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THE CARTAN MATRIX OF A CERTAIN CLASS OF FINITE SOLVABLE GROUPS

Dedicated to Professor Yukio Tsushima for his 60th birthday

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1. Introduction

Let G be a finite group, F an algebraically closed field of characteristic $p > 0$, B a block of the group algebra FG and C_B the Cartan matrix of B. In [14] we conjectured that if G is p-solvable, then $k(B) \leq \rho(B)$, where $k(B)$ is the number of ordinary irreducible characters in B and $\rho(B)$ is the Perron-Frobenius eigenvalue (i.e. the largest eigenvalue) of C_B. This conjecture is stronger than the Brauer's $k(B)$ conjecture i.e. $k(B) \leq |D|$, where D is a defect group of B, when G is p-solvable. We obtained Theorem A in [14] (also see the later page) that is a relation between $k(B)$ and the Cartan integers of B and in several cases we verified $k(B) \leq \rho(B)$ by using it. Theorem A seems to suggest that if there is a possibility that this conjecture fails, it might be when diagonal entries of C_B are extremely larger than the other entries. In particular if C_B has many zero entries, it could be the case as the group SL(2, p)(see Example in [14]), because $\rho(B)$ must be a small value by Lemma 3.1(2) in [5]. So we are interested in the Cartan matrix of p-solvable groups with many zero entries. When G is p-closed, actually we have the following examples. Let E_{p^r} be an elementary abelian p-group of order p^r. Let $p = 3$ and $G = D_8 \ltimes E_9, G = S_{16} \ltimes E_9$, and p=2 and $G = Fr_{21} \ltimes E_8$, where D_8, S_{16} is a dihedral, semi dihedral group of order 8, 16, respectively, and Fr_{21} is a Frobenius group of order 21. The Cartan matrix of these groups has zero entries.

In this paper by making use of Ninomiya's result [10] we give the Cartan matrix of a certain class of solvable groups having many zero entries which are p-closed or of p-length 2, and in these groups the above groups are contained as special cases. Then we show that the conjecture $k(B) \leq \rho(B)$ still holds in these groups.

Let $GF(p^n)$ be the finite field with p^n elements, $A(p^n)$ the additive group of $GF(p^n)$ which is isomorphic to an elementary abelian p-group of order p^n, and $M(p^n)$ the multiplicative group of $GF(p^n)$ which is isomorphic to a cyclic group of order $p^n - 1$. Then $M(p^n)$ acts on $A(p^n)$ by ordinary multiplication $a \cdot x = ax$ for $a \in M(p^n), x \in A(p^n)$. Let $X(p^n)$ be the affine group of $GF(p^n)$ i.e. the semi direct product $M(p^n) \ltimes A(p^n)$ (cf. p.32 in [2]). Then $X(p^n)$ is a complete Frobenius group whose Frobenius kernel is a Sylow p-subgroup, and it is known that the Cartan matrix C of $FX(p^n)$ is of the form $C = \begin{pmatrix} 2 & 1 & \dots & 1 \\ 1 & 2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & 2 \end{pmatrix}$, that is a typical example of the group of multiplicity one (see Theorem 4.2 in [7], also see [13] and [12]).

Let $< \sigma >$ be the Galois group of $GF(p^n)$ over $GF(p)$ of order n, then $< \sigma >$ naturally acts on $X(p^n)$ by $\sigma(ax) = \sigma(a)\sigma(x)$ for $a \in M(p^n), x \in A(p^n)$. So we denote by $G(p^n)$ the semi direct product $< \sigma > \ltimes X(p^n)$. The group $G(p^n)$ is isomorphic to the group of affine semi-linear mapping over $GF(p^n)$ (see [3], II.1.18d, p.151)

We consider the case $n = pq$, where q is a prime number different from p. Let us set $G = G(p^{pq})$ and denote the subgroup of G isomorphic to $X(p^{pq})$ by H. Since $O_{p'}(G)$ and $O_{p'}(H)$ are trivial, G and H have only the principal block by a theorem of Fong ([1, Chap.X, Theorem 1.5]).

2. The Cartan matrix of K

Let $< \sigma >$ be the Galois group of $GF(p^{pq})$ over $GF(p)$ of order pq, and $\tau = \sigma^p$ of order q. Let us denote by $K =< \tau > \ltimes H$ a subgroup of G that is a normal subgroup of G of index p containing H. Let ζ be a generator of $M(p^{pq})$ of order $p^{pq}-1$. Then $\sigma(\zeta) = \zeta^p$, as σ is the Frobenius map of $GF(p^{pq})$ over $GF(p)$. Therefore $\tau(\zeta) = \sigma^p(\zeta) = \zeta^{p^p}$.

Let us set the set of irreducible Brauer characters of H by $\mathrm{IBr}(H) = \left\{\tilde{\phi}_i \mid 0 \leq i \leq p^{pq} - 2\right\}$, where $\tilde{\phi}_i(\zeta) = \epsilon^i$ for a primitive $p^{pq} - 1$ th root ϵ of 1 in the complex number field.

We first calculate irreducible Brauer characters of H fixed by τ.

$$\begin{aligned} \tilde{\phi}_i^\tau = \tilde{\phi}_i &\iff \tilde{\phi}_i^\tau(\zeta) = \tilde{\phi}_i(\zeta) \qquad \text{for } 0 \leq i \leq p^{pq} - 2 \\ &\iff \epsilon^{p^p i} = \epsilon^i \\ &\iff (p^p - 1)i \equiv 0 \pmod{p^{pq} - 1} \\ &\iff i = 0, t = \frac{p^{pq} - 1}{p^p - 1}, 2t, \dots, (p^p - 2)t. \end{aligned}$$

So there are $p^p - 1$ irreducible Brauer characters of H fixed by τ. Therefore remaining $p^{pq} - 1 - (p^p - 1) = p^{pq} - p^p$ characters are not τ-fixed. We can set $p^{pq} - p^p = rq$ for some positive integer r by a Fermat's theorem. Then we reset
$\mathrm{IBr}(H) = \{\tilde{\varphi}_1, \dots, \tilde{\varphi}_{p^p-1}, \tilde{\varphi}_{11}, \dots, \tilde{\varphi}_{1q}, \dots, \tilde{\varphi}_{r1}, \dots, \tilde{\varphi}_{rq}\}$, where $\tilde{\varphi}_i$ is τ-fixed for $1 \leq i \leq p^p - 1$, and $\tilde{\varphi}_{ij} = \tilde{\varphi}_{i1}^{\tau^{j-1}}$ for $1 \leq i \leq r,\ 1 \leq j \leq q$.

