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EQUIVARIANT CASSON INVARIANT FOR KNOTS AND THE NEUMANN–WAHL FORMULA

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In 1985, Casson introduced a new 3-manifold invariant for integer homology spheres, via $SU(2)$–representation spaces of their fundamental groups. This invariant has solved some difficult problems in 3-manifold Topology and was found to have a relation to Gauge Theory, by work of Taubes in [17]. More recently, there have been various generalizations of Casson's invariant, for example Walker's extention to rational homology spheres. But more importantly for this article, in [3], Cappell, Lee and Miller have defined an equivariant Casson invariant, $\lambda^{(k/n)}(Y^3)$, for closed 3-manifolds Y^3 with a semi-free $\mathbb{Z}_n$–action, where n is an odd prime and k an integer such that $0 \leq k \leq (n-1)/2$. A particularly important case is that of n-fold cyclic branched coverings of S^3 along a knot K, denoted $V_n(K)$. In that case, the equivariant Casson invariants may be related to equivariant knot signatures of K, by developping a surgery formula for the invariants as done in [3].

In parallel, the author and B. Steer developped in [6] a Floer Homology for knots in S^3, a knot invariant consisting of four abelian groups, denoted $HF^{(\alpha)}(S^3, K)$, depending on a parameter $0 \leq \alpha \leq 1/2$. The Euler characteristic of this Floer Homology was related to equivariant knot signatures of K, via work of Herald in [9]. In fact, when $\alpha = k/n$ for n an odd prime, we have $\chi(HF_*^{(k/n)}(S^3, K)) = \lambda^{(k/n)}(V_n(K))$, so that the Floer Homology for knots may be seen as a generalisation of the equivariant Casson invariant of cyclic branched covers of S^3 along knots. In this article, we combine both points of view to define an equivariant Casson invariant for 3-manifolds which arise as 4-fold cyclic branched covers of S^3 along a knot K. In the case of cyclic branched covers with an even branching index, the construction in [3] does not apply. The original motivation for this work was therefore to complement the work done in [3]. The resulting invariant brings new light to the Neumann–Wahl formula, relating Casson's invariant of a 3-dimensional link of complex singularity to the signature of its Milnor fibre, as explained in Section 4, where the equivariant Casson invariant is expressed in terms of the Jones polynomial of K, using a result of Mullins. The article ends with a generalization to other cyclic branched coverings of S^3 with

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even ramification index.

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1. 3-orbifolds and cyclic actions on branched covers

Let $V_n(K)$ be the n-fold cyclic branched covering of S^3 along K. There is a $\mathbb{Z}_n$–action on $V_n(K)$ extending covering transformations of the unbranched n-fold cover of the knot complement $S^3 - N_K$, where N_K is an open tubular neighbourhood of K in S^3. This action is free away from K, but fixes the knot. There are two ways of looking at the quotient space under this action. One is to put an orbifold structure on this quotient and regard it as a 3-orbifold, as explained in [2] or in [6]. In this case, at the level of fundamental groups, one may prove the existence of an exact sequence

$$1 \to \pi_1(V_n(K)) \to \pi_1^V(S^3, K, n) \to \mathbb{Z}_n \to 1, \tag{1}$$

where $\pi_1^V(S^3, K, n)$ denotes the orbifold fundamental group $\pi_1^V(S^3, K, n) = \pi_1(S^3 - N_K)/ < \mu^n >$, for μ a meridian in $\pi_1(S^3 - N_K)$. It is not difficult to show that $\pi_1^V(S^3, K, n)$ is isomorphic to $\pi_1(V_n(K)) \rtimes \mathbb{Z}_n$, the semi-direct product induced by the $\mathbb{Z}_n$–action on $V_n(K)$. Another way of looking at the quotient space is to put a manifold structure on it, as done in [3]; the resulting 3-manifold is then simply S^3. We shall also encounter other quotients where the action of a cyclic group fixes some knot in a 3-manifold. The fact that the quotient space may be regarded as an orbifold or as a manifold will be important in our setting.

In this article, we shall look at $\mathbb{Z}_2$–actions on $V_4(K)$, similar to the actions explained above. Let $\pi_4 \colon V_4(K) \to S^3$ be the 4-fold cyclic branched cover of S^3 with branch sets K and $\pi_4^{-1}(K)$ respectively downstairs and upstairs. We could alternatively consider first the 2-fold cover of S^3, $\pi_2 \colon V_2(K) \to S^3$, with branch sets K and $\pi_2^{-1}(K)$ and then $\pi_{2,4} \colon V_4(K) \to V_2(K)$, the 2-fold cover of $V_2(K)$ with branch sets $\pi_2^{-1}(K)$ and $\pi_{2,4}^{-1}(\pi_2^{-1}(K))$: $\pi_2 \circ \pi_{2,4} \colon V_4(K) \longrightarrow V_2(K) \longrightarrow S^3$. By construction $\pi_4 = \pi_2 \circ \pi_{2,4}$ and also $\pi_4^{-1}(K) = \pi_{2,4}^{-1}(\pi_2^{-1}(K))$. The 3-manifold $V_2(K)$ is always a rational homology sphere in which $\pi_2^{-1}(K)$ is nullhomologous, so we may consider the 3-orbifold $(V_2(K), \pi_2^{-1}(K), 2)$ defined in a similar way as above. Then the quotient spaces involved, regarded as 3-orbifolds are:

