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EXISTENCE OF PENCILS WITH PRESCRIBED SCROLLAR INVARIANTS OF SOME GENERAL TYPE

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0. Introduction

Let C be an irreducible smooth projective non-hyperelliptic curve of genus g defined over the field $\boldsymbol{C}$ of complex numbers. Let g^1_k be a complete base-point free special linear system on C. The scrollar invariants of g^1_k are defined as follows. Let C be canonically embedded in $\boldsymbol{P}^{g-1}$ and let X be the union of the linear spans $\langle D\rangle$ with $D \in g^1_k$. This defines a set of integers $e_1 \geq \ldots \geq e_{k-1} \geq 0$ such that X is the image of the projective bundle $\boldsymbol{P}(e_1; \ldots ; e_{k-1}) = \boldsymbol{P}(O_{P^1}(e_1) \oplus \ldots \oplus O_{P^1}(e_{k-1}))$ using the tautological bundle (see e.g. [2]; [7]). Those integers $e_1; e_2; \ldots ; e_{k-1}$ are called the scrollar invariants of g^1_k.

Those scrollar invariants determine (and are determined by) the complete linear systems associated to multiples of the linear system g^1_k. For $1 \leq i \leq k-1$ the invariant e_i is one less than the number of non-negative integers j satisfying $\dim(|K_C - jg^1_k|) - \dim(|K_C - (j+1)g^1_k|) \geq i$. Here K_C denotes a canonical divisor on C. Let $m = e_{k-1}+2$. Then m is defined by the following conditions: $\dim(|(m-1)g^1_k|) = m-1$ and $\dim(|mg^1_k|) > m$. In case $|mg^1_k|$ is birationally very ample then the scrollar invariants satisfy the inequalities $e_i \leq e_{i+1} + m$ for $1 \leq i \leq k-2$ (see [3]). In case $k=3$ this number $m = e_2$ determines also the other scrollar invariant e_1. It is the starting point for so-called Maroni-theory for linear systems on trigonal curves (see [4]; [5]). Scrollar invariants for 4-gonal curves are intensively studied in [1]; [3] and for 5-gonal curves in [6].

For $(j-1)m - 1 < x \leq jm - 1$ with $j \leq k-1$ the inequalities between the scrollar invariants imply $\dim(|xg^1_k|) \geq \frac{j(j-1)}{2}m - 1 + (x - (j-1)m + 1)j$. Equality (if not in conflict with the Riemann-Roch Theorem) can be expected being the most general case for a fixed value of m. The inequalities also imply $\dim(|(k-1)mg^1_k|) = \dim(|((k-1)m-1)g^1_k|) + k$. This implies that $|((k-1)m-1)g^1_k|$ is not special. Using the dimension bound one obtains $g \leq [(k^2-k)m - 2k + 2]/2$. (This easy but interesting consequence from the inequalities is not mentioned by Kato and Ohbuchi.) In this paper we prove the following theorem.

Theorem. *For all nonnegative integers* k; m *and* g *satisfying* $k \geq 3$; $m \geq 2$ *and* $k-1 \leq g \leq [(k^2-k)m-2k+2]/2$ *there exists a smooth curve* C *of genus g possessing* *a complete base point free linear system g^1_k satisfying the following property. For each nonnegative integer x with $x \leq (k-1)m-1$ define the nonnegative integer j such that $(j-1)m-1 < x \leq jm-1$. Then $\dim(|xg^1_k|) = max\left(\left\{\frac{j(j-1)}{2}m - 1 + (x-(j-1)m+1)j; kx-g\right\}\right)$. Also $|mg^1_k|$ is birationally very ample.*

The curves C are obtained using special plane curves degenerating to special types of rational curves. First we construct those rational curves Γ_0 using some linear system g^1_k on $\boldsymbol{P}^1$. In order to prove the theorem we study canonical adjoint curves of Γ_0 containing all points belonging to a given number of divisors from g^1_k.

SOME NOTATIONS. On a smooth surface X; if Γ_1 and Γ_2 are two effective divisors intersecting at $x \in X$ (no common component containing x) then we write $i(\Gamma_1.\Gamma_2; x)$ for the intersection multiplicity of Γ_1 and Γ_2 at x. We write $(\Gamma_1.\Gamma_2)$ for the intersection number of Γ_1 and Γ_2. We also write K_X for a canonical divisor on X.