Then we have $\mathrm{IBr}(K) = \{\varphi_{11}, \ldots, \varphi_{1q}, \ldots, \varphi_{p^p-1,1}, \ldots, \varphi_{p^p-1,q}, \psi_1, \ldots, \psi_r\}$ by Clifford's theorem, where the restriction $\varphi_{ij|H} = \tilde{\varphi}_i$ to H for any j and the induced character $\tilde{\varphi}_{ij}^K = \psi_i$ to K for any j (see e.g.[4, Chap. 6, (6.19)Corollary]).

As is stated in section one, the Cartan matrix $C(H)$ of FH is $I_{p^p-1+rq} + J_{p^p-1+rq}$, where I_s is the unit matrix of degree s and J_s is the matrix of degree s all of whose entries are 1. The first $p^p - 1$ columns are indexed by $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{p^p-1}$, and the next rq columns are indexed by $\tilde{\varphi}_{11}, \ldots, \tilde{\varphi}_{1q}, \ldots, \tilde{\varphi}_{r1}, \ldots, \tilde{\varphi}_{rq}$, where $r = (p^{pq} - p^p)/q$.

Now since $O_{p'}(K)$ is trivial, K has only the principal block, and the inertial group of the principal block FH in K is K. We denote the Cartan invariant of K, H by e.g. $c(\psi_i, \psi_j)$, $\tilde{c}(\tilde{\varphi}_i, \tilde{\varphi}_j)$, respectively.

Lemma 1(Ninomiya, Proposition 15 in [10]). *Under the above notation, there are the following relation between the Cartan integers of K and those of H.*

(i) $c(\psi_i, \psi_j) = \sum_{k=1}^{q} \tilde{c}(\tilde{\varphi}_{i1}, \tilde{\varphi}_{jk}) = \cdots = \sum_{k=1}^{q} \tilde{c}(\tilde{\varphi}_{iq}, \tilde{\varphi}_{jk})$ for $1 \le i, j \le r$,

(ii) $c(\varphi_{i1}, \psi_j) = \cdots = c(\varphi_{iq}, \psi_j) = \tilde{c}(\tilde{\varphi}_i, \tilde{\varphi}_{j1}) = \cdots = \tilde{c}(\tilde{\varphi}_i, \tilde{\varphi}_{jq})$ for $1 \le i \le p^p - 1$, $1 \le j \le r$,

(iii) $\sum_{k=1}^{q} c(\varphi_{i1}, \varphi_{jk}) = \cdots = \sum_{k=1}^{q} c(\varphi_{iq}, \varphi_{jk}) = \tilde{c}(\tilde{\varphi}_i, \tilde{\varphi}_j)$ for $1 \le i, j \le p^p - 1$.

Lemma 2. *The Cartan matrix $C(K)$ of FK is the following:*

$$\begin{array}{c|cccc|c}
 & \varphi_1 & \varphi_2 & \cdots & \varphi_{p^p-1} & \psi \\ \hline
\varphi_1' & 2I_q & I_q & \cdots & I_q & \\
\varphi_2' & I_q & 2I_q & \ddots & \vdots & J_{(p^p-1)\times r} \\
\vdots & \vdots & \ddots & \ddots & I_q & \\
\varphi_{p^p-1}' & I_q & \cdots & I_q & 2I_q & \\ \hline
\psi' & & & J_{r\times(p^p-1)} & & I_r + qJ_r
\end{array}$$

where $r = \frac{p^{pq}-p^p}{q}$, φ_i means a row $\varphi_{i1}, \ldots, \varphi_{iq}$, ψ means a row $\psi_1, \ldots, \psi_r$, and φ_i', ψ' is its transpose, respectively. Furthermore I_s is the unit matrix of degree s, and J_s, $J_{s\times t}$ is the $s \times s, s \times t$ matrix all of whose entries are 1, respectively.

Proof. Since H is a normal subgroup of K with index q, we have

(2.1) $\sum_{k=1}^{q} c(\varphi_{ij}, \varphi_{ik}) = 2$ for $1 \le i \le p^p - 1, 1 \le j \le q$, (2.2) $\sum_{k=1}^{q} c(\varphi_{ij}, \varphi_{lk}) = 1 \quad \text{for} \quad 1 \leq i \neq l \leq p^p - 1, 1 \leq j \leq q$

by Lemma 1 (iii). (2.1) shows that $c(\varphi_{ij}, \varphi_{ij}) = 2$ and other entries $c(\varphi_{ij}, \varphi_{ik}) = 0$ for $1 \leq j \neq k \leq q$. (2.2) shows that we may take $c(\varphi_{ij}, \varphi_{lj}) = 1$ and other entries $c(\varphi_{ij}, \varphi_{lk}) = 0$ for $j \neq k$. We also have

(2.3) $c(\varphi_{ij}, \psi_k) = 1 \quad \text{for} \quad 1 \leq i \leq p^p - 1, 1 \leq j \leq q, 1 \leq k \leq r$

by Lemma 1 (ii), and

(2.4) $c(\psi_i, \psi_j) = \sum_{k=1}^{q} \tilde{c}(\tilde{\varphi}_{i1}, \tilde{\varphi}_{jk}) = \begin{cases} q+1 & \text{if } i = j \\ q & \text{if } i \neq j \end{cases}$

by Lemma 1 (i). □

3. The Cartan matrix of G

Let $\rho = \sigma^q$ of order p. Then $G = <\rho> \ltimes K = <\sigma> \ltimes H$. At first we calculate irreducible Brauer characters of K fixed by ρ.