$$V_4(K)/\mathbb{Z}_2 = (V_2(K), \pi_2^{-1}(K), 2) \text{ and } V_2(K)/\mathbb{Z}_2 = (S^3, K, 2),$$

or as manifolds, they are:

$$V_4(K)/\mathbb{Z}_2 \cong V_2(K) \text{ and } V_2(K)/\mathbb{Z}_2 \cong S^3.$$

2. $\mathbb{Z}_2$–actions on $SU(2)$–character varieties

In this section we adapt material found in [3] to the case of a $\mathbb{Z}_2$–action on $V_4(K)$. Let $\mathbb{Z}_2$ act on the 4-fold cyclic branched cover of S^3 along K, $V_4(K)$. This induces an action at the level of the fundamental group:

$$\mathbb{Z}_2 \times \pi_1(V_4(K)) \to \pi_1(V_4(K)) \ , \ (g, \alpha) \mapsto g \cdot \alpha.$$

Consequently, at the level of representations the action is given as follows: for $g \in \mathbb{Z}_2$ and a representation $\rho\colon \pi_1(V_4(K)) \to SU(2)$, we obtain

$$\rho_g\colon \pi_1(V_4(K)) \to SU(2) \ , \ \rho_g(\alpha) = \rho(g \cdot \alpha).$$

For any matrix $A \in SU(2)$, we have $A \cdot \rho_g \cdot A^{-1} = (A \cdot \rho \cdot A^{-1})_g$, so the $\mathbb{Z}_2$–action descends to the character variety $\mathcal{R}(V_4(K))$. Notice that the subset of irreducibles, $\mathcal{R}_*(V_4(K))$, is invariant under this action. Let $\mathcal{R}_*^{\mathbb{Z}_2}(V_4(K))$ be the fixed-point set of this action. For $\rho \in \mathcal{R}_*^{\mathbb{Z}_2}(V_4(K))$ and $g \in \mathbb{Z}_2$ there is a matrix $\Phi \in SU(2)$ such that $\rho_g = \Phi \cdot \rho \cdot \Phi^{-1}$.

Lemma 2.1. *Φ is determined up to $\pm I \in SU(2)$.*

Proof. Let $\Phi' \in SU(2)$ be another matrix such that $\rho_g = \Phi' \cdot \rho \cdot {\Phi'}^{-1}$. Then

$$\Phi' \cdot \rho \cdot {\Phi'}^{-1} = \Phi \cdot \rho \cdot \Phi^{-1} \implies (\Phi^{-1} \cdot \Phi') \cdot \rho \cdot ({\Phi'}^{-1} \cdot \Phi) = \rho.$$

As ρ is irreducible, its centralizer in $SU(2)$ consists only of $\{\pm I\} \subset SU(2)$ and so $\Phi' = \pm\Phi$.

For $g \in \mathbb{Z}_2$, we have $g^2 = 1$ so that $\rho = \rho_{g^2} = \Phi \cdot \rho_g \cdot \Phi^{-1} = \Phi^2 \cdot \rho \cdot \Phi^{-2}$. As in the proof of Lemma 2.1, ρ being irreducible, we obtain $\Phi^2 = \pm I$. There are then two cases to consider:
(I) $\Phi^2 = I \implies \Phi = I$ or $\Phi = -I$.
(II) $\Phi^2 = -I \implies \Phi^4 = I$.

REMARK. Unlike in [3], where the cyclic branched cover had an odd branching index, it is impossible in our case to fix the sign in $\Phi' = \pm\Phi$ so that $\Phi^2 = I$. This turns out to be an important difference between [3] or [5] and the present article.

While in case (I), Φ is in the center $\{\pm I\}$ of $SU(2)$, in case (II), we have up to conjugation:

$$\Phi = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

We can use this to decompose the fixed-point set $\mathcal{R}_*^{\mathbb{Z}_2}(V_4(K))$ according to cases (I) and (II) above:

$$(2) \qquad \mathcal{R}_*^{\mathbb{Z}_2}(V_4(K)) = \mathcal{R}_*^1(V_4(K)) \sqcup \mathcal{R}_*^1(V_4(K)) \sqcup \mathcal{R}_*^i(V_4(K)).$$

Notice that we take two copies of $\mathcal{R}_*^1(V_4(K))$: one for each I and $-I$, as Φ is determined up to $\pm I$. We now charactrize $\mathcal{R}_*^1(V_4(K))$ and $\mathcal{R}_*^i(V_4(K))$.

Proposition 2.2. $\mathcal{R}_*^1(V_4(K)) = \mathcal{R}_*(V_2(K))$.

