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A NOTE ON TWISTED LINEAR ACTIONS OF SL(n,C) AND SL(n,H)

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1. Introduction

In this paper, we shall study twisted linear actions of $SL(n,\boldsymbol{C})$ on the $(2n+k-1)$-sphere and of $SL(n,\boldsymbol{H})$ on the $(4n+k-1)$-sphere. A twisted linear action was introduced by F.Uchida [1].

There have been uncountably many topologically distinct analytic actions of $SL(n,\boldsymbol{R})$ on the $(n+k-1)$-sphere [2]. He studied the orbit of $L(n)$ that is a subgroup of $SL(n,\boldsymbol{R})$ to obtain the above result. We shall show that Uchida's method is useful to our problem.

In particular, we shall show that there are uncountably many topologically distinct analytic actions and uncountably many C^1-differentiably distinct but topologically equivalent analytic actions of $SL(n,\boldsymbol{C})$ on the $(2n+k-1)$-sphere.

2. Twisted linear actions

Here, we recall the definition of twisted linear actions.

Let $u=(u_i)$ and $v=(v_i)$ be column vectors in $\boldsymbol{R}^n$. As usual, we define their inner product by $u\cdot v=\Sigma_i u_i v_i$ and the length of u by $\|u\|=\sqrt{u\cdot u}$. Let $M=(m_{ij})$ be a square matrix of degree n. We say that M satisfies the condition (T) if the quadratic form

$$x\cdot Mx=\sum_{i,j} m_{ij}x_i x_j$$

is positive definite. It is easy to see that M satisfies (T) if and only if

$$(T')\qquad \frac{d}{dt}\|\exp(tM)x\|>0\qquad \textit{for each}\ x\in\boldsymbol{R}^n_0=\boldsymbol{R}^n-\{0\},t\in\boldsymbol{R}\,.$$

If M satisfies (T′), then

$$\lim_{t\to+\infty}\|\exp(tM)x\|=+\infty\qquad \textit{and}\qquad \lim_{t\to-\infty}\|\exp(tM)x\|=0$$

for each $x \in \boldsymbol{R}_0^n$, and hence there exists a unique real valued function τ on $\boldsymbol{R}_0^n$ such that

$$\|\exp(\tau(x)M)x\| = 1 \qquad for \quad x \in \boldsymbol{R}_0^n.$$

Therefore, we can define an analytic mapping π^M of $\boldsymbol{R}_0^n$ onto the unit $(n-1)$-sphere S^{n-1} by

$$\pi^M(x) = \exp(\tau(x)M)x \qquad for \quad x \in \boldsymbol{R}_0^n,$$

if M satisfies the condition (T).

Let G be a Lie group, $\rho : G \to GL(n, \boldsymbol{R})$ a matricial representation, and M a square matrix of degree n satisfying (T). We call (ρ, M) a TC-pair of degree n, if $\rho(g)M = M\rho(g)$ for each $g \in G$.

For a TC-pair (ρ, M) of degree n, we can define an analytic mapping

$$\xi : G \times S^{n-1} \to S^{n-1} \ by \ \xi(g,x) = \pi^M(\rho(g)x),$$

and we see that ξ is an analytic G-action on S^{n-1}. We call $\xi = \xi^{(\rho,M)}$ a twisted linear action of G on S^{n-1} determined by the TC-pair (ρ, M), and we say that ξ is associated to the matricial representation ρ.

For a given Lie group G, we introduce certain equivalence relations on TC-pairs. Let (ρ, M) and (σ, N) be TC-pairs of degree n. We say that (ρ, M) is algebraically equivalent to (σ, N), if there exist $A \in GL(n, \boldsymbol{R})$ and a positive real number c satisfying

$$(*) \qquad cN = AMA^{-1} \ and \ \sigma(g) = A\rho(g)A^{-1} \ for\ each \ g \in G.$$

We say that (ρ, M) is C^r-equivalent to (σ, N), if there exists a C^r-diffeomorphism f of S^{n-1} onto itself such that the following diagram is commutative :

$$\begin{array}{ccc} G \times S^{n-1} & \xrightarrow{1\times f} & G \times S^{n-1} \\ \downarrow{\scriptstyle \xi^{(\rho,M)}} & & \downarrow{\scriptstyle \xi^{(\sigma,N)}} \\ S^{n-1} & \xrightarrow{f} & S^{n-1}. \end{array}$$

We call f a G-equivariant C^r-diffeomorphism.