1. Construction of the plane rational curve

Choose a general linear system g^1_k on $\boldsymbol{P}^1$ and a general divisor $F \in g^1_k$. Choose a general effective divisor E of degree mk on $\boldsymbol{P}^1$. Consider the linear system g^2_{mk} containing $(m-1)F + g^1_k$ and E.

Claim 1.1. *g^2_{mk} is a simple base point free linear system on $\boldsymbol{P}^1$.*

Proof. The linear system g^2_{mk} has no base points: $mF \in (m-1)F + g^1_k \subset g^2_{mk}$ and $E \cap F = \emptyset$. For $P \in E$ and $D_P \in g^1_k$ containing P one has $E \cap D_P = \{P\}$ (the intersection as schemes is reduced), therefore also $E \cap ((m-1)F + D_P) = \{P\}$. Since $(m-1)F + D_P \in g^2_{mk}$ this implies that g^2_{mk} is simple. □

Claim 1.2. *The space parametrizing such linear systems g^2_{mk} on $\boldsymbol{P}^1$ is irreducible of dimension $mk + 2k - 3$.*

Proof. Effective divisors of degree d on $\boldsymbol{P}^1$ are parametrized by a projective space $\boldsymbol{P}^d$. Linear systems g^1_k (resp. g^2_{mk}) on $\boldsymbol{P}^1$ are parametrized by a grassmannian $G(1;k)$ of lines in $\boldsymbol{P}^k$ (resp. $G(2;mk)$ of planes in $\boldsymbol{P}^{mk}$). On $G(1;k) \times \boldsymbol{P}^k$ we have the incidence subvariety I defined as $(g^1_k; F) \in I$ if and only if $F \in g^1_k$. Clearly I is irreducible of dimension $\dim(G(1;k)) + 1 = 2k-1$. The linear systems g^2_{mk} on $\boldsymbol{P}^1$ constructed above belong to the image of the rational map $\tau : I \times \boldsymbol{P}^{mk} \to G(2;mk)$ defined by $\tau((g^1_k; F); E) = \langle (m-1)F + g^1_k; E \rangle$.

Suppose for $((g^1_k; F); E) \in I \times \boldsymbol{P}^{mk}$ general, there exists another element $((h^1_k; G); E') \in I \times \boldsymbol{P}^{mk}$ with $\tau((g^1_k; F); E) = \tau((h^1_k; G); E')$ but $(g^1_k; F) \neq (h^1_k; G)$. Because

$(m-1)F+g^1_k$ and $(m-1)G+h^1_k$ are both lines in g^2_{mk}, one has $[(m-1)F+g^1_k]\cap[(m-1)G+h^1_k]\neq\emptyset$. Assume $(m-1)F+g^1_k=(m-1)G+h^1_k$. Because g^1_k has no fixed points, this implies $F=G$ and so $g^1_k=h^1_k$; a contradiction. Choose $D\in h^1_k$ with $(m-1)G+D\notin(m-1)F+g^1_k$. Then $g^2_{mk}=\langle(m-1)F+g^1_k;(m-1)G+D\rangle$. Because g^2_{mk} is base point free (Claim 1.1) we find $F\cap G=\emptyset$. But then $[(m-1)F+g^1_k]\cap[(m-1)G+h^1_k]\neq\emptyset$ implies $m=2$; $F\in h^1_k$; $G\in g^1_k$ and so $g^1_k=h^1_k=\langle F;G\rangle$. For $D\in g^1_k$ one finds $F+D$; $G+D\in g^2_{2k}$ so $g^1_k+D\subset g^2_{2k}$. It follows that $g^2_{2k}=\{D_1+D_2:D_1;D_2\in g^1_k\}$. This contradicts g^2_{2k} being simple (Claim 1.1). So we find for a general $((g^1_k;F);E)\in I\times\boldsymbol{P}^{mk}$ one has $\tau((g^1_k;F);E)=\tau((h^1_k;G);E')$ if and only if $(g^1_k;F)=(h^1_k;G)$ and $E'\in\langle(m-1)F+g^1_k;E\rangle$. Therefore the general non-empty fiber of τ has dimension 2. So, the image of τ has dimension $mk+2k-3$.

Associated to g^2_{mk} there exist morphisms $\phi:\boldsymbol{P}^1\to\boldsymbol{P}^2$. Fix such a morphism and let Γ be the image.

