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TWIN TRIANGULAR DERIVATIONS

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1. Preliminaries

Let G_a denote the additive group of complex numbers, X a variety over $\mathbf{C}$ and σ $:G_a \times X \to X$ a regular (sometimes referred to as rational, polynomial, or algebraic) action of G_a on X. The action is said to admit an equivariant trivialization if there is a variety Y and a G_a equivariant isomorphism $X \cong Y \times \mathbf{C}$, with the group acting trivially on Y and by addition on the second factor. In that case, the action is conjugate to a global translation and Y is a geometric quotient.

If X is quasiaffine, then σ induces a G_a action on the ring $\mathbf{C}[X]$ of globally defined regular functions on X. This action can be "differentiated" to obtain a locally nilpotent derivation of $\mathbf{C}[X]$. Conversely a locally nilpotent derivation of an affine C algebra can be exponentiated to a G_a action on $\mathbf{C}[X]$.

It has been shown by several people [12, 11, 2, 6] that every fixed point free triangular G_a action on complex affine three space is conjugate to a translation, indeed there is an gquivariant isomorphism $\mathbf{C}^3 \cong \mathbf{C}^2 \times \mathbf{C}$ as above. For affine spaces of higher dimension, the situation is not as nice. Winkelmann [13] and the authors [3] gave examples of fixed point free triangular actions on $\mathbf{C}^4$ which are not proper and the authors gave an example of a proper action on $\mathbf{C}^5$ which isn't even locally trivial [4]. In each of those cases, the ring of G_a invariants happened to be an affine $\mathbf{C}$ algebra, hence the coordinate ring for an affine variety Y. The aforementioned results could be obtained by observing bad behavior of fibers of the morphism $\pi{:}\mathbf{C}^n \to Y$ over singular points of Y. The goal of this work is to present evidence that singularities of the variety associated to the ring of invariants may be the critical factor determining local triviality of fixed point free G_a action.

The main technical tool involves the concept of geometric irreducibility in codimension one (GICO) of a morphism of algebraic schemes. This concept was introduced by Miyanishi and was instrumental in proving some of his deep results on algebraic characterizations of affine three space [9, 10]. Since our concern is with morphisms of complex affine varieties, GICO can be expressed as follows:

DEFINITION 1. Let $\phi : X \to Y$ be a morphism of affine varieties with coordinate rings $\mathbf{C}[X]$ and $\mathbf{C}[Y]$ respectively. Then ϕ is GICO over Y provided that for any height one prime ideal p of $\mathbf{C}[Y]$ and prime ideal P of $\mathbf{C}[X]$ minimal over p$\mathbf{C}[X]$ defining a codimension one subvariety T of X, the field $\mathbf{C}(\overline{\phi(T)})$ is algebraically closed in $\mathbf{C}(T)$.

In the above definition $\overline{\phi(T)}$ denotes the Zariski closure of the image of $\phi(T)$.

This concept will be applied in the context of a G_a actions on $X = \mathbf{C}^4$ with Y the affine variety with coordinate ring $\mathbf{C}[Y] = \mathbf{C}[X]^{G_a}$, the ring of G_a invariants. In the cases we consider, the ring of invariants will turn out to be finitely generated. In general however, rings of G_a invariants on a factorial affine variety X are always factorially closed subrings of $\mathbf{C}[X]$, so that we need be concerned only with the relation between the quotient field of $\mathbf{C}[X]^{G_a}/(p)$ and the quotient field of $\mathbf{C}[X]/p\mathbf{C}[X]$ for principal prime ideals (p) of $\mathbf{C}[X]^{G_a}$. Let D denote the locally nilpotent derivation of $\mathbf{C}[X]$ generating the action and $h \in im(D) \cap \mathbf{C}[X]^{G_a}$. if $h \in (p)$ it is immediate that GICO will not be violated at p.Thus there are only finitely many prime ideals of $\mathbf{C}[X]^{G_a}$ that could cause problems. We will call a G_a action GICO if the morphism $\pi : X \to Y$ is GICO .

A derivation D of $\mathbf{C}[x, y, z, \omega]$ will be called twin triangular if:

$$D(\omega) = 0, D(z) \in \mathbf{C}[\omega], D(y) \in \mathbf{C}[z, \omega], D(x) \in \mathbf{C}[z, \omega]$$

The examples of badly behaved G_a actions on $\mathbf{C}^4$ mentioned previously are all generated by simple cases of twin triangular derivations.