Proof. In this case, Φ being in the center of $SU(2)$, we obtain

$$\rho_g = \Phi \cdot \rho \cdot \Phi^{-1} = \rho.$$

Hence under the $\mathbb{Z}_2$–action we have $\rho(g \cdot \alpha) = \rho(\alpha)$ for $\alpha \in \pi_1(V_4(K))$. The representation ρ therefore descends to a representation $\bar{\rho}$ of the fundamental group of the quotient manifold $V_2(K)$. Conversely, any representation $\sigma \in \mathcal{R}_*(V_2(K))$ lifts to an element in $\mathcal{R}_*(V_4(K))$ and we have $\mathcal{R}_*(V_2(K)) = \mathcal{R}_*^1(V_4(K))$. □

For $\mathcal{R}_*^i(V_4(K))$, consider the semi-direct product using the $\mathbb{Z}_2$–action on $\pi_1(V_4(K))$ such that $\Phi^4 = I$ and $\rho_g = \Phi \cdot \rho \cdot \Phi^{-1}$. We let

$$\rho_\Phi \colon \pi_1(V_4(K)) \rtimes \mathbb{Z}_4 \to SU(2) \ , \ \rho_\Phi(\alpha, g^i) = \rho(\alpha) \cdot \Phi^i.$$

This representation is easily seen to be well-defined and independent of the representative ρ chosen. This defines a mapping:

$$\Psi \colon \mathcal{R}_*^i(V_4(K)) \to \mathcal{R}_*(V_4(K) \rtimes \mathbb{Z}_4).$$

The map Ψ is injective, but it may not be surjective. Indeed, it is possible that for some $\sigma \in \mathcal{R}_*(V_4(K) \rtimes \mathbb{Z}_4)$, the restriction $\sigma_{|\pi_1(V_4(K))}$ be reducible, in which case $\sigma \notin \Psi(\mathcal{R}_*^i(V_4(K)))$. To facilitate the definition of the equivariant Casson invariant, we shall therefore impose the following:

Condition 2.3. $H_1(V_4(K), \mathbb{Z}) = 0$.

This condition implies that $\Psi \colon \mathcal{R}_*^i(V_4(K)) \to \mathcal{R}_*(V_4(K) \rtimes \mathbb{Z}_4)$ is surjective, as the only reducible in $\mathcal{R}(V_4(K))$ is the trivial representation which cannot be the restriction of any $\sigma \in \mathcal{R}_*(V_4(K) \rtimes \mathbb{Z}_4)$.

We can then make the connection to the orbifold $(S^3, K, 4)$ as the orbifold exact sequence (1) gives that $\pi_1(V_4(K)) \rtimes \mathbb{Z}_4 \simeq \pi_1^V(S^3, K, 4)$ and therefore representations in $\mathcal{R}_*(V_4(K)) \rtimes \mathbb{Z}_4$ are in bijection with

$$\mathcal{R}_*^{(1)}(S^3, K, 4) = \{\rho \in \mathcal{R}_*(S^3, K, 4) \mid \rho(\mu) = \begin{pmatrix} e^{i2\pi \cdot 1/4} & 0 \\ 0 & e^{-i2\pi \cdot 1/4} \end{pmatrix}\} :$$

Proposition 2.4. $\mathcal{R}^i_*(V_4(K)) = \mathcal{R}^{(1)}_*(S^3, K, 4)$.

3. Floer Homology and the Equivariant Casson Invariant

In this section we shall use Gauge Theory to construct an equivariant Casson invariant for the $\mathbb{Z}_2$–action on $V_4(K)$, by organizing $\mathcal{R}^{\mathbb{Z}_2}_*(V_4(K))$ into a bigraded Floer complex whose Euler characteristic will be the equivariant Casson invariant. This will involve the Floer homology for 3-orbifolds developped in [6].

Let us recall some aspects of the construction of Floer Homology for 3-orbifolds from [6]. We shall be brief here and refer to that article for details and more information on Gauge Theory for orbifolds. Let K be a knot in a homology sphere Y^3. One can construct the 3-orbifold (Y^3, K, n). Let E_k be the $SU(2)$–bundle over (Y^3, K, n) with isotropy representation $\tau_k\colon \mathbb{Z}_n \to SU(2)$ such that $tr\ \tau_k(1) = 2cos(2\pi k/n)$, and let $\mathcal{M}_{flat}(E_k)$ denote the moduli space of flat connections on E_k. This space is known to be homeomorphic to the subset in $\mathcal{R}(Y^3, K, n)$ consisting of representations ρ such that $\rho(\mu) = 2cos(2\pi k/n)$ where μ is a meridian for K in $\pi_1^V(Y^3, K, n)$. In particular, we have:

Lemma 3.1. $\mathcal{M}_{flat}(E_0) \cong \mathcal{R}(Y^3)$.