Claim 1.3. *Γ is a plane curve of degree mk. The divisor F induces a singular point s on Γ of multiplicity $(m-1)k$. The other singular points of Γ are ordinary nodes.*

Proof. Since g^2_{mk} is simple and base point free (Claim 1.1) the plane curve Γ has degree mk. There is a 1-dimensional subsystem of g^2_{mk} containing $(m-1)F$. This 1-dimensional subsystem corresponds to a pencil of lines on $\boldsymbol{P}^2$ containing some fixed point s. For a general line L containing s there are k intersections each one of multiplicity 1 with Γ outside s. This implies $i(\Gamma.L;s)=(m-1)k$, hence Γ has multiplicity $(m-1)k$ at s. Assume s' is another singular point of Γ.

First assume s' has multiplicity $\mu\geq 3$. The pencil of lines on $\boldsymbol{P}^2$ containing s' induces a linear subsystem $F'+g^1_{mk-\mu}\subset g^2_{mk}$. From $s\neq s'$ it follows that $F\cap F'\neq\emptyset$. The line $\langle ss'\rangle$ on $\boldsymbol{P}^2$ gives rise to $(m-1)F+D\in g^2_{mk}$ with $D\in g^1_k$. We find $D=F'+D'$ for some effective divisor D'. Let E' be the divisor corresponding to a general line through s', then $E'=F'+E''$ for some $E''\in g^1_{mk-\mu}$. Since $g^2_{mk}=\langle(m-1)F+g^1_k;E'\rangle$ we find that g^2_{mk} belongs to the image of the morphism $\tau‘:I'\times\boldsymbol{P}^{mk-\mu}\to G(2;mk)$ defined by $\tau‘((g^1_k;F;F');E'')=\langle(m-1)F+g^1_k;F'+E''\rangle$ with $I'\subset I\times\boldsymbol{P}^{\mu}$ defined by $(g^1_k;F;F')\in I'$ if and only if $D\geq F'$ for some $D\in g^1_k$ (here I is as in the proof of Claim 1.2). The choice of E'' implies that $\tau‘$ has non-empty fibers of dimension at least 1. Since $\dim(I'\times\boldsymbol{P}^{mk-\mu})=2k+mk-\mu$ we obtain $2k-1+mk-\mu\leq(m+2)k-3$. This contradicts $\mu\geq 3$. It follows that s' has multiplicity 2.

Using the same notations we have $\mu=2$; $\deg(F')=2$. Assume $F'=2P_0$. Then P_0 is a ramification point of g^1_k.

This implies that g^2_{mk} belongs to the image of the morphism $\tau“:I''\times\boldsymbol{P}^{mk-2}\to G(2;mk)$ defined by $\tau“((g^1_k;F;P_0);E'')=\langle(m-1)F+g^1_k;2P_0+E''\rangle$, with $I''\subset I\times\boldsymbol{P}^1$ defined by $(g^1_k;F;P_0)\in I''$ if and only if P_0 is a ramification point of g^1_k. Again, the non-empty fibers have dimension at least 1. Since $\dim(I''\times\boldsymbol{P}^{mk-2})=2k-1+mk-2$, we find a contradiction to $\dim(im\tau)=(m+2)k-3$.

We obtain $F' = P_0 + Q_0$ with $P_0 \neq Q_0$. Assume L_0 is a line through s' such that L_0 induces $2(P_0 + Q_0) + E'''$ for some effective divisor E''' of degree mk-4. Hence, we assume that s' is a tacnode. This implies g^2_{mk} belongs to the image of the rational map $\tau''' : I''' \times \boldsymbol{P}^{mk-4} \to G(2; mk)$ defined by $\tau'''((g^1_k; F; F'); E''') = \langle (m-1)F + g^1_k; 2F' + E''' \rangle$ with $I''' \subset I \times \boldsymbol{P}^2$ defined by $(g^1_k; F; F') \in I'''$ if and only if $D \geq F'$ for some $D \in g^1_k$. Because $\dim(I''' \times \boldsymbol{P}^{mk-4}) = (m+2)k - 4 < (m+2)k - 3$, once more we obtain a contradiction. This implies that s' is an ordinary node.

Because $mF \in g^2_{mk}$ there exists a line T on $\boldsymbol{P}^2$ through s inducing mF. This line T intersects Γ only at s, hence $i(T, \Gamma; s) = mk$. We can consider the singularity of Γ at s as follows. It consists of exactly k locally irreducible branches (we use $F \in \boldsymbol{P}^k$ is general), each one having multiplicity $m - 1$ at s and having T as "tangent line" intersecting the branch with multiplicity m at s. From now on we fix s and T. □

Claim 1.4. *We obtain a family of plane curves of dimension* $(m + 2)k - 1$.