Lemma 2.1. *If (ρ, M) is algebraically equivalent to (σ, N), then (ρ, M) is C^ω-equivalent to (σ, N).*

Proof. It has been proved in the paper [1], but we give a proof for completeness. Suppose that there exist $A \in GL(n, \boldsymbol{R})$ and a positive real number c satisfying (*). Define analytic mappings h_A and k_A of S^{n-1} into itself by

$$h_A(x) = \pi^N(Ax) \quad and \quad k_A(y) = \pi^M(A^{-1}y).$$

Then the composites $h_A k_A$ and $k_A h_A$ are the identity mapping on S^{n-1} by the condition $cN = AMA^{-1}$, and hence h_A is a C^ω-diffeomorphism . Furtheremore, the equality

$$h_A(\xi^{(\rho,M)}(g,x)) = \xi^{(\sigma,N)}(g,h_A(x))$$

holds for each $g \in G$ and $x \in S^{n-1}$, by the condition (*). □

Let us define mappings ι_i by the forms

$$\begin{aligned}
\iota_1 \quad &: M_n(\boldsymbol{C}) \to M_{2n}(\boldsymbol{R}) \\
&X + iY \mapsto \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \quad for\ X, Y \in M_n(\boldsymbol{R}) \\
\iota_2 \quad &: M_n(\boldsymbol{H}) \to M_{2n}(\boldsymbol{C}) \\
&X + jY \mapsto \begin{pmatrix} X & -\overline{Y} \\ Y & \overline{X} \end{pmatrix} \quad for\ X, Y \in M_n(\boldsymbol{C}) \\
\iota_3 \quad &= \iota_1 \circ \iota_2 \\
\iota_4 \quad &: \boldsymbol{C}^n \to \boldsymbol{R}^{2n} \\
&x + iy \mapsto \begin{pmatrix} x \\ y \end{pmatrix} \qquad for\ x, y \in \boldsymbol{R}^n \\
\iota_5 \quad &: \boldsymbol{H}^n \to \boldsymbol{C}^{2n} \\
&x + jy \mapsto \begin{pmatrix} x \\ y \end{pmatrix} \qquad for\ x, y \in \boldsymbol{C}^n \\
\iota_6 \quad &= \iota_4 \circ \iota_5
\end{aligned}$$

3. First examples

Here we shall study twisted linear actions of $G = SL(n, \boldsymbol{C})$ on the $(2n+k-1)$-sphere associated to the representation ρ defined by $\rho(A) = \iota_1(A) \oplus I_k$.

Let A and B be square matrices of degrees n and k, respectively. We denote by $A \oplus B$ the matrix

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

of degree $n + k$. We obtain the following lemma.

Lemma 3.1. *Let $n \geq 2$ and $k \geq 2$. Let $\overline{M}$ be a real square matrix of degree $2n + k$. Then*

$$\overline{M}(\iota_1(A) \oplus I_k) = (\iota_1(A) \oplus I_k)\overline{M}$$

for each $A \in SL(n, \boldsymbol{C})$*, if and only if* $\overline{M} = \iota_1(cI_n) \oplus M$ *for some real square matrix* M *of degree* k *and a complex number* c*. Furthermore,* $\overline{M} = \iota_1(cI_n) \oplus M$ *satisfies the condition* (T)*, if and only if* c *has positive real part and* M *satisfies the condition* (T)*.*

Let M be a real square matrix of degree k satisfying the condition (T). Let c be a complex number of which real part is equal to 1. Denote by $\chi^{c,M}$ the twisted linear $SL(n, \boldsymbol{C})$-action on the $(2n+k-1)$-sphere determined by the TC-pair $(\rho, \iota_1(cI_n) \oplus M)$. Then $\chi^{c,M}$ is written in the form

$$\chi^{c,M}(A, \iota_4(u) \oplus v) = \iota_4(e^{c\theta} Au) \oplus e^{\theta M} v$$

for a real number θ which is uniquely determined by the condition

$$\|\iota_4(e^{c\theta} Au)\|^2 + \|e^{\theta M} v\|^2 = 1,$$

where u is a column vector in $\boldsymbol{C}^n$ and v is a column vector in $\boldsymbol{R}^k$ satisfying $\|\iota_4(u)\|^2 + \|v\|^2 = 1$.

Let us define closed subgroups $L(n)$, $N(n)$ and $N_c(n)$ of $SL(n, \boldsymbol{C})$ by the forms

$$L(n) = \left\{ \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \right\}, \qquad N(n) = \left\{ \begin{pmatrix} \lambda & * \\ 0 & * \end{pmatrix} : \lambda \in \boldsymbol{C} - \{0\} \right\},$$

$$N_c(n) = \left\{ \begin{pmatrix} \lambda & * \\ 0 & * \end{pmatrix} : (\log|\lambda|)\mathrm{Im}(c) = \mathrm{Arg}\lambda \right\}.$$

Denote by $F(c, M)$ the fixed point set of $L(n)$ with respect to the twisted linear action $\chi^{c,M}$. Then we obtain the following lemma.