Proof. Indeed, representations $\rho\colon \pi_1^V(Y^3, K, n) \to SU(2)$ corresponding to an element in $\mathcal{M}_{flat}(E_0)$ are such that $\rho(\mu) = I$ and are thus by construction representations from $\pi_1(Y^3) = \pi_1(Y^3 - N_K)/ < \mu >$ to $SU(2)$. □

We may suppose that all flat connections in $\mathcal{M}_{flat}(E_k)$ are non-degenerate, since if they are not we can simply perturb the flatness equation to get a non-degenerate moduli space, as done in [6]. If $0 < k < n/2$, given an $A \in \mathcal{M}_{flat}(E_k)$ and the reducible flat connection $\theta_k \in \mathcal{M}_{flat}(E_k)$, one may define, as in [6], the Floer index

$$\mu(A) \equiv ind\ D_{\mathbb{A}}\ (mod\ 4), \tag{3}$$

where $\mathbb{A} = A_t$ is a connection over $(Y^3, K, n) \times \mathbb{R}$ limiting to θ_k and A at the ends of the cylinder, and $D_{\mathbb{A}}$ is the linearized ASD operator on this cylinder. The Floer complex consists of four free abelian groups $C_i^{(k/n)}(Y^3, K)$, generated by flat connections A such that $\mu(A) = i$. For the purpose of this article we shall not be concerned with the Floer Homology of (Y^3, K, n), but rather only the Euler characteristic of the Floer complex: $\chi(C_*^{(k/n)}(Y^3, K))$.

REMARK. In the case where Y^3 is not a homology sphere but only a rational homology sphere, if the knot K is assumed to be nullhomologous in Y^3, one can still define the Floer complex, now taking into account the various reducible flat connections.

This being said, let us build a bigraded Floer complex. Suppose for the moment that $V_4(K)$ satisfies Condition 2.3. This readily implies that $V_2(K)$ is also an integer homology sphere. By Proposition 2.2, $\mathcal{R}_*^1(V_4(K))$ is the same as $\mathcal{R}_*(V_2(K))$ which itself corresponds to $\mathcal{M}_{flat}^*(V_2(K))$. In turn, if we let E_0 be the $SU(2)$–bundle with trivial isotropy over $(V_2(K), \pi_2^{-1}(K), 2)$, then $\mathcal{M}_{flat}(V_2(K)) = \mathcal{M}_{flat}(E_0)$. We can therefore use the 3-orbifold Floer complex to make the following:

DEFINITION 3.2. Let

$$C_{0,*}^{\mathbb{Z}_2}(V_4(K)) = C_*^{(0)}(V_2(K), \pi_2^{-1}(K)) \oplus C_*^{(0)}(V_2(K), \pi_2^{-1}(K)).$$

The Euler characteristic of $C_{0,*}^{\mathbb{Z}_2}(V_4(K))$ is defined to be

$$\chi(C_{0,*}^{\mathbb{Z}_2}(V_4(K))) = 2 \cdot \chi(C_*^{(0)}(V_2(K), \pi_2^{-1}(K))).$$

REMARK. We take two copies of $C_*^{(0)}(V_2(K), \pi_2^{-1}(K))$ to be consistent with the presence of two copies of $\mathcal{R}_*^1(V_4(K))$ in Equation (2).

On the other hand, we also have the subset $\mathcal{R}_*^i(V_4(K))$ which is equal to $\mathcal{R}_*^{(1)}(S^3, K, 4)$. By parallel transport, this is also equal to $\mathcal{M}_{flat}^*(E_1)$ for the $SU(2)$-bundle E_1 over $(S^3, K, 4)$.

DEFINITION 3.3. $C_{1,*}^{\mathbb{Z}_2}(V_4(K)) = C_*^{(1/4)}(S^3, K)$. The Euler characteristic of $C_{1,*}^{\mathbb{Z}_2}(V_4(K))$ is

$$\chi(C_{1,*}^{\mathbb{Z}_2}(V_4(K))) = \chi(C_*^{(1/4)}(S^3, K)).$$

Combining the two yields the definition of our invariant:

DEFINITION 3.4. The bigraded equivariant Floer complex of $V_4(K)$ is

$$C_{0,*}^{\mathbb{Z}_2}(V_4(K)) \oplus C_{1,*}^{\mathbb{Z}_2}(V_4(K)).$$

The Euler characteristic of this complex is the $\mathbb{Z}_2$–equivariant Casson invariant of $V_4(K)$: $\lambda^{\mathbb{Z}_2}(V_4(K)) = \chi(C_{0,*}^{\mathbb{Z}_2}(V_4(K)) \oplus C_{1,*}^{\mathbb{Z}_2}(V_4(K)))$.

We wish to give an expression of this invariant $\lambda^{\mathbb{Z}_2}(V_4(K))$. This is given in the following:

Theorem 3.5. *Let $\lambda(V_2(K))$ denote the Casson invariant of the 2-fold cyclic covering $V_2(K)$ and $\sigma(K)$ be the knot signature of K. Then,*

$$\lambda^{\mathbb{Z}_2}(V_4(K)) = 4 \cdot \lambda(V_2(K)) - \frac{1}{2} \cdot \sigma(K).$$

Proof. We have,

$$\begin{aligned}\lambda^{\mathbb{Z}_2}(V_4(K)) &= \chi(C^{\mathbb{Z}_2}_{0,*}(V_4(K)) \oplus C^{\mathbb{Z}_2}_{1,*}(V_4(K))) \\ &= \chi(C^{\mathbb{Z}_2}_{0,*}(V_4(K))) \;-\; \chi(C^{\mathbb{Z}_2}_{1,*}(V_4(K))).\end{aligned}$$