Lemma 3. *The following are equivalent for* $1 \leq i \leq p^p - 1$.

(i) $\varphi_{i1}, \ldots, \varphi_{iq}$ *are all* ρ*-fixed.*

(ii) $\tilde{\varphi}_i$ *is* ρ*-fixed, in particular* $\tilde{\varphi}_i$ *is* σ*-fixed.*

Proof. (i) $\rightarrow$ (ii). If ${\varphi_{ij}}^{\rho} = \varphi_{ij}$ for some j, then ${\varphi_{ij}}^{\rho}{}_{|H} = \varphi_{ij|H}$. Since ${\varphi_{ij}}^{\rho}(\zeta) = \varphi_{ij}(\rho(\zeta)) = \tilde{\varphi}_i(\rho(\zeta)) = \tilde{\varphi}_i^{\rho}(\zeta)$ and $\varphi_{ij}(\zeta) = \tilde{\varphi}_i(\zeta)$, we have $\tilde{\varphi}_i^{\rho}(\zeta) = \tilde{\varphi}_i(\zeta)$, and this means ${\tilde{\varphi}_i}^{\rho} = \tilde{\varphi}_i$.

(ii)$\rightarrow$ (i). If $\tilde{\varphi}_i^{\rho} = \tilde{\varphi}_i$, then for any $1 \leq j \leq q,\ 0 \leq k \leq q-1$, and $0 \leq l \leq p^{pq} - 1$,

$$\begin{aligned} {\varphi_{ij}}^{\rho}(\tau^k \zeta^l) &= \varphi_{ij}(\rho \tau^k \rho^{-1}) \varphi_{ij}(\rho(\zeta^l)) \quad \text{since } \varphi_{ij} \text{ is a linear character of } K \\ &= \varphi_{ij}(\tau^k) \tilde{\varphi}_i(\rho(\zeta^l)) \\ &= \varphi_{ij}(\tau^k) \tilde{\varphi}_i(\zeta^l) \quad \text{since } \tilde{\varphi}_i^{\rho} = \tilde{\varphi}_i \\ &= \varphi_{ij}(\tau^k) \varphi_{ij}(\zeta^l) \\ &= \varphi_{ij}(\tau^k \zeta^l) \quad \text{since } \varphi_{ij} \text{ is a linear character of } K. \end{aligned}$$

□

Here

$$\begin{aligned} \tilde{\phi}_i \text{ is } \sigma-\text{fixed} &\iff \tilde{\phi}_i^{\sigma}(\zeta) = \tilde{\phi}_i(\zeta) \\ &\iff \epsilon^{pi} = \epsilon^{i}, \quad \text{where } \epsilon \text{ is the } p^{pq}-1 \text{ th root of 1 in the complex number field} \\ &\iff (p-1)i \equiv 0 \pmod{p^{pq}-1} \\ &\iff i = 0,\ u = \frac{p^{pq}-1}{p-1},\ 2u, \ldots, (p-2)u. \end{aligned}$$

Therefore there are $p-1$ σ-fixed irreducible Brauer characters of H, and we reset them $\tilde{\varphi}_1, \dots, \tilde{\varphi}_{p-1}$ and remaining $p^p - 1 - (p-1) = p^p - p$ characters are τ-fixed but not ρ-fixed. Then we also reset them $\tilde{\eta}_{11}, \dots, \tilde{\eta}_{1p}, \dots, \tilde{\eta}_{n1}, \dots, \tilde{\eta}_{np}$, where $n = p^{p-1} - 1$ and $\tilde{\eta}_{ij} = \tilde{\eta}_{i1}^{\rho^{j-1}}$ for $1 \le j \le p$. As $\tilde{\eta}_{ij}$ is τ-fixed, there are q irreducible Brauer characters $\eta_{ij,k}$ of K such that $\eta_{ij,k|H} = \tilde{\eta}_{ij}$ for $1 \le k \le q$. So it is natural to arrange $\eta_{ij,k}$ so that $\eta_{ij,k} = \eta_{i1,k}{}^{\rho^{j-1}}$ for $1 \le j \le p$ by Lemma 3. Therefore we rearrange again $\eta_{ij,k}$ so that $\gamma_{ik} = \eta_{i1,k}{}^G$ is irreducible such that

$$\gamma_{ik|K} = \eta_{i1,k} + \eta_{i2,k} + \cdots + \eta_{ip,k} \quad \text{for } 1 \le i \le n, 1 \le k \le q.$$

Lemma 4. *The following are equivalent for* $1 \le i \le r$.
(i) $\tilde{\varphi}_{ij}$ *is not* τ*-fixed but* $\tilde{\varphi}_{ij}^K = \psi_i$ *is* ρ*-fixed.*
(ii) *Neither of* $\tilde{\varphi}_{i1}, \dots, \tilde{\varphi}_{iq}$ *is* τ*-fixed but they are all* ρ*-fixed.*

Proof. (ii) $\longrightarrow$ (i) is clear. (i) $\longrightarrow$ (ii). Since $\psi_{i|H} = \tilde{\varphi}_{i1} + \cdots + \tilde{\varphi}_{iq}$ and $\psi_i{}^\rho = \psi_i$, we have $\tilde{\varphi}_{i1}^\rho + \cdots + \tilde{\varphi}_{iq}^\rho = \tilde{\varphi}_{i1} + \cdots + \tilde{\varphi}_{iq}$. Then as ρ is of order p, there is at least one ρ-fixed $\tilde{\varphi}_{ij}$. We denote it again by $\tilde{\varphi}_{i1}$. Then $(\tilde{\varphi}_{i1}^\tau)^\rho = (\tilde{\varphi}_{i1}^\rho)^\tau = \tilde{\varphi}_{i1}^\tau$ and then ρ fixes all $\tilde{\varphi}_{i1}^{\tau^k}$ for $0 \le k \le q-1$. □