Proof. This follows from Claim 1.2 taking into account that $\dim(Aut(\boldsymbol{P}^1)) = 3$; $\dim(Aut(\boldsymbol{P}^2)) = 8$ and fixing s and T imposes 3 independent conditions on Γ. □

2. Blowing_up the projective plane

Let $\pi_1 : X_1 \to \boldsymbol{P}^2$ be the blowing-up of $\boldsymbol{P}^2$ at s; let E_1 be the exceptional divisor. Let T_1 (resp. Γ_1) be the strict transform of T (resp. Γ) on X_1. Let L be the inverse image of a line on $\boldsymbol{P}^2$. Then $T_1 \in |L - E_1|$; $\Gamma_1 \in |kmL - k(m-1)E_1|$. Let $s_1 = E_1 \cap T_1$.

The linear system $|L - E_1|$ induces g^1_k on $\boldsymbol{P}^1$ and T_1 induces F. Since the images of points of F under the morphism $\boldsymbol{P}^1 \to \Gamma_1$ are contained in E_1, it follows that $i(T_1, \Gamma_1; s_1) = k$. Hence the k different points of F correspond to k different irreducible branches of Γ_1 at s_1. Hence Γ_1 has a singular point of multiplicity k at s_1. Also $E_1 \cap \Gamma_1 = \{s_1\}$ and since $(E_1.\Gamma_1) = k(m-1)$ it follows that $i(E_1, \Gamma_1; s_1) = (m-1)k$. Because $T_1 + E_1$ induces mF on $\boldsymbol{P}^1$, it follows that E_1 intersects each branch of Γ_1 at s_1 with multiplicity $m - 1$ at s_1.

Let $\pi_2 : X_2 \to X_1$ be the blowing-up of X_1 at s_1. Let E_2 be the exceptional divisor. We continue to write L for the inverse image of a general line on $\boldsymbol{P}^2$. Let E_{12} (resp. $T_2; \Gamma_2$) be the strict transforms of E_1 (resp. $T; \Gamma$) on X_2. We also write E_1 to denote the inverse image of E_1 on X_2. Then $E_{12} \in |E_1 - E_2|$; $T_2 \in |L - E_1 - E_2|$; $\Gamma_2 \in |kmL - k(m-1)E_1 - kE_2|$. Let $s_2 = E_2 \cap E_{12}$. One has $(T_2.\Gamma_2) = 0$ hence $T_2 \cap \Gamma_2 = \emptyset$. In case $m = 2$ we find $(\Gamma_2.E_{12}) = 0$ hence $\Gamma_2 \cap E_{12} = \emptyset$.

Assume $m > 2$. From $(\Gamma_2.E_2) = k$ it follows that each branch of Γ_2 corresponding to a point of F is smooth and intersects E_2 transversally at one point. Because $E_2 + E_{12}$ induces $(m-1)F$ on $\boldsymbol{P}^1$ it follows that those points of F map to s_2 and E_{12} intersects each branch with multiplicity $m - 2$ at s_2. It follows that Γ_2 has multiplicity k at s_2.

We continue to make blowings-up. For each $i \leq m$ we obtain the blowing-up $\pi_i : X_i \to X_{i-1}$ with exceptional divisor E_i. On X_i we continue to write L to denote the inverse image of a general line on $\boldsymbol{P}^2$. We write Γ_i (resp. $E_{i-1,i}$; T_i) to denote the strict transform of Γ (resp. E_{i-1}; T) on X_i. Also, for $j \leq i-2$ we write $E_{j,i}$ for the strict transform of $E_{j,i-1}$. Let $s_i = E_i \cap E_{i-1,i}$. In case $i < m$ the multiplicity of Γ_i at s_i is k. At s_i the curve Γ_i has k smooth locally irreducible branches. Also $E_{1,i}$ intersects each branch with multiplicity $m-i$ at s_i. On X_m we also write E_i for the inverse image of E_i on X_i (for $i < m$). We obtain $\Gamma_m \in \boldsymbol{P} := |kmL - k(m-1)E_1 - kE_2 - \ldots - kE_m|$, $T_m \in |L - E_1 - E_2|$, $E_{1m} \in |E_1 - E_2 - \ldots - E_m|$ and $E_{im} \in |E_i - E_{i+1}|$ for $2 \leq i \leq m-1$.