Lemma 3.2. *With respect to the twisted linear action* $\chi^{c,M}$*,*

$$F(c, M) = \{\iota_4(ae_1) \oplus v : a \in \boldsymbol{C}, |a|^2 + \|v\|^2 = 1\}$$

where $e_1 = {}^t(1, 0, \cdots, 0) \in \boldsymbol{C}^n$*. The isotropy group at* $0 \oplus v$ *coincides with* $SL(n, \boldsymbol{C})$*, the one at* $\iota_4(ae_1) \oplus 0$ *coincides with* $N_c(n)$*, and if* $a\|v\| \neq 0$*, then the one at* $\iota_4(ae_1) \oplus v$ *coincides with* $L(n)$*.*

Let $N(L(n))$ be the normalizer of $L(n)$. Then $N(L(n)) = N(n)$ and the factor group $N(L(n))/L(n)$ is naturally isomorphic to the multiplicative group $\boldsymbol{C} - \{0\}$. Let us investigate the induced $N(L(n))/L(n)$ action on $F(c, M)$ via $\chi^{c,M}$. Leaving fixed any point $\iota_4(ae_1) \oplus v$ of $F(c, M)$ satisfying $a\|v\| \neq 0$, we have real valued analytic function $\theta = \theta(\lambda)$, determined by

$$\chi^{c,M}\left(\begin{pmatrix} \lambda & * \\ 0 & * \end{pmatrix}, \iota_4(ae_1) \oplus v \right) = \iota_4(e^{c\theta} \lambda ae_1) \oplus e^{\theta M} v$$

and $|e^{c\theta}\lambda a|^2 + \|e^{\theta M}v\|^2 = 1$. Then $\theta(\lambda) = \theta(|\lambda|)$ and

$$\frac{d}{d|\lambda|}\theta < 0 < \frac{d}{d|\lambda|}(e^{\theta}|\lambda|)$$

for $\lambda \in \boldsymbol{C}$-$\{0\}$. Furthermore, we obtain

$$\lim_{|\lambda|\to\infty} \theta(|\lambda|) = -\infty, \quad \lim_{|\lambda|\to\infty} e^{\theta}|\lambda| = \frac{1}{|a|}, \quad \lim_{|\lambda|\to\infty} \|e^{\theta M}v\| = 0,$$

and

$$\lim_{|\lambda|\to 0+} e^{\theta}|\lambda| = 0, \quad \lim_{|\lambda|\to 0+} e^{\theta M}v = \pi^M(v).$$

Denote by $S^{2n+k-1}(c, M)$ the sphere with the twisted linear $SL(n, \boldsymbol{C})$-action $\chi^{c,M}$.

Theorem 3.3. *Let M, N be any real square matrices of degree k satisfying the condition (T). Let $c, c' \in \boldsymbol{C}$, $\mathrm{Re}(c) = \mathrm{Re}(c') = 1$ and $\mathrm{Im}(c) \neq \mathrm{Im}(c')$. Then there is no $SL(n, \boldsymbol{C})$-equivariant homeomorphism from $S^{2n+k-1}(c, M)$ to $S^{2n+k-1}(c', N)$.*

Proof. Assume that f is equivariant homeomorphism from $S^{2n+k-1}(c, M)$ to $S^{2n+k-1}(c', N)$. By considering the restricted $L(n)$-action , $f(F(c, M)) = f(F(c', N))$.

Furthermore, considering points of $F(c, M)$ and $F(c', N)$ whose isotropy groups coincide with neither $SL(n, \boldsymbol{C})$ nor $L(n)$, we obtain $f(\{\iota_4(ae_1) \oplus 0\}) = \{\iota_4(ae_1) \oplus 0\}$. Then isotropy groups of corresponding points induce a contradiction. □

Theorem 3.4. *Let M,N be any square matrices of degree k satisfying the condition (T). Let c be a complex number whose real part is equal to 1. Then there exists an $SL(n, \boldsymbol{C})$-equivariant homeomorphism f of $S^{2n+k-1}(c, M)$ onto $S^{2n+k-1}(c, N)$.*

Proof. By the above investigation, we can construct uniquely an $N(L(n))/L(n)$-equivariant homeomorphism f_0 of $F(c, M)$ onto $F(c, N)$ satisfying the following conditions

$$f(\iota_4(ae_1) \oplus v) = \iota_4(ae_1) \oplus v \qquad \text{for } |a| = 1 \quad \text{or} \quad \frac{1}{\sqrt{2}},$$

$$f_0(0 \oplus e^{\theta M}v) = 0 \oplus e^{\theta' N} \quad \text{for} \quad \|v\| = \frac{1}{\sqrt{2}}, \quad \|e^{\theta M}v\| = 1 \text{ and } \|e^{\theta' N}v\| = 1,$$

$$f_0(\chi^{c,M}(A, \iota_4(ae_1) \oplus v)) = \chi^{c,N}(A, \iota_4(ae_1) \oplus v) \quad \text{for} \quad |a| = \frac{1}{\sqrt{2}} \text{ and } A \in N(n).$$