Here

$$\begin{aligned} \tilde{\phi}_i \text{ is } \rho - \text{fixed} &\iff \tilde{\phi}_i^\rho(\zeta) = \tilde{\phi}_i(\zeta) \\ &\iff \epsilon^{p^q i} = \epsilon^i, \quad \text{since } \rho = \sigma^q \\ &\iff (p^q - 1)i \equiv 0 \pmod{p^{pq} - 1} \\ &\iff i = 0,\ s = \frac{p^{pq} - 1}{p^q - 1},\ 2s, \dots, (p^q - 2)s. \end{aligned}$$

So there are $p^q - 1$ ρ-fixed irreducible Brauer characters of H. Among them $p-1$characters are σ-fixed, then there are $p^q - 1 - (p-1) = p^q - p$ characters of H which are ρ-fixed but not τ-fixed. So there are $m = (p^q - p)/q$ irreducible Brauer characters of K which are ρ-fixed but $\tilde{\varphi}_{ij}$s are not τ-fixed. We denote again the above m characters of K by $\psi_1, \dots, \psi_m$. Thus the following comes from Lemma 4.

Lemma 5. *The following are equivalent for* $1 \le i \le r$.
(i) $\tilde{\varphi}_{ij}$ *is not* τ*-fixed and* $\tilde{\varphi}_{ij}^K = \psi_i$ *is not* ρ*-fixed.*
(ii) $\tilde{\varphi}_{ij}$ *is neither* τ*-fixed nor* ρ*-fixed.*

As is mentioned in section two, there are r irreducible Brauer characters of K induced by $\tilde{\varphi}_{ij}$ such that $\tilde{\varphi}_{ij}$ is not τ-fixed. Therefore there are $r-m$ irreducible Brauer characters of K neither of which is ρ-fixed such that $\tilde{\varphi}_{ij}$ is not τ-fixed. We denote again them by $\varphi_1, \dots, \varphi_{r-m}$. Here, $m = (p^q - p)/q, r - m = (p^{pq} - p^p - p^q + p)/q$.

We denote the row $\varphi_{i1},\ldots,\varphi_{iq}$ by $\boldsymbol{\varphi_i}$ for $1 \le i \le p-1$, $\psi_1,\ldots,\psi_m$ by $\boldsymbol{\psi}$, $\eta_{i1,k},\ldots,\eta_{ip,k}$ by $\boldsymbol{\eta_{i,k}}$ for $1 \le i \le n$, $1 \le k \le q$, and $\varphi_1,\ldots,\varphi_{r-m}$ by $\boldsymbol{\varphi}$.

Lemma 6. *Under the above notaion, we rearrange rows and columns of $C(K)$ indexing by $\boldsymbol{\varphi_i},\ldots,\boldsymbol{\varphi_{p-1}},\boldsymbol{\psi},\boldsymbol{\eta_{1,1}},\ldots,\boldsymbol{\eta_{1,q}},\ldots,\boldsymbol{\eta_{n,1}},\ldots,\boldsymbol{\eta_{n,q}},\boldsymbol{\varphi}$. Then we have the Cartan matrix $C(K)$ as follows.*

$$
\begin{array}{ccc|c|ccccccc|c}
\boldsymbol{\varphi_1} & \cdots & \boldsymbol{\varphi_{p-1}} & \boldsymbol{\psi} & \boldsymbol{\eta_{1,1}} & \cdots & \boldsymbol{\eta_{1,q}} & \cdots & \boldsymbol{\eta_{n,1}} & \cdots & \boldsymbol{\eta_{n,q}} & \boldsymbol{\varphi} \\ \hline
2I_q & \cdots & I_q & & & & & & & & & \\
\vdots & \ddots & \vdots & J_1' & A_1 & \cdots & A_q & \cdots & A_1 & \cdots & A_q & J_2' \\
I_q & \cdots & 2I_q & & & & & & & & & \\ \hline
 & {}^tJ_1' & & B_1 & & & & J_3' & & & & qJ_4' \\ \hline
 & {}^tA_1 & & & B_2 & \cdots & 0 & & J_p & \cdots & 0 & \\
 & \vdots & & & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots & \\
 & {}^tA_q & & & 0 & \cdots & B_2 & & 0 & \cdots & J_p & \\
 & \vdots & & {}^tJ_3' & & \vdots & & \ddots & & \vdots & & J_5' \\
 & {}^tA_1 & & & J_p & \cdots & 0 & & B_2 & \cdots & 0 & \\
 & \vdots & & & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots & \\
 & {}^tA_q & & & 0 & \cdots & J_p & & 0 & \cdots & B_2 & \\ \hline
 & {}^tJ_2' & & q\,{}^tJ_4' & & & & {}^tJ_5' & & & & B_3
\end{array}
$$

where I_s is the unit matrix of degree s, J_s is the $s\times s$ matrix all of whose entries are 1, and $J_1', J_2', J_3', J_4', J_5'$ is also the matrix all of whose entries are 1 and the size of it is $(p-1)q\times m$, $(p-1)q\times(r-m)$, $m\times pqn$, $m\times(r-m)$, $pqn\times(r-m)$, respetively. A_i is the $(p-1)q\times p$ matrix whose $i, 2i, \ldots, (p-1)i$ th rows are all $(1,1,\ldots,1)$ for $1 \le i \le q$, and other rows are all $(0,0,\ldots,0)$. Furthermore $B_1 = I_m + qJ_m$, $B_2 = I_p + J_p$, and $B_3 = I_{r-m} + qJ_{r-m}$.