Claim 2.1. *Γ_m has ordinary nodes as its only singularities. The intersection points of Γ_m and E_m are smooth points on Γ_m.*

Proof. Because of Claim 1.3 it is enough to prove that the intersection points of Γ_m and E_m are smooth points on Γ_m. The inverse image on $\boldsymbol{P}^1$ of the intersection as schemes of Γ_m and E_m is the divisor F, hence a general divisor of degree k on $\boldsymbol{P}^1$. If that intersection would not be smooth then 2 different points in F would have the same image on Γ_m. Because of monodromy on $\boldsymbol{P}^1$, in that case all k points on F need to have the same image on Γ_m, hence $\Gamma_m \cap E_m$ is a single multiple point s_m of Γ_m. Since $(\Gamma_m.E_{1m}) = 0$ and $(\Gamma_m.E_{m-1,m}) = 0$ it follows that $s_m \notin \{E_{1,m} \cap E_m; E_{m-1,m} \cap E_m\}$. Let $\pi_{m+1} : X_{m+1} \to X_m$ be the blowing-up of X_m at s_m. Let E_{m+1} be the exceptional divisor of π_{m+1} and let Γ_{m+1} be the strict transform of Γ_m. We find $\Gamma_{m+1} \in \boldsymbol{P}_{m+1} := |kmL - k(m-1)E_1 - kE_2 - \ldots - kE_{m+1}|$. If Γ_{m+1} is not smooth at each point of $\Gamma_{m+1} \cap E_{m+1}$ then as before we find $s_{m+1} \in \Gamma_{m+1}$ such that Γ_{m+1} has multiplicity k at s_{m+1}. In that case we blow-up Γ_{m+1} at s_{m+1} and so on.

For some $m' \geq 1$ we obtain $X_{m+m'}$ and $\Gamma_{m+m'} \in \boldsymbol{P}_{m+m'} := |kmL - k(m-1)E_1 - kE_2 - \ldots - kE_{m+m'}|$ such that $\Gamma_{m+m'}$ has ordinary nodes as its only singularities. The arithmetic genus of $\Gamma_{m+m'}$ is equal to $[(km-1)(km-2) - (k(m-1)-1)k(m-1) - (m+m'-1)(k-1)k]/2$. This has to be at least 0, hence $(m-m')k^2 + (m'-m)k - 2k + 2) \geq 0$. This condition implies $m' \leq m$.

In $\boldsymbol{P}_{m+m'}$ we find that the locus of irreducible rational nodal curves has a component of dimension at least $mk + 2k - 1 - m'$. (This follows from Claim 1.4 taking into account the choice of s_{m+i} on E_{m+i} for $0 \leq i \leq m'$.) The number of nodes of $\Gamma_{m+m'}$ is equal to the arithmetic genus of $\Gamma_{m+m'}$ being $\delta = [(m-m')(k^2-k) - 2k + 2]/2$. Because $m' \leq m$ we find $(K_{X_{m+m'}}.\Gamma_{m+m'}) = -3km + k(m-1) + k(m+m'-1) = (m'-m-2)k < 0$. From Lemma 2.2 in [8] it follows that $\dim(\boldsymbol{P}_{m+m'}) \geq mk + 2k - 1 - m' + \delta$. Also from the end of the proof of Lemma 2.2 in [8] we also obtain $\dim(\boldsymbol{P}_{m+m'}) = \delta - (K_{X_{m+m'}}.\Gamma_{m+m'}) - 1 = \delta + (m+2-m')k - 1$. This would imply $(m+2-m')k - 1 \geq mk + 2k - 1 - m'$, hence $m' \geq m'k$. Since $m' \geq 1; k \geq 2$ this is a contradiction. This completes the proof of the claim. □

3. Canonically adjoint curves

In order to study canonically adjoint curves for curves belonging to $\boldsymbol{P}$ we consider the linear system $\boldsymbol{P}_0' = |(km-3)L-(k(m-1)-1)E_1-(k-1)E_2-\ldots-(k-1)E_m|$.

Claim 3.1. *$\boldsymbol{P}_0' = \boldsymbol{P}_0$+(fixed components) with $\boldsymbol{P}_0 = |(km-2-m)L-(k(m-1)-m)E_1-(k-2)E_2-\ldots-(k-2)E_m|$.*

Proof. From $T_m.\boldsymbol{P}_0' = -1$ it follows that T_m is a fixed component of $\boldsymbol{P}_0'$. Deleting T_m from $\boldsymbol{P}_0$ we obtain $|(km-4)L-(k(m-1)-2)E_1-(k-2)E_2-(k-1)E_3-\ldots-(k-1)E_m|$. In case $m=2$ this finishes the proof of the claim.