If the G_a action induced by a twin triangular derivation D is fixed point free and $D(z)$ has no multiple roots, then the action will be shown to be GICO (provided of course the ring of invariants is finitely generated). It then follows from a result of Miyanishi that the only obstructions to local triviality of the action are $\mathbf{C}[Y]$ not finitely generated or singularities in Y when $\mathbf{C}[Y]$ is finitely generated. In two special situations, $D(z) = \omega$, or $D(x), D(y) \in \mathbf{C}[z], \mathbf{C}[Y]$ is shown to be generated by four elements, so that Y is isomorphic to a hypersurface in $\mathbf{C}^4$.

In these cases, an explicit polynomial defining the hypersurface is given and an elementary method is presented to distinguish the following two possibilities:

1. If Y is singular, then the topological orbit space is not Hausdorff, and hence the action is not proper (and not locally trivial).
2. If Y is smooth, then $Y \cong \mathbf{C}^3$ and the action is conjugate to a global translation.

2. Lemmas about plane curves

Some possibly well known facts about plane curves parametrized by polynomials are collected here for later use.

Lemma 2.1. *Let $f, g \in \mathbf{C}[z]$ have no common zero and set*

$$F = \int_0^z f(t)dt, G = \int_0^z g(t)dt.$$

Then $\mathbf{C}(F,G) = \mathbf{C}(z)$.

Proof. Since the subfield of $\mathbf{C}(z)$ generated by F and G contains nonconstant polynomials in z, [8, Proposition, p.50] forces $\mathbf{C}(F,G) = \mathbf{C}(h)$ for some polynomial $h \in \mathbf{C}[z]$. Moreover,$F, G \in \mathbf{C}[h]$. Writing $F = P_1(h(z)), G = P_2(h(z))$, the chain rule shows that any root of h' is a common root of f and g. Thus h' is a constant, so that $\mathbf{C}(F,G) = \mathbf{C}(h) = \mathbf{C}(z)$. □

Lemma 2.2. *Let* $F, G \in \mathbf{C}[z]$ *define a morphism* (F,G) *from* $\mathbf{C}^1$ *to the plane curve* X. *If* (F,G) *is birational, then it is surjective.*

Proof. Let X be the closure of image of (F,G), and $\overline{X}$ its normalization. We obtain a morphism $h : \mathbf{C}^1 \to \overline{X}$ factoring (F,G) which by hypothesis is birational, hence an open immersion. But $\overline{X}$, being smooth and rational, is isomorhpic to $\mathbf{C}^1$ with finitely many points deleted, so that h is an isomorphism. Since h and the natural map $\overline{X} \to X$ are surjective, so is (F,G). □

The conclusion of Lemma 2.2 holds under the hypothesis of Lemma 2.1.

Lemma 2.3. *Let* $(f,g), (F,G)$, *and* X *be as above. If* x *is a singular point of* X, *there are distinct* $t_1, t_2 \in \mathbf{C}^1$ *so that* $(F,G)(t_i) = x$.

Proof. By the implicit function theorem, for each $t \in \mathbf{C}^1$, there is a neighborhood B_t so that $(F,G)|_{B_t}$ is a diffeomorphism onto its image. As a consequence, if z is any point in X for which $(F,G)^{-1}z$ is single valued, X is nonsingular at z. □

3. Twin triangular actions are GICO

Theorem 3.1. *Let* D *be a locally nilpotent derivation of* $\mathbf{C}[x,y,z,w]$ *defined by*

$$D(\omega) = 0, D(z) = r(\omega), D(y) = p(z,\omega), D(x) = q(z,\omega).$$

Assume that the kernel of D *is finitely generated and* r, p, q *have no common zeros in* $\mathbf{C}^4$ *(i.e. that the associated* G_a *action is fixed point free and the ring of* G_a *invariants is finitely generated). If* $r(\omega)$ *has no multiple roots then the action is GICO .*

Proof. Denote by C the polynomial ring $\mathbf{C}[x,y,z,w]$ and by C_0 the ring of G_a invariants in C. Let p be a height one prime ideal of C_0. If p does not contain $\omega - c$ for any root c of $r(\omega)$, let $S = C_0 - p$, and note that $S^{-1}C$ is isomorphic to a one variable polynomial ring over $S^{-1}C_0$. Since taking residues mod (p) preserves the polynomiality of the extension, we may assume that $p = (\omega - c)$ and, for simplicity, that $p = (\omega)$. It remains to show that the quotient field of $C_0/(\omega)$ is algebraically closed in $C/(\omega)$.

Since ω is an invariant, D induces a locally nilpotent derivation on $A = C/(\omega)$ with ring of constants A_0 isomorphic to a polynomial ring in two variables generated by z and $xp(z,0) - yq(z,0)$. Note that the quotient field of A_0 is algebraically closed in $\mathbf{C}(x,y,z)$, and that $C_0/(\omega)$ is isomorphic to a subring of A_0. We show that this ring extension is birational.