First consider $\chi(C^{\mathbb{Z}_2}_{0,*}(V_4(K)))$. Given a generator $A \in C^{\mathbb{Z}_2}_{0,*}(V_4(K))$, as this is by construction an element in $\mathcal{M}_{flat}(E_0)$, for E_0 the $SU(2)$–bundle with trivial isotropy over $(V_2(K), \pi_2^{-1}(K), 2)$, it corresponds by Lemma 3.1 to a flat connection $\bar{A} \in \mathcal{M}_{flat}(V_2(K))$, which will be a generator in $C_*(V_2(K))$. But we know by work of Taubes in [17] that $\chi(C_*(V_2(K))) = 2 \cdot \lambda(V_2(K))$, so we need to show that

$$\chi(C_*(V_2(K))) = \chi(C^{(0)}_*(V_2(K), \pi_2^{-1}(K), 2)). \tag{4}$$

In the case $k = 0$, Equation (3) is amended to

$$\mu(A) \equiv ind\ D_{\mathbb{A}}\ (mod\ 8),$$

and as the connections involved have no holonomy around the singular locus, so

$$\mu(A) \equiv ind\ D_{\mathbb{A}} = ind\ D_{\bar{\mathbb{A}}} \equiv \mu(\bar{A})\ (mod\ 8),$$

which directly implies (4) and thus gives $\chi(C^{\mathbb{Z}_2}_{0,*}(V_4(K))) = 4 \cdot \lambda(V_2(K))$. On the other hand, $\chi(C^{\mathbb{Z}_2}_{1,*}(V_4(K))) = \chi(C^{(1/4)}_*(S^3, K))$ by definition, and Theorem 4.9 in [6] with $k = 1$ and $n = 4$ gives $\chi(C^{(1/4)}_*(S^3, K)) = 1/2 \cdot \sigma(K)$, completing the proof. □

In the case where $V_4(K)$ does not satisfy Condition 2.3, the manifold $V_2(K)$ is nevertheless a rational homology sphere and the chain complex in Definition 3.2, $C^{(0)}_*(V_2(K), \pi_2^{-1}(K))$, still exists, but the Euler characteristic has to be normalized to take into account reducibles:

$$\chi(C^{\mathbb{Z}_2}_{0,*}(V_4(K))) = \frac{2}{|H_1(V_2(K))|} \cdot \chi(C^{(0)}_*(V_2(K), \pi_2^{-1}(K))).$$

On the other hand, Definition 3.3 still makes sense, as the Floer complex $C^{(1/4)}_*(S^3, K)$ exists whether or not $H_1(V_4(K), \mathbb{Z}) = 0$. This being said, Theorem 3.5 remains true in this more general setting if in the statement $\lambda(V_2(K))$ is understood to be the Casson–Walker invariant.

4. Equivariant Casson Invariant and the Neumann–Wahl Formula

Let $f\colon \mathbb{C}^2 \to \mathbb{C}$ be a polynomial having an isolated singularity at 0 with link of the singularity $\{f^{-1}(0)\} \cap S^3$, a knot K in S^3. Such a knot is called an algebraic knot. Then the polynomial $g(x, y, z) = f(x, y) + z^n$ also has an isolated singularity at 0 so let $\Sigma^3 = \{g^{-1}(0)\} \cap S^5$ be the corresponding 3–dimensional link of singularity. The Milnor fibre of Σ is $M^4 = \{g^{-1}(\delta)\} \cap B^6$, for $\delta > 0$ small enough. M^4 is a smooth, parallelizable 4-manifold with boundary Σ^3. If Σ is a rational homology sphere, two invariants may be considered here. An invariant of Σ is the Casson–Walker invariant, $\lambda(\Sigma)$, while another one is the signature of M^4, $\sigma(M^4)$. In the case where Σ^3 is an integer homology sphere, Neumann and Wahl proved that the two invariants are essentially the same:

Theorem 4.1 ([15] Proposition 2.5).

$$\lambda(\Sigma^3) = \frac{1}{8} \cdot \sigma(M^4).$$

In fact, Neumann and Wahl essentially used the particular case of Brieskorn homology spheres $\Sigma(p, q, r)$, for which Fintushel and Stern proved the formula above, and the additivity of the Casson invariant under splicing to obtain Theorem 4.1. At the end of the introduction of [15], they ask the natural question whether their result holds for rational homology spheres, by substituting the Casson invariant by the Casson–Walker invariant. The aim of this section is to relate the Neumann–Wahl formula to the equivariant Casson invariant $\lambda^{\mathbb{Z}_2}(V_4(K))$. This can be regarded as an analogue of work done by the author and N. Saveliev in [5], where a geometric proof of the Fintushel–Stern formula was obtained using the Floer Homology for knots introduced in the last section. In this section, we will also exhibit links of isolated complex singularities which are rational homology spheres for which the Neumann–Wahl formula is not true, hence answering negatively to the question of Neumann and Wahl. Moreover, the relation to the equivariant Casson invariant enables us to place the formula in a more general context than that of links of singularities.