Assume $m>2$. The intersection number with E_{2m} is -1, hence E_{2m} is a fixed component. Deleting E_{2m} we obtain $|(km-4)L-(k(m-1)-2)E_1-(k-1)E_2-(k-2)E_3-(k-1)E_4-\ldots-(k-1)E_m|$. Continuing in this way one finds fixed components $E_{3m},\ldots,E_{m-1,m}$. Deleting them, one obtains $|(km-4)L-(k(m-1)-2)E_1-(k-1)E_2-\ldots-(k-1)E_{m-1}-(k-2)E_m|$. Now T_m is a fixed component. Deleting T_m one obtains $|(km-5)L-(k(m-1)-3)E_1-(k-2)E_2-(k-1)E_3-\ldots-(k-1)E_{m-1}-(k-2)E_m|$. In case $m=3$ this proves the claim. In case $m>3$ one has $E_{2m},\ldots E_{m-2,m},T_m$ again as fixed components. Deleting them this proves the claim for $m=4$; in case $m>4$ one continues.

For curves Γ' of $\boldsymbol{P}$ we need to investigate canonical adjoint curves containing intersections of Γ' with elements from $|L-E_1|$ (in terms of linear systems : containing a sum of divisors from g_k^1). For a general element R of $|L-E_1|$ the intersection of R with an element Γ_m of $\boldsymbol{P}$ not containing E_{1m} are k different points. The intersection multiplicity with an element of $\boldsymbol{P}_0$ is $k-2<k$. Therefore an element of $\boldsymbol{P}_0$ containing this intersection of Γ_m and R contains R as a component. Taking x general elements $R_1,\ldots,R_x$ in $|L-E_1|$, the elements of $\boldsymbol{P}_0$ containing $(R_1\cup\ldots\cup R_x)\cap\Gamma_m$ have $R_1,\ldots,R_x$ as components. Deleting $R_1,\ldots,R_x$ we obtain $\boldsymbol{P}_x' = |(km-2-m-x)L-(k(m-1)-m-x)E_1-(k-2)E_2-\ldots-(k-2)E_m|$. □

Claim 3.2. *Write $x=lm+y$ with $-1\le y\le m-2$. Then $\boldsymbol{P}_x' = \boldsymbol{P}_x$+(fixed components) with $\boldsymbol{P}_x = |(km-(l+1)m-y-2)L-(k(m-1)-(l+1)m+l-y)E_1-(k-l-2)E_2-\ldots-(k-l-2)E_m|$. (We do not claim that $\boldsymbol{P}_x$ has no more fixed components.)*

Proof. First, take $0\le x\le m-2$, hence $x=y$ and $l=0$. Then $\boldsymbol{P}_x' = \boldsymbol{P}_x$ and there is nothing to prove.

Next, take $x=m-1$, hence $l=1$; $y=-1$. Then $(E_{1m}.\boldsymbol{P}_{m-1}') = -1$ (the intersection number of E_{1m} with elements of $\boldsymbol{P}_{m-1}'$), therefore E_{1m} is a fixed component of $\boldsymbol{P}_{m-1}'$. Deleting E_{1m} we obtain $\boldsymbol{P}_{m-1}$.

More general, for any $x \geq m$ the curve E_{1m} is a fixed component of $\boldsymbol{P}'_x$. Deleting E_{1m} we obtain $|(km-2-m-x)L-(k(m-1)-m-x+1)E_1-(k-3)E_2-\ldots-(k-3)E_m|$. In case $m \leq x<2m-1$ this is $\boldsymbol{P}_x$. For $x=2m-1$ (hence $l=2$; $y=-1$) the intersection number of E_{1m} with that linear system is -1, hence E_{1m} is a fixed component. Deleting E_{1m} one obtains $\boldsymbol{P}_{2m-1}$. Continuing in this way one proves the claim. □

REMARK 3.3. Taking $x=(k-2)m+m-2$ (hence $l=k-2$; $y=m-2$) one finds $\boldsymbol{P}_{(k-2)m+m-2}=0$. For $x \geq (k-2)m+m-1$ one finds $\boldsymbol{P}_x=\emptyset$.