A calculation shows that C_0 contains ω, $c_2 = r(\omega)y - \int_0^z p(t,\omega)dt$, and $c_3 = r(\omega)x - \int_0^z q(t,\omega)dt$. By hypothesis and the assumption that $r(0) = 0$, we have that $p(z,0)$ and $q(z,0)$ have no common zeros. By Lemma 2.1, $\int_0^z p(t,\omega)dt|_{\omega=0}$ and $\int_0^z q(t,\omega)dt|_{\omega=0}$ generate the field $\mathbf{C}(z)$. But these are just the negatives of the residues mod(ω) of c_2 and c_3.

The algorithm of [7] produces another invariant in C_0 as follows. Let U, V be indeterminants and $R(U,V)$ a polynomial relation of minimal total degree satisfied by the residues mod (ω) of c_2, c_3. Another invariant is obtained by dividing $R(c_2,c_3)$ by the highest possible power of ω. We claim that ω is that highest power. Consider the coefficient of y. A typical term $\lambda U^n V^m$ of $R(U,V)$ evaluated at c_2 and c_3 yields

$$\lambda n \left[-\int_0^z p(t,\omega)dt\right]^{n-1} \left[-\int_0^z q(t,\omega)dt\right]^m r(\omega)y$$

as its only term involving only the first power of y. It follows from this that the coefficient of y in $R(c_2,c_3)$ is

$$\left.\frac{\partial R}{\partial U}\right|_{U=-\int_0^z p(t,\omega)dt,\, V=-\int_0^z q(t,\omega)dt} r(\omega).$$

The first factor can be neither 0 nor a multiple of ω since either of these cases would show that $\frac{\partial R}{\partial U}$ is a relation of lower total degree. Since ω is a simple root of $r(\omega)$ by hypothesis, ω is the highest power dividing the coefficient of y. Similarly ω is the highest power dividing the coefficient of x.

Any term which has total degree greater than one in x and y will be divisible by $r^2(\omega)$. Thus, dividing by ω and setting $\omega = 0$ in the result, yields an expression of the form $h(z)y + j(z)x + k(z)$ which corresponds to an element in A_0. Note that if

$$\operatorname{Det} \begin{vmatrix} h(z) & j(z) \\ -q(z,0) & p(z,0) \end{vmatrix}$$

is unequal to 0, then both x and y would lie in the quotient field of A_0, which is obviously false. Thus $\mathbf{C}(z, h(z)y + j(z)x) = \mathbf{C}(z, p(z,0)x - q(z,0)y)$. $\square$

4. Special twin trianglar actions

This section is concerned with the following two special types of twin triangular actions generated by derivations

- **A** :$D(\omega) = 0, D(z) = \omega, D(y) = p(z, \omega), D(x) = q(z, \omega)$ and
- **B** :$D(\omega) = 0, D(z) = r(\omega), D(y) = p(z), D(x) = q(z)$

This class of actions generalizes all of the badly behaved triangular G_a actions mentioned in the introduction. In fact all of those are special cases of type **B**

Proposition 4.1. *If D is of type* **A** *or* **B** *and the associated G_a action is fixed point free, then the ring of invariants C_0 is generated by four polynomials.*

Proof. Following the algorithm of [7], we show that the four invariants $c_1 = \omega, c_2, c_3$, and $c_4 = \frac{R(c_2,c_3)}{c_1}$, described in Theorem 3.1, generate the ring of inbariants. In case **A** it is clear that $C_1 \equiv \mathbf{C}[c_1, c_2, c_3, c_4] \subset C_0 \subset \mathbf{C}[c_1, c_2, c_3, c_4, \frac{1}{c_1}]$. To construct new invariants one looks for the ideal I_1 of all polynomials $P(U, V, Z, W)$ such that $P(c_1, c_2, c_3, c_4) \in \omega\mathbf{C}[c_1, c_2, c_3, c_4]$,(i.e. for relations among the c_i modulo ω. For each generator Q_i, $1 \leq i \leq m$, of I_1, we have $Q(c_1, c_2, c_3, c_4) = f_i\omega, f_i \in \mathbf{C}[x, y, z, w]$, and the so determined f_i are new invariants. The algorithm continues with $C_2 \equiv C_1[\{f_i | 1 \leq i \leq m\}]$ and terminates when no new relatinos mod ω are obtained.

Writing $P(U, V, Z, W)$ in the initial step as a polynomial in U, it is clear that new invariants can be obtained only from the constant term. But writing the constant term as a polynomial in W, we see that each coefficient must be a multiple of $R(V, Z)$, since $c_4|_{\omega=0}$ is transcendental over $\mathbf{C}(c_2|_{\omega=0}, c_3|_{\omega=0}$. Thus no new invariants are obtained after the first step.