Next we consider the following diagram

$$\begin{array}{ccc} SU(n)\times F(c,M) & \xrightarrow{\psi_1} & S^{2n+k-1} \\ \Big\downarrow{\scriptstyle 1\times f_0} & & \Big\downarrow{\scriptstyle f} \\ SU(n)\times F(c,N) & \xrightarrow{\psi_2} & S^{2n+k-1} \end{array},$$

where

$$\psi_1(K,x)=\chi^{c,M}(K,x)=(\iota_1(K)\oplus I_k)x,$$

$$\psi_2(K,x)=\chi^{c,N}(K,x)=(\iota_1(K)\oplus I_k)x.$$

By the construction of f_0 , we see that $\psi_1(K,x) = \psi_1(K',x')$ if and only if $\psi_2(K,f_0(x))=\psi_2(K',f_0(x'))$ and hence we obtain a unique bijection f of S^{2n+k-1} onto itself satisfying

$$f\circ\psi_1=\psi_2\circ(1\times f_0)\ .$$

Then f is a homeomorphism, because ψ_1 and ψ_2 are closed continuous mappings.

Finally, we show that f is $SL(n,\boldsymbol{C})$-equivariant. Let $A\in SL(n,\boldsymbol{C})$, $K\in SU(n)$ and $x\in F(c,M)$. Then there are $B\in SU(n)$, $U\in N(n)$ such that $AK=BU$, and hence

$$\begin{aligned} f(\chi^{c,M}(A,\psi_1(K,x))) &= f(\chi^{c,M}(AK,x))=f(\chi^{c,M}(BU,x)) \\ &= f(\psi_1(B,\chi^{c,M}(U,x)))=\psi_2(B,f_0(\chi^{c,M}(U,x))) \\ &= \psi_2(B,\chi^{c,N}(U,f_0(x)))=\chi^{c,N}(BU,f_0(x)) \\ &= \chi^{c,N}(AK,f_0(x))=\chi^{c,N}(A,\psi_2(K,f_0(x))) \\ &= \chi^{c,N}(A,f(\psi_1(K,x))). \end{aligned}$$

Consequently, we see that f is an $SL(n,\boldsymbol{C})$-equivariant homeomorphism of $S^{2n+k-1}(c,M)$ onto $S^{2n+k-1}(c,N)$. □

Theorem 3.5. *Let M, N be square matrices of degree k satisfying the condition* (T). *Let c be a complex number whose real part is equal to 1. If there exists an* $SL(n,\boldsymbol{C})$*-equivariant* C^1*-diffeomorphism of* $S^{2n+k-1}(c,M)$ *onto* $S^{2n+k-1}(c,N)$*, then*

$$N=PMP^{-1}$$

for some $P\in GL(k,\boldsymbol{R})$.

Proof. By the existence of such an equivariant C^1-diffeomorphism f, we obtain an $N(L(n))/L(n)$-equivariant C^1-diffeomorphism $f_1: F(c,M)\to F(c,N)$. Considering points of whose isotropy groups coincide with neither $SL(n,\boldsymbol{C})$ nor $L(n)$, we obtain $f_1(\{\iota_4(ae_1) \oplus 0\}) = \{\iota_4(ae_1) \oplus 0\}$. Then there is $a \in \boldsymbol{C}$ such that $|a| = 1$ and $f_1(\iota_4(e_1) \oplus 0) = \iota_4(ae_1) \oplus 0$. Let $l_b : F(c, N) \to F(c, N)$ denote the mapping defined by

$$l_b(xe_1 \oplus v) = \chi^{c,N}\left(\begin{pmatrix} b & * \\ 0 & * \end{pmatrix} , \; xe_1 \oplus v\right) = bxe_1 \oplus v \quad for\ |b| = 1 .$$

Then,l_b is an $N(L(n))/L(n)$-equivariant C^ω-diffeomorphism. Now, setting $f_0 = l_{a^{-1}} \circ f_1$, we obtain an $N(L(n))/L(n)$-equivariant C^1-diffeomorphism $f_0 : F(c, M) \to F(c, N)$ satisfying the following condition

$$f_0(\iota_4(e_1) \oplus 0) = \iota_4(e_1) \oplus 0.$$

Then we obtain an isomorphism

$$df_0 : T_{\iota_4(e_1)\oplus 0}F(c, M) \to T_{\iota_4(e_1)\oplus 0}F(c, N)$$

of tangential representation spaces of the group $N_c(n)/L(n)$.