Finally we have irreducible Brauer characters of G as follows. Since $G \triangleright K$ whose index is p, and φ_{ij} is ρ-fixed, there exists a unique $\alpha_{ij} \in \mathrm{IBr}(G)$ such that $\alpha_{ij|K} = \varphi_{ij}$ for $1 \le i \le p-1$, $1 \le j \le q$ ([1, Chap.III, Corollary 3.16]). Also since ψ_i is ρ-fixed, there is a unique $\beta_i \in \mathrm{IBr}(G)$ such that $\beta_{i|K} = \psi_i$ for $1 \le i \le m$. Next, since $\eta_{i1,k}$ is not ρ-fixed, we have $\gamma_{ik} = {\eta_{i1,k}}^G \in \mathrm{IBr}(G)$, and $\gamma_{ik|K} = \eta_{i1,k} + \cdots + \eta_{ip,k}$ for $1 \le i \le n$, $1 \le k \le q$. Also since φ_i is not ρ-fixed, we have $\theta_1,\ldots,\theta_{\frac{r-m}{p}} \in \mathrm{IBr}(G)$ such that $\theta_i = {\varphi_j}^G$ for some j and $\theta_{i|K} = \varphi_{j_1} + \cdots + \varphi_{j_p}$ for some $j_1,\ldots,j_p$.

Lemma 7. (Ninomiya, Proposition 7 in [10]). *Suppose $G \triangleright K$ whose index is p. Let b be a block of FK and B a unique block of FG covering b. Assume the inertial group $T_G(b) = G$. Let $\mathrm{IBr}(B) = \{\theta_1,\ldots,\theta_r,\alpha_1,\ldots,\alpha_t\}$ and $\mathrm{IBr}(b) = \{\tilde\theta_{11},\ldots,\tilde\theta_{1p},\ldots,\tilde\theta_{r1},\ldots,\tilde\theta_{rp},\tilde\alpha_1,\ldots,\tilde\alpha_t\}$, where the inertial group $T_G(\tilde\theta_{ij}) = K$ for*

$1 \le i \le r,\ 1 \le j \le p$, and $T_G(\tilde{\alpha}_i) = G$ for $1 \le i \le t$, respectively. Furthermore, $\theta_{i|K} = \tilde{\theta}_{i1} + \cdots + \tilde{\theta}_{ip}$ for $1 \le i \le r$, and $\alpha_{i|K} = \tilde{\alpha}_i$ for $1 \le i \le t$.
We denote the Cartan integer of C_B, C_b for example by $c(\theta_i, \alpha_j), \tilde{c}(\tilde{\theta}_{ij}, \tilde{\alpha}_k)$, respectively. Then we have the following relation between the Cartan integers of C_B and C_b.

(i) $c(\theta_i, \theta_j) = \sum_{k=1}^{p} \tilde{c}(\tilde{\theta}_{i1}, \tilde{\theta}_{jk}) = \cdots = \sum_{k=1}^{p} \tilde{c}(\tilde{\theta}_{ip}, \tilde{\theta}_{jk})$ for $1 \le i, j \le r$,

(ii) $c(\theta_i, \alpha_j) = \sum_{k=1}^{p} \tilde{c}(\tilde{\theta}_{ik}, \tilde{\alpha}_j)$ for $1 \le i \le r,\ 1 \le j \le t$,

(iii) $c(\alpha_i, \alpha_j) = p\tilde{c}(\tilde{\alpha}_i, \tilde{\alpha}_j)$ for $1 \le i, j \le t$.

Let $\boldsymbol{\alpha_i}$ be the row $\alpha_{i1}, \ldots, \alpha_{iq}$ for $1 \le i \le p-1$, $\boldsymbol{\beta}$ be the row $\beta_1, \ldots, \beta_m$, $\boldsymbol{\gamma_i}$ be the row $\gamma_{i1}, \ldots, \gamma_{iq}$ for $1 \le i \le n$, and $\boldsymbol{\theta}$ be the row $\theta_1, \ldots, \theta_{\frac{r-m}{p}}$. We arrange rows and columns of $C(G)$ indexing by $\boldsymbol{\alpha_1}, \ldots, \boldsymbol{\alpha_{p-1}}, \boldsymbol{\beta}, \boldsymbol{\gamma_1}, \ldots, \boldsymbol{\gamma_n}, \boldsymbol{\theta}$. Then we have the following.

Theorem 8. *Under the above notation, the Cartan matrix $C(G)$ of FG is the following.*

$$
\begin{array}{cccc|c|cccc|c}
\boldsymbol{\alpha_1} & \boldsymbol{\alpha_2} & \cdots & \boldsymbol{\alpha_{p-1}} & \boldsymbol{\beta} & \boldsymbol{\gamma_1} & \boldsymbol{\gamma_2} & \cdots & \boldsymbol{\gamma_n} & \boldsymbol{\theta} \\
\hline
2pI_q & pI_q & \cdots & pI_q & & pI_q & pI_q & \cdots & pI_q & \\
pI_q & 2pI_q & \ddots & \vdots & & pI_q & pI_q & \cdots & pI_q & \\
\vdots & \ddots & \ddots & pI_q & pJ'_1 & \vdots & \vdots & & \vdots & pJ'_2 \\
pI_q & \cdots & pI_q & 2pI_q & & pI_q & pI_q & \cdots & pI_q & \\
\hline
 & & p\,{}^tJ'_1 & & B_1 & & & pJ'_3 & & pqJ'_4 \\
\hline
pI_q & pI_q & \cdots & pI_q & & (p+1)I_q & pI_q & \cdots & pI_q & \\
pI_q & pI_q & \cdots & pI_q & & pI_q & (p+1)I_q & \ddots & \vdots & \\
\vdots & \vdots & & \vdots & p\,{}^tJ'_3 & \vdots & \ddots & \ddots & pI_q & pJ'_5 \\
pI_q & pI_q & \cdots & pI_q & & pI_q & \cdots & pI_q & (p+1)I_q & \\
\hline
 & & p\,{}^tJ'_2 & & pq\,{}^tJ'_4 & & & p\,{}^tJ'_5 & & B_2
\end{array},
$$

where I_s is the unit matrix of degree s, $J'_1, J'_2, J'_3, J'_4, J'_5$ is the $(p-1)q \times m, (p-1)q \times (r-m)/p, m \times nq, m \times (r-m)/p, nq \times (r-m)/p$ matrix all of whose entries are 1, respectively. Furthermore, $B_1 = pI_m + pqJ_m$ and $B_2 = I_{\frac{r-m}{p}} + pqJ_{\frac{r-m}{p}}$, where J_s is the $s \times s$ matrix all of whose entries are 1.

