H^1(x_0,x_0+2\pi)} \\ &= C_2\|u^-\|_{H^1} \end{aligned} \tag{2.14}$$

for some constant $C_2 > 0$ independent of u, where $p(x) = \max\{\alpha, a(x)\}$. Combining (2.14) with Lemma 2, we have

$$K_2\|u^-\|_{H^1}^2 = K_2\|v\|_{H^1}^2 = K_2\|v\|_{H^1(x_0,x_0+2\pi)}^2 \le C_2\|u^-\|_{H^1}.$$

Consequently, there exists a constant $C_3 > 0$ such that for each $0 < t < 1$, we have

$$\|u^-\|_{H^1} \le C_3 \tag{2.15}$$

for all possible solutions u to (2.5) with zeros in $[0, 2\pi]$. In order, we consider the case that u^- has no zero in $[0, 2\pi]$. In this case, we have $u = -u^-$ and $u < 0$ on $[0, 2\pi]$. Since $a(x) \leq 1/4 < 1$ and u is a periodic solution of period 2π to (2.5), one may now proceed as in the proof of Theorem 1 to show the existence of a bound on $\|u\|_{H^1} = \|u^-\|_{H^1}$. Integrating (2.5) over $[0, 2\pi]$, we have

$$(1-t)\alpha \int_0^{2\pi} u(x)dx + t \int_0^{2\pi} g(x, u(x))dx = t \int_0^{2\pi} h(x)dx, \tag{2.16}$$

and then using $g = g_1 + g_2$, the boundedness of u^-, the boundary condition of (2.5) and (2.16), we also have

$$\begin{aligned}
&(1-t)\alpha \int_{u(x)>0} u(x)dx + t \int_{u(x)>0} g_1(x, u(x))dx \\
&= t \int_0^{2\pi} h(x)dx - t \int_{u(x)\leq 0} g_1(x, u(x))dx \\
&\quad -t \int_0^{2\pi} g_2(x, u(x))dx - (1-t)\alpha \int_{u(x)\leq 0} u(x)dx \\
&\leq \|h\|_{L^1} + \|a\|_{L^1}\|u^-\|_C + \|e\|_{L^1} + 2\pi\alpha\|u^-\|_C \\
&\leq C_4
\end{aligned} \tag{2.17}$$

for some constant $C_4 > 0$ independent of u. It follows that there exists a constant $C_5 > 0$ such that

$$\int_0^{2\pi} |(1-t)\alpha u(x) + tg(x, u(x)) - th(x)|dx \leq C_5 \tag{2.18}$$

for all possible solutions u to (2.5) and $0 < t < 1$. But

$$\begin{aligned}
&u'' + tf(u)u' \\
&= -(1-t)\alpha u - tg(x, u) + th \text{ in } (0, 2\pi).
\end{aligned} \tag{2.19}$$

Hence, from (2.18), (2.19) and Lemma 3 that there exists a constant $C_6 > 0$ such that

$$\|\tilde{u}\|_{H^1} \leq K_3 C_5 \leq C_6 \tag{2.20}$$

for all possible solutions u to (2.5) and $0 < t < 1$. We may now use (2.20) as (2.8) in the proof of Theorem 1 to show the existence $\|u\|_{H^1} \leq C_7$ for some constant $C_7 > 0$ independent of u. Hence the proof of this theorem is complete. □

By slightly modifying proofs of Theorems 1 and 2, we can obtain new solvability conditions in which the nonlinearity g satisfies the following condition:

(F) There exist constants $\widetilde{r_0}$, γ, $\delta \geq 0$ and $\widetilde{c}$, $\widetilde{d} \in L^1(0, 2\pi)$ such that for a.e. $x \in (0, 2\pi)$ and all $u \geq \widetilde{r_0}$

$$g(x, u)u \geq \widetilde{c}(x)|u|^{1-\gamma},$$

and for a.e. $x \in (0, 2\pi)$ and all $u \leq -\widetilde{r_0}$

$$g(x, u)u \geq \widetilde{d}(x)|u|^{1-\delta};$$

and conditions (2.1)-(2.4) may be replaced by one of the following conditions:

$$\int_0^{2\pi} g_-^{\delta}(x)dx < \int_0^{2\pi} h(x)dx = 0 < \int_0^{2\pi} g_+^{\gamma}(x)dx; \tag{2.21}$$

$$\int_0^{2\pi} g_-^{\delta}(x)dx < 0 = \int_0^{2\pi} h(x)dx \leq \int_0^{2\pi} c(x)dx; \tag{2.22}$$

$$\int_0^{2\pi} d(x)dx \leq \int_0^{2\pi} h(x)dx = 0 < \int_0^{2\pi} g_+^{\gamma}(x)dx; \tag{2.23}$$

where $g_-^{\delta}(x) = \limsup_{u\to-\infty} g(x, u)|u|^{\delta}$ and $g_+^{\gamma}(x) = \liminf_{u\to\infty} g(x, u)|u|^{\gamma}$.

Theorem 3. *Let* $g : (0, 2\pi) \times \mathbf{R} \to \mathbf{R}$ *be a Caratheodory function satisfying* (F) *and* (H), *then for each* $h \in L^1(0, 2\pi)$ *the problem* (1.1) *has a solution* $u \in W^{2,1}(0, 2\pi)$, *provided that one of* (2.21)-(2.23) *holds.*

Proof. In proving Theorem 1, conditions (2.1)-(2.4) are used only to produce contradiction in the final part of the proof. Thus we can proceed exactly the same way as in the proof of Theorem 1 up to the point where $\beta > 0$ is considered and (2.9) is satisfied. In this case, we may assume similarly that $u_n \geq \widetilde{r_0}$ on $[0, 2\pi]$ independent of n. It follows from (F) and the fact that $v_n \geq \beta/2$ on $[0, 2\pi]$ for all $n \geq n_0$, we have

$$g(x, u_n(x))\|u_n\|_{H^1}^{\gamma} = g(x, u_n(x))|u_n(x)|^{\gamma}|v_n(x)|^{-\gamma} \geq -|\widetilde{c}(x)|\left(\frac{\beta}{2}\right)^{-\gamma}$$

for a.e. $x \in (0, 2\pi)$ and all $n \geq n_0$. Applying Fatou's lemma to the left hand side of the inequality

$$\|u_n\|_{H^1}^{\gamma}\int_0^{2\pi} g(x, u_n(x))dx < \|u_n\|_{H^1}^{\gamma}\int_0^{2\pi} h(x)dx = 0,$$

we have

$$\beta^{-\gamma}\int_0^{2\pi} g_+^{\gamma}(x)dx \leq 0,$$

or equivalently

$$\int_0^{2\pi} g_+^{\gamma}(x)dx \leq 0,$$

which contradicts the second inequality in either (2.21) or (2.23). Hence the proof is complete. □

Similarly we can obtain the following theorem in which the condition (H) may be replaced by (G).

Theorem 4. *Let* $g : (0, 2\pi) \times \mathbf{R} \to \mathbf{R}$ *be a Caratheodory function satisfying* (F) *and* (G), *then for each* $h \in L^1(0, 2\pi)$ *the problem* (1.1) *has a solution* $u \in W^{2,1}(0, 2\pi)$, *provided that one of* (2.21)-(2.23) *holds.*

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