Here we consider the representation space $T_{\iota_4(e_1)\oplus 0}F(c, M)$. Put $F(M)_+ = \{\iota_4(ae_1) \oplus v \in F(c, M) : |a| \neq 0\}$, and define

$$\psi^{c,M} : F(M)_+ \to S^1 \times \boldsymbol{R}^k \quad by \quad \psi^{c,M}(\iota_4(ae_1) \oplus v) = (e^{(-\log|a|)c}a, e^{(-\log|a|)M}v).$$

Then $\psi^{c,M}$ is a C^ω-diffeomorphism satisfying $\psi^{c,M}(\iota_4(e_1)\oplus 0) = (1, 0)$. Let $\lambda^{c,M}$ denote the mapping on $F(c, M)$ defined by

$$\lambda^{c,M}(\iota_4(ae_1) \oplus v) = \chi^{c,M}\left(\begin{pmatrix} \lambda & * \\ 0 & * \end{pmatrix}, \iota_4(ae_1) \oplus v\right) \quad for \quad \begin{pmatrix} \lambda & * \\ 0 & * \end{pmatrix} \in N_c(n).$$

Then, since $(\log|\lambda|)\mathrm{Im}(c) = \mathrm{Arg}\lambda$,

$$\begin{aligned} \psi^{c,M}(\lambda^{c,M}(\iota_4(ae_1) \oplus v)) &= \psi^{c,M}(\iota_4(e^{c\theta}\lambda ae_1) \oplus e^{\theta M}v) \\ &= (e^{(-\log|\lambda a|)c}\lambda a \ , \ e^{(-\log|\lambda a|)M}v) \\ &= (e^{(-\log|a|)c}a \ , \ e^{(-\log|\lambda|)M}e^{(-\log|a|)M}v). \end{aligned}$$

Define $L_M : \boldsymbol{R}^k \to \boldsymbol{R}^k$ by $L_M(w) = e^{(-\log|\lambda|)M}w$ and $h_M : S^1 \times \boldsymbol{R}^k \to S^1 \times \boldsymbol{R}^k$ by $h_M = id \times L_M$. Then we obtain the commutative diagram

$$\begin{array}{ccc} F(M)_+ & \xrightarrow{\psi^{c,M}} & S^1 \times \boldsymbol{R}^k \\ \downarrow \lambda^{c,M} & & \downarrow h_M \\ F(M)_+ & \xrightarrow{\psi^{c,M}} & S^1 \times \boldsymbol{R}^k. \end{array}$$

Next we consider the following commutative diagram

$$
\begin{array}{ccccccc}
T_1(S^1)\bigoplus T_0(\boldsymbol{R}^k) & & & \xrightarrow{\quad dh_M \quad} & & & T_1(S^1)\bigoplus T_0(\boldsymbol{R}^k) \\
& \nwarrow^{d\psi^{c,M}} & & & & \nearrow^{d\psi^{c,M}} & \\
\Big\downarrow & & T_{\iota_4(e_1)\oplus 0}\, F(M)_+ & \xrightarrow{d\lambda^{c,M}} & T_{\iota_4(e_1)\oplus 0}\, F(M)_+ & & \Big\downarrow \\
q & & \Big\downarrow df_0 & & \Big\downarrow df_0 & & q \\
& & T_{\iota_4(e_1)\oplus 0}\, F(N)_+ & \xrightarrow{d\lambda^{c,N}} & T_{\iota_4(e_1)\oplus 0}\, F(N)_+ & & \\
& \swarrow_{d\psi^{c,N}} & & & & \searrow_{d\psi^{c,N}} & \\
T_1(S^1)\bigoplus T_0(\boldsymbol{R}^k) & & & \xrightarrow{\quad dh_N \quad} & & & T_1(S^1)\bigoplus T_0(\boldsymbol{R}^k)\ ,
\end{array}
$$

where q is the linear mapping so that this diagram is commutative. Considering $d\psi^{c,M}$, df_0 and $d\psi^{c,N}$, we see that there exists $P \in GL(k, \boldsymbol{R})$ such that

$$q(x) = \begin{pmatrix} 1 & * \\ 0 & P \end{pmatrix} x \, .$$

Therefore,

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{(-\log|\lambda|)N} \end{pmatrix} \begin{pmatrix} 1 & * \\ 0 & P \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & P \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{(-\log|\lambda|)M} \end{pmatrix} .$$

Hence, we obtain

$$N = PMP^{-1} \, .$$

□

4. Second examples

Here we shall study twisted linear actions of $G = SL(n, \boldsymbol{H})$ on the $(4n+k-1)$-sphere associated to the representation ρ defined by $\rho(A) = \iota_3(A) \oplus I_k$.

Lemma 4.1. *Let $n \geq 2$ and $k \geq 2$. Let $\overline{M}$ be a square real matrix of degree 4n+k. Then*

$$\overline{M}(\iota_3(A) \oplus I_k) = (\iota_3(A) \oplus I_k)\overline{M}$$

for each $A \in SL(n, \boldsymbol{H})$, *if and only if* $\overline{M} = cI_{4n} \oplus M$ *for some real square matrix M of degree k and a real number c. Futhermore,* $\overline{M} = cI_{4n} \oplus M$ *satisfies the condition* (T), *if and only if c is positive and M satisfies the condition* (T).