Proof. It is immediate from Lemma 7 by noting that $\boldsymbol{\varphi_i}$ and $\boldsymbol{\psi}$ are ρ-fixed part, and $\boldsymbol{\eta_{i,k}}$ and $\boldsymbol{\varphi}$ are not ρ-fixed part. □

4. Relation between $k(G)$ and $\rho(G)$

Let $\rho(B)$ be the Perron-Frobenius eigenvalue of the Cartan matrix C_B of a block B of FG. We raised a conjecture in [14] that if G is p-solvable, then $k(B) \leq \rho(B)$. We shall show the above conjecture is true for our group $G = G(p^{pq})$. Since G has only the principal block, we write $k(G), l(G), C(G)$ and $\rho(G)$ instead of $k(FG), l(FG)$, C_{FG} and $\rho(FG)$, respectively.

As is seen in section three,

$$l(G) = (p-1)q + m + nq + \frac{r-m}{p}.$$

Since H is a complete Frobenius group, there is a unique ordinary irreducible character $\tilde{\chi}$ of H of degree $p^{pq} - 1$. As G/H is cyclic of order pq and $\tilde{\chi}$ is σ-fixed, $\tilde{\chi}$ is extendible to G (Chap.III, Theorem 2.14 in [1] or Chap.6, (6.17) in [4]) and there are pq ordinary irreducible characters $\chi_1, \ldots, \chi_{pq}$ of G such that $\chi_{i|H} = \tilde{\chi}$ for $1 \leq i \leq pq$.

Let us set $R = A(p^{pq})$ be the subgroup of $G = G(p^{pq})$ which is isomorphic to an elementary abelian p-group of order p^{pq}. Since K/R is a p'-group, the number of ordinary irreducible characters in K whose kernel contains R coinsides with $l(K)$. The group K has $(p-1)q$ ρ-fixed irreducible Brauer characters φ_{ij} in which $\varphi_{ij|H} = \tilde{\varphi}_i$ for $1 \leq j \leq q$ and $\tilde{\varphi}_i$ is τ-fixed, and furthermore m ρ-fixed $\psi_1, \ldots, \psi_m$ in which $\tilde{\varphi}_{ij}^K = \psi_i$ and $\tilde{\varphi}_{ij}$ is not τ-fixed. So they are regarded as the ordinary irreducible characters of K whose kernel contains R. Since they are ρ-fixed, the number of ordinary irreducible extending characters of them to G is p times as large as the number of ρ-fixed irreducible Brauer chracters of K. Therefore we have

$$k(G) = p(p-1)q + pm + nq + \frac{r-m}{p} + pq = p^2q + pm + nq + \frac{r-m}{p}.$$

Let c_i be the i th row sum of $C(G)$, then $\sum_{i=1}^{l(G)} c_i/l(G) \leq \rho(G)$ by Lemma 3.1(2) in [5]. Now we shall show by a direct calculation that

$$l(G)k(G) \leq \sum_{i=1}^{l(G)} c_i.$$

Now,

$$\begin{aligned} l(G)k(G) &= pm^2 + q^2n^2 + (p+1)qmn \\ &+ \{(p+1)\frac{r-m}{p} + p^2q\}m \end{aligned}$$

$$
\begin{aligned}
&+ \quad \{(p+q)\frac{r-m}{p} + q^2(p^2+p-1)\}n \\
&+ \quad \frac{(r-m)^2}{p^2} + q(p^2+p-1)\frac{r-m}{p} + p^2q^2(p-1).
\end{aligned}
$$

Next, we give a table of a block-wise sum of $C(G)$ as follows;

α_1	$\cdots$	α_{p-1}	β	γ_1	$\cdots$	γ_n	θ
$2pq$	$\cdots$	pq	pqm	pq	$\cdots$	pq	$pq \times \frac{r-m}{p}$
pq	$\ddots$	pq	pqm	pq	$\cdots$	pq	$pq \times \frac{r-m}{p}$
$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
pq	$\cdots$	$2pq$	pqm	pq	$\cdots$	pq	$pq \times \frac{r-m}{p}$
pqm	$\cdots$	pqm	pqm^2+pm	pqm	$\cdots$	pqm	$pqm \times \frac{r-m}{p}$
pq	$\cdots$	pq	pqm	$pq+q$	$\cdots$	pq	$pq \times \frac{r-m}{p}$
pq	$\cdots$	pq	pqm	pq	$\ddots$	pq	$pq \times \frac{r-m}{p}$
$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
pq	$\cdots$	pq	pqm	pq	$\cdots$	$pq+q$	$pq \times \frac{r-m}{p}$
$pq \times \frac{r-m}{p}$	$\cdots$	$pq \times \frac{r-m}{p}$	$pqm \times \frac{r-m}{p}$	$pq \times \frac{r-m}{p}$	$\cdots$	$pq \times \frac{r-m}{p}$	$(pq \times \frac{r-m}{p} + 1)\frac{r-m}{p}$

and a further block-wise sum is the following;

α	β	γ	θ
$(p-1)p^2q$	$(p-1)pqm$	$(p-1)pqn$	$(p-1)pq \times \frac{r-m}{p}$
$(p-1)pqm$	pqm^2+pm	$pqmn$	$pqm \times \frac{r-m}{p}$
$(p-1)pqn$	$pqmn$	pqn^2+qn	$pqn \times \frac{r-m}{p}$
$(p-1)pq \times \frac{r-m}{p}$	$pqm \times \frac{r-m}{p}$	$pqn \times \frac{r-m}{p}$	$(pq \times \frac{r-m}{p} + 1)\frac{r-m}{p}$