For this section, we shall be concerned with 3-dimensional links of singularities of a polynomial of the form $g(x, y, z) = f(x, y) + z^2$. It is well-known that the resulting 3-manifold is a rational homology sphere which is a 2-fold cyclic branched cover of S^3 along the algebraic knot $K = \{f^{-1}(0)\} \cap S^3$. Moreover, in this context the Milnor fibre appears as a 2-fold cover of B^4 along some surface spanning the knot K. See [11] for further details on this.

More generally, for any knot K in S^3 with $S^3 = \partial B^4$ and $K = \partial F$, for F a Seifert surface of K pushed in B^4, let M_2 be the 2-fold branched covering of B^4 along F. At the level of homology, one has the decomposition $H_2(M_2, \mathbb{C}) = H_0 \oplus H_1$, corresponding to an eigenspace decomposition using the $\mathbb{Z}_2$ action on M_2 inducing an action on $H_2(M_2, \mathbb{C})$ by square roots of unity. This defines $\sigma_0(M_2)$ and $\sigma_1(M_2)$ as signatures of the intersection form on $H_2(M_2, \mathbb{C})$, restricted to H_0 and H_1 respectively. Lemma 12.4 in [11] proves the following:

Proposition 4.2. *$\sigma_0(M_2) = 0$ and $\sigma_1(M_2) = \sigma(K)$. In particular, $\sigma(M_2)$ is independent of the Seifert surface chosen to define M_2.*

Proposition 4.2 implies, in the case of the link of singularity Σ^3 and its Milnor fibre M^4 constructed above, that

$$\sigma(M^4) = \sigma(K). \tag{5}$$

The Neumann–Wahl formula is then expressed as

$$\lambda(V_2(K)) = \frac{1}{8} \cdot \sigma(K). \tag{6}$$

If $V_2(K)$ is not an integer homology sphere or a link of singularity, then Equation (6), if true, is a generalization of the Neumann–Wahl formula. Now, Theorem 3.5 for $V_4(K)$ implies that the equivariant Casson invariant $\lambda^{\mathbb{Z}_2}(V_4(K))$ is the obstruction for the generalized Neumann–Wahl formula to hold:

Corollary 4.3. $\lambda(V_2(K)) = 1/8 \cdot \sigma(M_2) \Longleftrightarrow \lambda^{\mathbb{Z}_2}(V_4(K)) = 0$.

For Corollary 4.3 to be of any use, one has to be able to compute the invariant $\lambda^{\mathbb{Z}_2}(V_4(K))$, to see whether the obstruction to the generalized Neumann–Wahl formula vanishes. Surprisingly, there is quite an easy way to do that. Let $V_K(t)$ be the Jones polynomial of the knot K, as defined by Jones in [10]. Then our equivariant Casson invariant may be expressed as:

Theorem 4.4.

$$\lambda^{\mathbb{Z}_2}(V_4(K)) = -\frac{1}{3} \cdot \frac{\frac{dV_K}{dt}(-1)}{V_K(-1)}.$$

Proof. Indeed, Mullins has shown in [14] that

$$\lambda(V_2(K)) = -\frac{1}{12} \cdot \frac{\frac{dV_K}{dt}(-1)}{V_K(-1)} + \frac{1}{8} \cdot \sigma(K).$$

Notice that this formula is slightly different than the one given in [14], as our convention for λ follows Casson, while Mullins follows Walker's convention. Combining this with Theorem 3.5 readily gives the equation appearing in the statement of the theorem. □

Clearly, Theorem 4.4 enables one to easily compute $\lambda^{\mathbb{Z}_2}(V_4(K))$ from tables of the Jones polynomial for various knots. We now make use of this to derive some consequences for the Neumann–Wahl formula and its generalization (6).

First, consider a Brieskorn homology sphere $\Sigma(2,p,q)$, where p,q are odd and coprime, with Milnor fibre $M(2,p,q)$. As $\Sigma(2,p,q)$ may be regarded as the 2-fold cover of S^3 along a torus knot of type (p,q), we can reprove the Fintushel–Stern formula (see Theorem 2.10 in [7] and also Theorem 3.5 in [5]) for $\Sigma(2,p,q)$:

Corollary 4.5. $\lambda(\Sigma(2,p,q)) = 1/8 \cdot \sigma(M(2,p,q))$.

Proof. Combining Equation (5), Corollary 4.3 and Theorem 4.4, we simply have to show that if K is a torus knot of type (p,q), for p,q odd and coprime, then

$$\frac{dV_K}{dt}(-1) = 0.$$

The Jones polynomial of torus knots was computed in [10], Proposition 11.9:

$$V_K(t) = t^{\frac{(p-1)(q-1)}{2}}(1 - t^{p+1} - t^{q+1} + t^{p+q})/(1 - t^2).$$

Factorizing $(1+t)$ in the second term, we get:

$$V_K(t) = t^{\frac{(p-1)(q-1)}{2}}(1 - t + t^2 - \dots - t^p - t^{q+1} + t^{q+2} - \dots + t^{p+q-1})/(1-t)$$