Let M be a square real matrix of degree k satisfying (T). Denote by χ^M the twisted linear $SL(n, \boldsymbol{H})$ action on the $(4n + k - 1)$-sphere determined by the TC-pair $(\rho, I_{4n} \oplus M)$. Then χ^M is written in the form

$$\chi^M(A, \iota_6(u) \oplus v) = \iota_6(e^\theta Au) \oplus e^{\theta M}v$$

for a real number θ which is uniquely determined by the condition

$$\|\iota_6(e^\theta Au)\|^2 + \|e^{\theta M}v\|^2 = 1,$$

where u is a column vector in $\boldsymbol{H}^n$ and v is a column vector in $\boldsymbol{R}^k$ satisfying $\|\iota_6(u)\|^2 + \|v\|^2 = 1$.

Let us define closed subgroups $L(n)$ and $N(n)$ of $SL(n, \boldsymbol{H})$ by the forms

$$L(n) = \left\{ \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \right\}, \qquad N(n) = \left\{ \begin{pmatrix} \lambda & * \\ 0 & * \end{pmatrix} : \lambda > 0 \right\}.$$

Denote by $F(M)$ the fixed point set of $L(n)$ with respect to the twisted linear action χ^M . Then we obtain the following lemma .

Lemma 4.2. *With respect to the twisted linear action* χ^M,

$$F(M) = \{\iota_6(ae_1) \oplus v : a \in \boldsymbol{H}, |a|^2 + \|v\|^2 = 1\}$$

where $e_1 = {}^t(1, 0, \cdots, 0) \in \boldsymbol{H}^n$. *The isotropy group at* $0 \oplus v$ *coincides with* $SL(n, \boldsymbol{H})$, *the one at* $\iota_6(ae_1) \oplus 0$ *coincides with* $N(n)$, *and if* $a\|v\| \neq 0$, *then the one at* $\iota_6(ae_1) \oplus v$ *coincides with* $L(n)$.

Let $N(L(n))$ be a normalizer of $L(n)$. Then the factor group $N(L(n))/L(n)$ is naturally isomorphic to the multiplicative group $\boldsymbol{H} - \{0\}$. Let us investigate the induced $N(L(n))/L(n)$ action on $F(M)$ via χ^M. Leaving fixed any point $\iota_6(ae_1) \oplus v$ of $F(M)$ satisfying $a\|v\| \neq 0$, we have real valued analytic function $\theta = \theta(\lambda)$, determined by

$$\chi^M\left(\begin{pmatrix} \lambda & * \\ 0 & * \end{pmatrix}, \iota_6(ae_1) \oplus v\right) = \iota_6(e^\theta \lambda ae_1) \oplus e^{\theta M}v$$

and $|e^\theta \lambda a|^2 + \|e^{\theta M}v\|^2 = 1$. Then $\theta(\lambda) = \theta(|\lambda|)$ and

$$\frac{d}{d|\lambda|}\theta < 0 < \frac{d}{d|\lambda|}(e^\theta|\lambda|)$$

for $\lambda \in \boldsymbol{H}$-$\{0\}$. Furthermore, we obtain

$$\lim_{|\lambda|\to\infty} \theta(|\lambda|) = -\infty, \quad \lim_{|\lambda|\to\infty} e^{\theta}|\lambda| = \frac{1}{|a|}, \quad \lim_{|\lambda|\to\infty} \|e^{\theta M} v\| = 0,$$

and

$$\lim_{|\lambda|\to 0+} e^{\theta}|\lambda| = 0.$$

Denote by $S^{4n+k-1}(M)$ the sphere with the twisted linear $SL(n, \boldsymbol{H})$ action χ^M.

Theorem 4.3. *Let M,N be any real square matrices of degree k satisfying the condition(T). Then there exists an $SL(n, \boldsymbol{H})$-equivariant homeomorphism f of $S^{4n+k-1}(M)$ onto $S^{4n+k-1}(N)$.*

Proof. By the above investigation, we can construct uniquely an $N(L(n))/L(n)$-equivariant homeomorphism f_0 of $F(M)$ onto $F(N)$ satisfying the following conditions

$$f_0(\iota_6(ae_1) \oplus v) = \iota_6(ae_1) \oplus v \qquad \text{for } |a| = 1 \quad \text{or} \quad \frac{1}{\sqrt{2}},$$

$$f_0(0 \oplus e^{\theta M} v) = 0 \oplus e^{\theta' N} \quad \text{for} \quad \|v\| = \frac{1}{\sqrt{2}}, \quad \|e^{\theta M} v\| = 1 \ \text{and} \ \|e^{\theta' N} v\| = 1,$$

$$f_0(\chi^M(A, \iota_6(ae_1) \oplus v)) = \chi^N(A, \iota_6(ae_1) \oplus v) \quad \text{for} \quad |a| = \frac{1}{\sqrt{2}} \ \text{and} \ A \in N(L(n)).$$