Thus we have

$$
\begin{aligned}
\sum_{i=1}^{l(G)} c_i \quad &= \quad pqm^2 + pqn^2 + 2pqmn \\
&+ \quad \{2pq\frac{r-m}{p} + q(2p^2-2p) + p\}m \\
&+ \quad \{2pq\frac{r-m}{p} + q(2p^2-2p+1)\}n \\
&+ \quad pq\frac{(r-m)^2}{p^2} + \{q(2p^2-2p)+1\}\frac{r-m}{p} + p^2q(p-1).
\end{aligned}
$$

Lemma 9.

$$
\frac{r-m}{p} > q^{p-1}(n+1).
$$

Proof. Since $m = (p^q - p)/q$, we have $p^q = qm + p$, and if we set $m = pa$ for some a, then $p^{q-1} = qa + 1$. So

$$\begin{aligned}\frac{r-m}{p} &= \frac{p^{pq} - p^p - p^q + p}{pq} \\ &= \frac{p^p(p^{p(q-1)} - 1) - p(p^{q-1} - 1)}{pq} \\ &= p^{p-1}\frac{(qa+1)^p - 1}{q} - a.\end{aligned}$$

Since

$$\frac{(qa+1)^p - 1}{q} = \frac{1}{q}\{q^p a^p + \binom{p}{1} q^{p-1}a^{p-1} + \cdots + \binom{p}{p-1} qa + 1 - 1\},$$

we have

$$\begin{aligned}\frac{r-m}{p} &= p^{p-1}\{q^{p-1}a^p + \binom{p}{1} q^{p-2}a^{p-1} + \cdots + \binom{p}{p-1} a\} - a \\ &> (pq)^{p-1} = (n+1)q^{p-1}, \quad \text{since } p^{p-1} = n+1.\end{aligned}$$ □

Comparing each term between $l(G)k(G)$ and $\sum_{i=1}^{l(G)} c_i$, it is easy to see that in $\sum_{i=1}^{l(G)} c_i$ the m^2, the mn, the only m, and the $(r-m)^2/p^2$ terms are larger than the ones in $l(G)k(G)$. By Lemma 9 the $(r-m)^2/p^2$term in $\sum_{i=1}^{l(G)} c_i$ is so large that the remaining $(r-m)^2/p^2$ term, when we subtract $l(G)k(G)$ from $\sum_{i=1}^{l(G)} c_i$, covers enough the minus in the $(r-m)/p$ term, the n^2, the only n and the pq terms. Thus we have the following.

Proposition 10. *Let $G = G(p^{pq})$ for a different prime number q from p, $C(G)$ be the Cartan matrix of FG and $\rho(G)$ be the Perron-Frobenius eigenvalue of $C(G)$. Then*

$$k(G) < \rho(G).$$

Theorem A ([14]). *Let G be a finite group and B a block of FG. For $l = l(B)$ we consider a permutation σ on l letters $\{1, 2, \ldots, l\}$. We set $l \backslash t := \{1, 2, \ldots, l\} - \{t\}$ for $1 \le t \le l$. Then we have*

$$k(B) \le \sum_{i=1}^{l} c_{ii} \; - \sum_{j \in l\backslash t} c_{j\sigma(j)}$$

for any cycle σ of length l and any choice of $1 \le t \le l$.

REMARK 11. We can also show Proposition 10 by taking a diagonal line, which is q columns apart from the main diagonal line, as a cycle of length $l(G)$ and verifying the inequality in Theorem A. But it is so complicated that we omit it. But Theorem A does not always work well to show directly that $k(B) \le \rho(B)$. For examle, let $G = D_8 \ltimes E_9$ and $p = 3$. Then

$$C(G) = \begin{pmatrix} 3 & 0 & 1 & 1 & 2 \\ 0 & 3 & 1 & 1 & 2 \\ 1 & 1 & 3 & 0 & 2 \\ 1 & 1 & 0 & 3 & 2 \\ 2 & 2 & 2 & 2 & 5 \end{pmatrix}.$$

Here $C(G)$ has 0 entries and the diagonal entry 5 is relatively large comparing the other non diagonal entries. If we choose 1,1,2,2 as the non diagonal four entries, which is the best choice, we have 11 as the value in the right hand side of the inequality in Theorem A. But $\rho(G)$ is 9 by Proposition 4.3 in [5], since G has a normal defect group. Another one is $G = Fr_{21} \ltimes E_8$ which is isomorphic to $G(2^3)$, and its Cartan matrix is obtained in the next section.

5. The Cartan matrix of $G(p^q)$ and $G(p^p)$

We briefly mention about the Cartan matrix of $G(p^q)$ and $G(p^p)$, where q is a prime number which is different from p, because we can show it by the same method as $G(p^{pq})$.

The group $G(p^q)$ is p-closed and its Cartan matrix has 0 entries as follows. Let σ be a generator of the Galois group of $GF(p^q)$ over $GF(p)$ of order q. There are $p-1$ σ-fixed irreducible Brauer characters $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{p-1}$ of $X(p^q)$. We set the other $p^q - p = rq$ characters by $\tilde{\varphi}_{ij}$ for $1 \le i \le r$, $1 \le j \le q$, where $\tilde{\varphi}_{ij} = \tilde{\varphi}_{i1}^{\sigma^{j-1}}$. Then there are $(p-1)q$ irreducible Brauer characters φ_{ij} of $G(p^q)$ such that $\varphi_{ij|X(p^q)} = \tilde{\varphi}_i$ for $1 \le i \le p-1$, $1 \le j \le q$, and r characters ψ_i such that $\psi_{i|X(p^q)} = \tilde{\varphi}_{i1} + \cdots + \tilde{\varphi}_{iq}$ for $1 \le i \le r$. Next we arrange rows and columns of $C(G(p^q))$ indexing by $\boldsymbol{\varphi_1},\ldots,\boldsymbol{\varphi_{p-1}},\boldsymbol{\psi}$, where $\boldsymbol{\varphi_i}$ is the row $\varphi_{i1}, \ldots, \varphi_{iq}$ and $\boldsymbol{\psi}$ is the row $\psi_1, \ldots, \psi_r$.