Given $0 \leq x \leq (k-2)m+m-2$ define the integer j by means of the inequalities $(j-1)m-1<x \leq jm-1$ with $j \leq k-1$.

Claim 3.4. $\dim(\boldsymbol{P}_x)=\frac{j(j-1)}{2}m-1+(x-(j-1)m+1)j-kx+\frac{(k-1)mk-2k+2}{2}-1.$

Proof. In case $x=(k-1)m-2$ we have to prove $\dim(\boldsymbol{P}_{(k-1)m-2})=0$. This follows from Remark 3.3.

Now, fix some $x<(k-1)m-2$ and assume the claim is proved for $x+1$ instead of x. Writing $x=lm+y$ with $-1 \leq y \leq m-2$, from the description in Claim 3.2 we find $\dim(\boldsymbol{P}_x) \geq [(km-(l+1)m-y+1))(km-(l+1)m-y-2)]/2-[(k(m-1)-(l+1)m+l-y+1)(k(m-1)-(l+1)m+l-y)]/2-[(m-1)(k-l-1)(k-l-2)]/2$. A computation shows us that we need to prove equality. Assume for x we have strict inequality. For $R \in |L-E_1|$ we have $(R.\boldsymbol{P}_x)=k-l-2$, hence R imposes at most $k-l-1$ conditions on $\boldsymbol{P}_x$. This implies $\dim(|\boldsymbol{P}_x-(L-E_1)|) \geq \dim(\boldsymbol{P}_x)-(k-l-1)$. In case $y=m-2$ one finds $((\boldsymbol{P}_x-(L-E_1)).E_{1m})<0$, hence $\dim(|\boldsymbol{P}_x-(L-E_1)-E_{1m}|) \geq dim(\boldsymbol{P}_x)-(k-l-1)$. But $|\boldsymbol{P}_x-(L-E_1)-E_{1m}|=\boldsymbol{P}_{x+1}$. One more computation shows that $\dim(\boldsymbol{P}_{x+1}) \geq dim(\boldsymbol{P}_x)-(k-l-1)$ gives a contradiction to the assumption that the claim holds for $\boldsymbol{P}_{x+1}$. In case $y<m-2$ then $|\boldsymbol{P}_x-(L-E_1)|=\boldsymbol{P}_{x+1}$ and again, a computation shows a contradiction.

On X_m we constructed the rational irreducible curve Γ_m belonging to $\boldsymbol{P}$. From Claim 2.1 we know that Γ_m is a nodal curve, so it has $g_0 := [(k^2-k)m-2k+2]/2$ ordinary nodes. We write s to denote a node of Γ_m. □

Claim 3.5. *We can arrange the nodes $s_1;\ldots;s_{g0}$ is such a way that the following property holds. First we introduce some notation: for $0 \leq \delta \leq g_0$ let $\boldsymbol{P}_x(s_1;\ldots;s_\delta)=\{\Gamma \in \boldsymbol{P}_x : s_i \in \Gamma$ for $1 \leq i \leq \delta\}$. Then $\boldsymbol{P}_x(s_1;\ldots;s_\delta)=\emptyset$ for $\delta>\dim(\boldsymbol{P}_x)$ and $\dim(\boldsymbol{P}_x(s_1;\ldots;s_\delta))=\dim(\boldsymbol{P}_x)-\delta$ if $\delta \leq \dim(\boldsymbol{P}_x)$.*

Proof. For $\delta=0$ there is nothing to prove.

Fix $\delta>0$ and assume the claim holds for $\delta-1$ instead of δ. So, we assume a suited arrangement $s_1;\ldots;s_{\delta-1}$ for a suitable part of the set of the nodes. We have to prove that the set of the remaining nodes of Γ_m contains a suited one to be numbered s_δ.

Numbers x satisfying $\delta - 1 > \dim(\boldsymbol{P}_x)$ impose no conditions on s_δ. Let x_0 be the minimal number such that $\delta - 1 \leq \dim(\boldsymbol{P}_x)$. We know that $\dim(\boldsymbol{P}_{x_0}(s_1; \dots; s_{\delta-1})) = \dim(\boldsymbol{P}_{x_0}) - (\delta - 1) \geq 0$. If each element of $\boldsymbol{P}_{x_0}(s_1; \dots; s_{\delta-1})$ would contain all the nodes of Γ_m then Γ_m possesses a canonically adjoint curve. Since Γ_m is a rational curve this is impossible. Hence, there exists a node s_0 such that $dim(\boldsymbol{P}_{x_0}(s_1; \dots; s_{\delta-1}; s_0)) = \dim(\boldsymbol{P}_{x_0}) - \delta$ (with $\boldsymbol{P}_{x_0}(s_1; \dots; s_{\delta-1}; s_0) = \emptyset$ if $\delta - 1 = dim(\boldsymbol{P}_{x_0})$). In case for all $x \leq x_0$ we find $\dim(\boldsymbol{P}_x(s_1; \dots; s_{\delta-1}; s_0)) = \dim(\boldsymbol{P}_x) - \delta$ then we can take $s_0 = s_\delta$.