Next we consider the following diagram

$$\begin{array}{ccc} Sp(n) \times F(M) & \xrightarrow{\psi_1} & S^{4n+k-1} \\ \downarrow 1 \times f_0 & & \downarrow f \\ Sp(n) \times F(N) & \xrightarrow{\psi_2} & S^{4n+k-1} \end{array},$$

where

$$\psi_1(K, x) = \chi^M(K, x) = (\iota_3(K) \oplus I_k)x,$$
$$\psi_2(K, x) = \chi^N(K, x) = (\iota_3(K) \oplus I_k)x.$$

By the construction of f_0, we see that $\psi_1(K, x) = \psi_1(K', x')$ if and only if $\psi_2(K, f_0(x)) = \psi_2(K', f_0(x'))$ and hence we obtain unique bijection f of S^{4n+k-1} onto itself satisfying

$$f \circ \psi_1 = \psi_2 \circ (1 \times f_0).$$

Then f is a homeomorphism, because ψ_1 and ψ_2 are closed continuous mappings.

Finally, we show that f is $SL(n,\boldsymbol{H})$-equivariant. Let $A \in SL(n,\boldsymbol{H})$, $K \in Sp(n)$ and $x \in F(M)$. Then there are $B \in Sp(n)$, $U \in N(n)$ such that $AK = BU$, and hence

$$\begin{aligned} f(\chi^M(A,\psi_1(K,x))) &= f(\chi^M(AK,x)) = f(\chi^M(BU,x)) \\ &= f(\psi_1(B,\chi^M(U,x))) = \psi_2(B,f_0(\chi^M(U,x))) \\ &= \psi_2(B,\chi^N(U,f_0(x))) = \chi^N(BU,f_0(x)) \\ &= \chi^N(AK,f_0(x)) = \chi^N(A,\psi_2(K,f_0(x))) \\ &= \chi^N(A,f(\psi_1(K,x))). \end{aligned}$$

Consequently, we see that f is an $SL(n,\boldsymbol{H})$-equivariant homeomorphism of $S^{4n+k-1}(M)$ onto $S^{4n+k-1}(N)$. □

Theorem 4.4. *Let M, N be square real matrices of degree k satisfying the condition (T). If there exists an $SL(n,\boldsymbol{H})$-equivariant C^1-diffeomorphism of $S^{4n+k-1}(M)$ onto $S^{4n+k-1}(N)$, then*

$$N = PMP^{-1}$$

for some $P \in GL(k,\boldsymbol{R})$.

Proof. By the existence of such an equivariant C^1-diffeomorphism f, we obtain an $N(L(n))/L(n)$-equivariant C^1-diffeomorphism $f_1 : F(M) \to F(N)$. Considering points of whose isotropy groups coincide with neither $SL(n,\boldsymbol{H})$ nor $L(n)$, we obtain $f_1(\{\iota_6(ae_1)\oplus 0\}) = \{\iota_6(ae_1)\oplus 0\}$. Then there is $a \in \boldsymbol{H}$ such that $|a| = 1$ and $f_1(\iota_6(e_1)\oplus 0) = \iota_6(ae_1)\oplus 0$. Let $l_b : F(N) \to F(N)$ denote the mapping defined by

$$l_b(xe_1 \oplus v) = \chi^N\left(\begin{pmatrix} b & * \\ 0 & * \end{pmatrix} , \ xe_1 \oplus v\right) \quad \textit{for } |b| = 1 \ , \ b \in H.$$

Then,l_b is an $N(n)/L(n)$-equivariant C^ω-diffeomorphism. Now, setting $f_0 = l_{a^{-1}} \circ f_1$, we obtain an $N(n)/L(n)$-equivariant C^1-diffeomorphism $f_0 : F(M) \to F(N)$ satisfying the following condition

$$f_0(\iota_6(e_1)\oplus 0) = \iota_6(e_1)\oplus 0.$$

Then we obtain an isomorphism

$$df_0 : T_{\iota_6(e_1)\oplus 0}F(M) \to T_{\iota_6(e_1)\oplus 0}F(N)$$

of tangential representation spaces of the isotropy group $N(n)/L(n)$.