Factorizing $(1-t)$ we have:

$$V_K(t) = t^{\frac{(p-1)(q-1)}{2}}(\sum_{k=0}^{\frac{p-1}{2}} t^{2k} - \sum_{k=1}^{\frac{p-1}{2}} t^{q+2k-1})$$

Taking the derivative and evaluating it at $t = -1$, one gets

$$\begin{aligned}\frac{d}{dt}V_K(-1) = -\frac{(p-1)(q-1)}{2}(\sum_{k=0}^{\frac{p-1}{2}} 1 - \sum_{k=1}^{\frac{p-1}{2}} \frac{(p-1)}{2}) + \sum_{k=0}^{\frac{p-1}{2}} -2k \\ + \sum_{k=1}^{\frac{p-1}{2}} q + 2k - 1 \\ = -\frac{(p-1)(q-1)}{2} + \sum_{k=1}^{\frac{p-1}{2}} q - 1\end{aligned}$$

$$= -\frac{(p-1)(q-1)}{2} + \frac{p-1}{2}(q-1)$$
$$= 0.$$

□

In light of Corollary 4.3 and Theorem 4.4, if $V_2(K)$ is an integer homology sphere obtained as a 2-fold cover of S^3 along an algebraic knot K, by the Neumann–Wahl formula, the Jones polynomial of K has the property that:

$$\frac{dV_K}{dt}(-1) = 0.$$

Unlike for the value of $V_K(t)$ at roots of unity, there seems to be no information in the literature on the topological significance of the derivative of $V_K(t)$ evaluated at $t = -1$. It may seem natural to think that the vanishing above of this derivative at $t = -1$ has something to do with the fact that the knot K is algebraic. It turns out this is not the case: we exhibit knots which illustrate this in the following two examples:

EXAMPLE 1. Let K be an algebraic knot for which $\frac{dV_K}{dt}(-1) = 0$, for example the torus knot of type $(3,5)$. Consider the composite knot $K\sharp K$. By the product formula for the Jones polynomial under the operation of connected sum, $V_{K\sharp K}(t) = V_K(t) \cdot V_K(t)$. It follows that

$$\frac{dV_{K\sharp K}}{dt}(-1) = 0,$$

so that $\lambda^{\mathbb{Z}_2}(V_4(K\sharp K)) = 0$ and the generalized Neumann–Wahl formula (6) holds for the 3-manifold $V_2(K\sharp K)$. However $K\sharp K$ is not algebraic as it is a well-known fact that algebraic knots are prime.

EXAMPLE 2. Let K be the trefoil, the torus knot of type $(2,3)$. This is, of course, an algebraic knot. In this case, the Jones polynomial is easily found to be

$$V_K(t) = t + t^3 - t^4.$$

Taking the derivative of $V_K(t)$ and evaluating it at $t = -1$ gives, unlike in the proof of Corollary 4.5,

$$\frac{dV_K}{dt}(-1) \neq 0.$$

As a consequence of Example 2, we can answer negatively the question of Neumann and Wahl regarding the eventual extention of their work, using the Casson–Walker invariant, to rational homology spheres which arise as 3-dimensional links of singularities:

Corollary 4.6. *The Brieskorn sphere* $\Sigma(2,2,3)$ *is a rational homology sphere with Milnor fibre* $M(2,2,3)$ *for which*

$$\lambda(\Sigma(2,2,3)) \neq \frac{1}{8} \cdot \sigma(M(2,2,3)).$$

Proof. Indeed, for the 4-fold cover along the trefoil K, $\lambda^{\mathbb{Z}_2}(V_4(K)) \neq 0$ by Example 2, and therefore Proposition 4.2 and Corollary 4.3 prove the assertion. □

5. The general case and concluding remarks

A natural generalization of Equation (6) for cyclic branched covers of branching index greater than 2 can be given as follows. Let $V_n(K)$ be the n-fold cyclic cover of S^3 along K, and M_n its extension to a cyclic covering of B^4 along a spanning surface for K. Such manifolds are generalizations of the 3 and 4-manifolds appearing in the Neumann–Wahl formula (Theorem 4.1), as the knot is not necessarily algebraic here. We say that the generalized Neumann-Wahl formula holds for the pair $V_n(K)$ and M_n if we have

$$\lambda(V_n(K)) = \frac{1}{8} \cdot \sigma(M_n). \tag{7}$$

The framework of equivariant Casson invariants introduced in sections 3 and 4 can be generalized to more general cyclic branched coverings, by starting the construction with a nullhomologous knot K in a rational homology sphere Y^3, instead of a knot in S^3. Forming the 2-fold and 4-fold cyclic covers $V_2(K)$ and $V_4(K)$, supposing these are again rational homology spheres, the analogue of Theorem 3.5 is

$$\lambda^{\mathbb{Z}_2}(V_4(K)) = 4 \cdot \lambda(V_2(K)) - \frac{1}{2} \cdot \sigma(K) - 4 \cdot \lambda(Y^3). \tag{8}$$