Assume $x' < x_0$ such that $\dim(\boldsymbol{P}_{x'}(s_1; \dots; s_{\delta-1}; s_0)) = \dim(\boldsymbol{P}_{x'}) - \delta + 1$ while $\dim(\boldsymbol{P}_{x'+1}(s_1; \dots; s_{\delta-1}; s_0)) = \dim(\boldsymbol{P}_{x'+1}) - \delta$. Using a general $R \in |L - E_1|$ and using the arguments from the proof of Claim 3.4 one finds a contradiction. □

4. Proof of the theorem

Now, we finish the proof of the theorem in the introduction. We start with the rational irreducible nodal curve Γ_m on X_m. We make an arrangement of the nodes as in Claim 3.5. The main result in Section 2 of [8] implies that there exists a 1-dimensional flat family $Y \to T$ of curves on X_m belonging to $\boldsymbol{P}$ such that the fiber over a special point t_0 of T is the curve Γ_m and a general fiber is a nodal curve Γ with exactly $g_0 - g$ nodes such that those nodes specialize to the nodes $s_1; \dots; s_{g_0-g}$ on Γ_m. Define $\boldsymbol{P}_{x,\Gamma} = \{D \in \boldsymbol{P}_x : D \text{ contains the nodes of } \Gamma\}$. Clearly $\dim(\boldsymbol{P}_{x,\Gamma}) \geq dim(\boldsymbol{P}_x) - (g_0 - g)$. For the special fiber Γ_m we have $\boldsymbol{P}_x(s_1; \dots; s_{g_0-g}) = \emptyset$ if $g_0 - g > \dim(\boldsymbol{P}_x)$ and $\dim(\boldsymbol{P}_x(s_1; \dots; s_{g_0-g})) = \dim(\boldsymbol{P}_x) - (g_0 - g)$ if $g_0 - g \leq \dim(\boldsymbol{P}_x)$. Semicontinuity implies $\boldsymbol{P}_{x,\Gamma} = \emptyset$ if $g_0 - g > \dim(\boldsymbol{P}_x)$ and $\dim(\boldsymbol{P}_{x,\Gamma}) = \dim(\boldsymbol{P}_x) - (g_0 - g)$ if $g_0 - g \leq \dim(\boldsymbol{P}_x)$. Let C be the normalization of Γ. It is a smooth curve of genus g. The linear system $|L - E_1|$ induces a linear system g^1_k on C without base points. Taking x general elements $R_1; \dots; R_x$ in $|L - E_1|$ corresponds to taking x general divisors in g^1_k. From the description of $\boldsymbol{P}_x$ in 3.2 we find that $\dim(\boldsymbol{P}_{x,\Gamma})$ is equal to the dimension of canonically adjoint curves Γ containing the intersection of Γ with $R_1 \cup \dots \cup R_x$, hence it is equal to $\dim(|K_C - xg^1_k|)$. In particular, if $\boldsymbol{P}_{x,\Gamma} = \emptyset$ then $|K_C - xg^1_k| = \emptyset$. So, we find $|K_C - xg^1_k| = \emptyset$ if $g_0 - g > \dim(\boldsymbol{P}_x)$; $\dim(|K_C - xg^1_k|) = \dim(\boldsymbol{P}_x) - (g_0 - g)$ if $g_0 - g \leq \dim(\boldsymbol{P}_x)$. Using 3.4 and the Riemann-Roch Theorem one finds $\dim(|xg^1_k|) = \max\{\frac{j(j-1)}{2}m - 1 + (x - (j-1)m + 1)j; kx - g\}$. In particular $\dim(|(m-1)g^1_k|) = m - 1$ and $\dim(|mg^1_k|) = m + 1$. Since Γ is obtained from C using a linear subsystem of $|mg^1_k|$, one also finds $|mg^1_k|$ is birationally very ample. This finishes the proof of the theorem. □

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