Here we consider the representation space $T_{\iota_6(e_1)\oplus 0}F(M)$. Put $F(M)_+ = \{\iota_6(ae_1) \oplus v \in F(M) : \mathrm{Re}(a) > 0\}$. We identify $\boldsymbol{R}^3$ with $\{{}^t(1,*) \in \boldsymbol{R}^4\}$ and define

$$\psi^M : F(M)_+ \to \boldsymbol{R}^3 \oplus \boldsymbol{R}^k \quad by \quad \psi^M(\iota_6(ae_1)\oplus v) = \iota_6(e^{-\log(\mathrm{Re}(a))}a) \oplus e^{-\log(\mathrm{Re}(a))M}v.$$

Then ψ^M is a C^ω-diffeomorphism satisfying $\psi^M(\iota_6(e_1) \oplus 0) = 0 \oplus 0$. Let $\lambda^M : F(M) \to F(M)$ denote the mapping defined by

$$\lambda^M(\iota_6(ae_1) \oplus v) = \chi^M\left(\begin{pmatrix} \lambda & * \\ 0 & * \end{pmatrix}, \iota_6(ae_1) \oplus v\right) \quad for \quad \begin{pmatrix} \lambda & * \\ 0 & * \end{pmatrix} \in N(n).$$

Then,

$$\begin{aligned} \psi^M(\lambda^M(\iota_6(ae_1) \oplus v)) &= \psi^M(\iota_6(e^\theta \lambda a e_1) \oplus e^{\theta M} v) \\ &= \iota_6(e^{(-\log \mathrm{Re}(\lambda a))}\lambda a) \oplus e^{(-\log \mathrm{Re}(\lambda a))M} v \\ &= \iota_6(e^{-\log \mathrm{Re}(a)}a) \oplus e^{(-\log \lambda)M} e^{(-\log \mathrm{Re}(a))M} v). \end{aligned}$$

Define $L_M : \boldsymbol{R}^k \to \boldsymbol{R}^k$ by $L_M(w) = e^{(-\log \lambda)M} w$ and $h_M : \boldsymbol{R}^3 \oplus \boldsymbol{R}^k \to \boldsymbol{R}^3 \oplus \boldsymbol{R}^k$ by $h_M = id \oplus L_M$. Then we obtain the commutative diagram

$$\begin{array}{ccc} F(M)_+ & \xrightarrow{\psi^M} & \boldsymbol{R}^3 \oplus \boldsymbol{R}^k \\ \Big\downarrow{\scriptstyle \lambda^M} & & \Big\downarrow{\scriptstyle h_M} \\ F(M)_+ & \xrightarrow{\psi^M} & \boldsymbol{R}^3 \oplus \boldsymbol{R}^k. \end{array}$$

Next we consider the following commutative diagram

$$\begin{array}{ccccc} T_0(\boldsymbol{R}^{3+k}) & & \xrightarrow{\quad dh_M \quad} & & T_0(\boldsymbol{R}^{3+k}) \\ & \nwarrow{\scriptstyle d\psi^M} & & \nearrow{\scriptstyle d\psi^M} & \\ & T_{\iota_6(e_1)\oplus 0}\, F(M)_+ & \xrightarrow{d\lambda^M} & T_{\iota_6(e_1)\oplus 0}\, F(M)_+ & \\ \Big\downarrow q & \Big\downarrow{\scriptstyle df_0} & & \Big\downarrow{\scriptstyle df_0} & \Big\downarrow q \\ & T_{\iota_6(e_1)\oplus 0}\, F(N)_+ & \xrightarrow{d\lambda^N} & T_{\iota_6(e_1)\oplus 0}\, F(N)_+ & \\ & \swarrow{\scriptstyle d\psi^N} & & \searrow{\scriptstyle d\psi^N} & \\ T_0(\boldsymbol{R}^{3+k}) & & \xrightarrow{\quad dh_N \quad} & & T_0(\boldsymbol{R}^{3+k}) \end{array} ,$$

where q is a linear mapping so that this diagram is commutative. Considering $d\psi^M$, df_0 and $d\psi^N$, there exists $P \in GL(k, \boldsymbol{R})$ such that

$$q(x) = \begin{pmatrix} I_3 & * \\ 0 & P \end{pmatrix} x \ .$$

Therefore,

$$\begin{pmatrix} I_3 & 0 \\ 0 & e^{(-log\lambda)N} \end{pmatrix} \begin{pmatrix} I_3 & * \\ 0 & P \end{pmatrix} = \begin{pmatrix} I_3 & * \\ 0 & P \end{pmatrix} \begin{pmatrix} I_3 & 0 \\ 0 & e^{(-log\lambda)M} \end{pmatrix} .$$

Hence,we obtain

$$N = PMP^{-1} \ .$$

□

References

[1] F.Uchida: *On a method to construct analytic actions of non-compact Lie groups on a sphere* , Tohoku Math. J. **39** (1987), 61-69.
[2] F.Uchida: *Certain aspects of twisted linear actions* , Osaka. J. Math. **25** (1988), 343-352.

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