This can be used to generalize inductively what was done before, in the following way. Let us first consider a 2^n-fold covering of S^3 along K, $V_{2^n}(K)$. Let M_{2^n} be the extention of this cyclic covering to a cyclic covering of B^4 along a spanning surface for K. We have a sequence of coverings which are all easily seen to be rational homology spheres:

$$V_{2^{n+1}}(K) \xrightarrow{\pi_n} V_{2^n}(K) \xrightarrow{\pi_{n-1}} V_{2^{n-1}}(K) \xrightarrow{\pi_{n-2}} \dots \xrightarrow{\pi_0} S^3$$

If we let $K^0 = K$ and for $k \geq 1$, K^k be the preimage of K under $\pi_{k-1} \circ \pi_{k-2} \circ \dots \circ \pi_0$, then $V_{2^k}(K^0) \cong V_2(K^{k-1})$ and $V_{2^{k+1}}(K^0) \cong V_4(K^{k-1})$. Hence, applying Equation (8), we get

$$\lambda^{\mathbb{Z}_2}(V_{2^{k+1}}(K)) = 4 \cdot \lambda(V_{2^k}(K)) - \frac{1}{2} \cdot \sigma(K^{k-1}) - \frac{1}{2} \cdot \sigma(K^{k-2}) - \lambda^{\mathbb{Z}_2}(V_{2^k}(K)).$$

Hence using (8) n times, we obtain

$$\lambda(V_{2^n}(K)) - \frac{1}{8} \cdot \sum_{k=1}^{n} \sigma(K^{k-1}) = \frac{1}{4} \cdot \sum_{k=1}^{n} \lambda^{\mathbb{Z}_2}(V_{2^{k+1}}(K)).$$

The signatures $\sigma(K^{k-1})$ may be expressed in terms of signatures of 4-dimensional branched coverings as explained in [12] (Definition 4.6 and Remark 4.8):

$$\sigma(K^{k-1}) = \sigma(M_{2^k}) - 2 \cdot \sigma(M_{2^{k-1}}),$$

so that

$$\sigma(M_{2^n}) = \sum_{k=1}^{n} 2^{n-k} \cdot \sigma(K^{k-1}).$$

The author would like to thank the referee for pointing out a mistake at this stage in the first version of this paper. A generalization of Corollary 4.3, providing obstructions to the generalized Neumann–Wahl formula for $V_{2^n}(K)$ and M_{2^n} proved by finite induction then reads as:

Theorem 5.1.

$$\lambda(V_{2^n}(K)) = \frac{1}{8} \cdot \sigma(M_{2^n}) \Longleftrightarrow \sum_{k=1}^{n} \lambda^{\mathbb{Z}_2}(V_{2^{k+1}}(K)) = 1/2 \cdot \sum_{k=1}^{n-1} (2^{n-k} - 1) \cdot \sigma(K^{k-1}).$$

Unlike in Corollary 4.3, there is not one obstruction, but n of them. Obviously each of these is therefore not a complete obstruction. Recall that in the Corollary, the obstruction was a complete one. It is therefore worth mentionning the following more general result:

Theorem 5.2. *Suppose that the pair $V_n(K)$ and M_n satisfy the generalized Neumann–Wahl formula and that $V_{2n}(K)$ is a rational homology sphere. Then*

$$\lambda(V_{2n}(K)) = \frac{1}{8}\sigma(M_{2n}) \Longleftrightarrow \lambda^{\mathbb{Z}_2}(V_{4n}(K)) = 4 \cdot \lambda(V_n(K)).$$

Proof. Let $K' \subset V_n(K)$ correspond to $K \subset S^3$ under the branched covering map. Then $V_2(K') \cong V_{2n}(K)$ and $V_4(K') \cong V_{4n}(K)$. By Equation (8), we have

$$\lambda^{\mathbb{Z}_2}(V_4(K')) = 4 \cdot \lambda(V_2(K')) - \frac{1}{2} \cdot \sigma(K') - 4 \cdot \lambda(V_n(K)).$$

Consequently,

$$\begin{aligned}
\lambda^{\mathbb{Z}_2}(V_{4n}(K)) &= 4 \cdot \lambda(V_{2n}(K)) - \frac{1}{2} \cdot (\sigma(K') + 8\lambda(V_n(K))) \\
&= 4 \cdot \lambda(V_{2n}(K)) - \frac{1}{2} \cdot (\sigma(K') + \sigma(M_n)) \\
&= 4 \cdot \lambda(V_{2n}(K)) - \frac{1}{2} \cdot (\sigma(M_{2n}) - \sigma(M_n)) \\
&= (4 \cdot \lambda(V_{2n}(K)) - 1/2 \cdot \sigma(M_{2n})) - 4 \cdot \lambda(V_n(K)).
\end{aligned}$$

□

It would be useful to have an expression for the equivariant Casson invariants appearing in Theorems 5.1 and 5.2, as was the case in Theorem 4.4. One of the natural approaches to this end would be to try to generalize Mullins' result from knots in S^3 to knots in arbitrary rational homology spheres, developping the appropriate skein theory. But maybe, in a more unifying way, it would be worthwhile to look at the general setting in the framework of Perturbative Chern–Simons Theory, as the invariants discussed in this article most likely have an interpretation in that context.

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