$$C(G(p^q)) = \begin{array}{cccc|c} \varphi_1 & \varphi_2 & \cdots & \varphi_{p-1} & \psi \\ \hline 2I_q & I_q & \cdots & I_q & \\ I_q & 2I_q & \ddots & \vdots & \\ \vdots & \ddots & \ddots & I_q & J_{(p-1)\times r} \\ I_q & \cdots & I_q & 2I_q & \\ \hline & & J_{r\times(p-1)} & & I_r + qJ_r \end{array},$$

where $r = (p^q - p)/q$, I_s is the unit matrix of degree s, and J_s, $J_{s\times t}$ is the $s\times s, s\times t$ matrix all of whose entries are 1, respectively.

Since $X(p^q)$ is a complete Frobenius group, there is a unique ordinary irreducible character $\tilde{\theta}$ of $X(p^q)$ which is σ-fixed. Then G has q more ordinary irreducible characters other than irreducible Brauer characters of G. So in this case we obtain $k(G) \leq \rho(G)$ by direct calculation with the following lemma, because $k(G) = pq + r$ and $\rho(G) = p^q$, and the equality holds if and only if $(p,q) = (2,3)$ *or* $(3,2)$. We should note that $G(2^3) \simeq Fr_{21} \ltimes E_8$ and $G(3^2) \simeq S_{16} \ltimes E_9$.

Lemma 12. *Let $p, q \geq 2$ be different prime numbers. Then $p^{q-1} - q^2 > 0$ except when $(p,q) = (2,3), (2,5)$ or $(3,2)$.*

Proof. Let $f(x) = p^{x-1} - x^2$ be a real valued function defined on x such that $x \geq 2$, and for a constant integer $p \geq 2$. Then $f'(x) = (\log p)p^{x-1} - 2x$, $f''(x) = (\log p)^2 p^{x-1} - 2$, and $f'''(x) = (\log p)^3 p^{x-1}$. So $f'''(x) > 0$ and then $f''(x)$ is monotonously increasing. Since $f''(5) = (\log p)^2 p^4 - 2$, $f''(5) > 0$ if $p \geq 2$. So if $x \geq 5$, then $f''(x) > 0$ for any $p \geq 2$. Then $f'(x)$ is monotonously increasing for $x \geq 5$ and for any $p \geq 2$. Since $f'(5) = (\log p)p^4 - 10$, $f'(5) > 0$ if $p \geq 2$. Therefore if $x \geq 5$, then $f'(x) > 0$ for any $p \geq 2$. Thus if $x \geq 5$, then $f(x)$ is monotonously increasing for any $p \geq 2$. We have $f(5) = p^4 - 25 > 0$ if $p \geq 3$, and $f(7) = p^6 - 49 > 0$ if $p \geq 2$. Therefore, if $x \geq 7$, then $f(x) > 0$ for any $p \geq 2$ and if $x \geq 5$, then $f(x) > 0$ for $p \geq 3$. So suppose $p = 2$. If $f(q) \leq 0$, then $q = 3$ or 5. Suppose $p = 3$. If $f(q) \leq 0$, then $q = 2$. □

REMARK 13. If m is any integer such that $(m,p) = 1$, then the Cartan matrix of the group $G(p^m)$ has zero entries by our consideration. At least, the part of the trivial irreducible Brauer character has zero entries.

We have also the Cartan matrix of $G(p^p)$ which is of p-length 2, but it has no 0 entries. Let σ be a generator of the Galois group of $GF(p^p)$ over $GF(p)$ of order p. There are $p-1$ σ-fixed irreducible Brauer characters $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{p-1}$ of $X(p^p)$. We set the other $p^p - p = rp$ characters by $\tilde{\varphi}_{ij}$ for $1 \leq i \leq r$, $1 \leq j \leq p$, where $\tilde{\varphi}_{ij} = \tilde{\varphi}_{i1}^{\sigma^{j-1}}$. Then there are $p-1$ irreducible Brauer characters α_i such that $\alpha_{i|X(p^p)} = \tilde{\varphi}_i$, for $1 \leq i \leq p-1$, and r characters ψ_i such that $\psi_{i|X(p^p)} = \tilde{\varphi}_{i1} + \cdots + \tilde{\varphi}_{ip}$ for $1 \leq i \leq r$. We set by $\boldsymbol{\alpha}$ the row $\alpha_1, \ldots \alpha_{p-1}$, and by $\boldsymbol{\psi}$ the row $\psi_1, \ldots, \psi_r$. Then we arrange rows and columns of $C(G(p^p))$ indexing by α and ψ.

$$C(G(p^p)) = \begin{array}{c|c} \alpha & \psi \\ \hline pI_{p-1} + pJ_{p-1} & pJ_{(p-1)\times r} \\ \hline pJ_{r\times(p-1)} & I_r + pJ_r \end{array},$$

where $r = p^{p-1}-1$, and I_s is the unit matrix of degree s, and J_s, $J_{s\times t}$ is the $s\times s, s\times t$ matrix all of whose entries are 1, respectively. The Cartan matrix $C(G(p^p))$ has already been obtained in [6] (also see [8], [11]).

In this case, $C(G)$ has no zero entries, and $l(G) = p-1+r$, $k(G) = p^2+r$, and

$$\sum_{i=1}^{l(G)} c_i = p^3 - p^2 + pr(2p + r - 2) + r.$$

Then we have also $l(G)k(G) < \sum_{i=1}^{l(G)} c_i$ and therefore $k(G) < \rho(G)$ holds.

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