Case **B** follows from case **A** by considering the restriction of the derivation D to $\mathbf{C}[x, y, z, r(\omega)]$. □

Denote $\int_0^s D(y)dz|_{\omega=0}$ by $P(s)$ and $\int_0^s D(x)dz|_{\omega=0}$ by $Q(s)$.

Corollary 4.2. *Given a fixed point free G_a action generated by a derivation of type* **A** *or* **B** *, denote by $R(V, Z)$ a relation of minimal total degree satisfied by $P(z)$ and $Q(z)$. Then the ring of invariants c_0 is isomorphic to $\mathbf{C}[x_1, x_2, x_3, x_4]/(R(x_2, x_3) - r(x_1)x_4)$, with $r(x_1) = x_1$ in case* **A**.

Proof. The four invaiants $c_i, = 1 \leq i \leq 4$ satisfy the relation. □

For the twin triangular actions of type **A** and **B**, the affine variety with coordinate ring C_0 is thus isomorphic to a hypersurface Y in $\mathbf{C}^4$.

Corollary 4.3. *With notations as in the previous corollary, Y is singular at and only at points (0,b,c,0) where (b,c) are singularities of the plane curve defined by R.*

Proof. This follows from examining the differential of $R(x_2, x_3) - r(x_1)x_4$ and the hypothesis that $r(\omega)$ has only simple roots. □

Theorem 4.4. *Let $\sigma: G_a \times \mathbf{C}^4 \to \mathbf{C}^4$ be a twin triangular action of type* **A** *or* **B**. *Then either*

1.*The action is conjugate to a translation with quotient isomorphic to* $\mathbf{C}^3$ *or*

2.*The action is not proper and therefore not even locally trivial.*

Proof. If the plane curve X defined by R is smooth, then $X \cong \mathbf{C}^1$ by Lemma 2.2. By the Abhyankar-Moh-Suzuki theorem [1] $R(x_2, x_3)$ is a variable of $\mathbf{C}[x_2, x_3]$, and therefore by [10, theorem 2], C_0 is isomorphic to a polynomial ring in three variables. Since, by Theorem 3.1, the action is GICO , it is locally trivial [9, Theorem 2]. But a locally trivial G_a action $\mathbf{C}^n$ with ring of invariants isomorphic to a polynomial ring in $n-1$ variables is equivariantly trivial [5, Theorem 3].

Assume now that X is singular at (a, b) and that $\omega = 0$ is a root of $r(\omega)$. According to Lemma 2.3 there are complex numbers $n_1 \neq n_2$ with $(a, b) = (P(n_i), Q(n_i))$. The G_a orbits of the two points $(d, c, n_i, 0)$ are $(d + tq(n_i, 0), c + tp(n_i, 0), n_i, 0)$ in case **A** and $(d + tq(n_i), c + tp(n_i), n_i, 0)$ in case **B** . Therefore they are disjoint in both cases. But for $\epsilon \neq 0$ the orbit of $(0, 0, n_i, \epsilon)$ is,in case **B** ,

$$\left(tq(n_1) + \frac{t^2}{2}q'(n_1)r(\epsilon) + ..., tp(n_i) + \frac{t^2}{2}p'(n_1)r(\epsilon) + ..., n_1 + tr(\epsilon), \epsilon\right).$$

Recall that each application of the derivation to z contributes a factor of $r(\epsilon)$.

Set $t = \frac{n_2 - n_1}{r(\epsilon)}$ to get the followig point in the orbit:

$$\left(\frac{1}{r(\epsilon)}(n_2 - n_1)q(n_1) + \frac{1}{r(\epsilon)}\frac{(n_2 - n_1)^2}{2}q'(n_1) + ...,\right.$$
$$\left.\frac{1}{r(\epsilon)}(n_2 - n_1)p(n_1) + \frac{1}{r(\epsilon)}\frac{(n_2 - n_1)^2}{2}p'(n_1) + ..., n_2, \epsilon\right).$$

Recall that $q = Q'$ and $p = P'$ and observe that the first two coordinates in the orbit above are, respectively, the Taylor expansions of $\frac{1}{r(\epsilon)}[Q(n_2) - Q(n_1)]$ and $\frac{1}{r(\epsilon)}[P(n_2) -$

$P(n_1)]$ centered at $z = n_1$. Thus $(0, 0, n_1, \epsilon)$ and $(0, 0, n_2, \epsilon)$ lie in the same orbit. In particular, the topological orbit space is not Hausdorff in the natural topology on $\mathbf{C}^4$, and the action is not proper.

The argument for case $\mathbf{A}$ is the same, expect that we must consider the Taylor expansions for $Q(n_2, 0) - Q(n_1, 0)$ and $P(n_2, 0) - P(n_1, 0)$ centered at $z = n_1, \omega = \epsilon$. $\square$

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