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COMMUTATOR ESTIMATES AND A SHARP FORM OF GÅRDING'S INEQUALITY

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0. Introduction

In the present paper we show a commutator estimate of pseudo-differential operators in the framework of $L^2(\mathbb{R}^n)$. As an application we give a sharp form of Gårding's inequality for sesqui-linear forms with coeffiencients in $\mathcal{B}^2$. There has been similar kinds of commutator estimates. In [5], Kumano-go and Nagase obtain a result on commutator estimates and used it to show a sharp form of Gårding's inequality for sesqui-linear form defined by elliptic differential operators of the form

$$B[u,v] = \sum_{|\alpha|\leq m, |\beta|\leq m} (a_{\alpha\beta}(x) D_x^\alpha u, D_x^\beta v)$$

where the coefficients $a_{\alpha\beta}(x)$ are $\mathcal{B}^2(\mathbb{R}^n)$ functions.

In [3], Koshiba shows a sharp form of Gårding's inequality for the form

$$B[u,v] = (p(X, D_x)u, v)$$

where the symbol $p(x,\xi)$ of the operator $p(X, D_x)$ is $\mathcal{B}^2$ smooth in space variable x and homogeneous in covariable ξ, and used the sharp form of Gårding's inequality to the study of the stability of difference schemes for hyperbolic initial problems. On the other hand in [2], N. Jacob shows Gårding's inequality for the form

$$B[u,v] = \sum_{i,j=1}^{m} \int_{\mathbb{R}^n} \overline{a_{i,j}(x) Q_j(D) u(x)} P_i(D) v(x) dx$$

where $P_i(D)$ and $Q_j(D)$ are pseudo-differential operators, and $a_{i,j}(x)$ are non-smooth functions. The symbol class of the present paper is similar to the one in [2].

In section 1, as a preliminary we give definitions and fundamental facts of pseudo-differential operators. In section 2 we treat commutator estimates and give the main theorem relative to the commutator estimate. Finally in section 3 we give the sharp form of Gårding's inequalities for our class of operators.

1. Preliminaries

Let $\alpha = (\alpha_1, ..., \alpha_n)$ and $\beta = (\beta_1, ..., \beta_n)$ be multi-integers. We denote

$$|\alpha| = \alpha_1 + ... + \alpha_n$$

We denote n-dimensional partial differential operators by

$$\partial_\xi = (\frac{\partial}{\partial \xi_1}, ..., \frac{\partial}{\partial \xi_n}) \quad \text{and} \quad D_x = \frac{1}{i}\partial_x = \frac{1}{i}(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n})$$

Then for a function $f(x, \xi)$, we denote

$$\partial_\xi^\alpha {D_x}^\beta f(x, \xi) = f_{(\beta)}^{(\alpha)}(x, \xi)$$

and

$$\partial_\xi^\alpha {D_x}^\beta {D_{x'}}^{\beta'} f(x, \xi, x') = f_{(\beta, \beta')}^{(\alpha)}(x, \xi, x')$$

for a function $f(x, \xi, x')$. We denote by $\mathcal{B}^k = \mathcal{B}^k(\mathbb{R}^n)$ the set of k-times continuously differentiable functions on $\mathbb{R}^n$ which are bounded with all upto k-th derivatives. We denote by $C_0^\infty(\mathbb{R}^n)$ the set of C^∞-smooth functions with compact support. Moreover $\mathcal{S}$ denotes the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$. Let λ be a real valued smooth function on $\mathbb{R}^n$ satisfying

$$\text{(i)} \qquad \lambda(\xi) \geq 1$$
$$\text{(ii)} \qquad |\lambda^{(\alpha)}(\xi)| \leq C_\alpha \lambda(\xi)^{1-|\alpha|}$$

for any α. Then we say that the function $\lambda(\xi)$ is a basic weight function(see [5]).

Let $\lambda(\xi)$ be a basic weight function. Then we say that a function $p(x, \xi, x')$ on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ belongs to $S_{\rho,\delta,\lambda}^m$ if

$$|p_{(\beta,\beta')}^{(\alpha)}(x, \xi, x')| \leq C_{\alpha,\beta,\beta'} \lambda(\xi)^{m-\rho|\alpha|+\delta|\beta+\beta'|}$$

for any multi-integers α, β and β'. For any $p(x, \xi, x')$ in $S_{\rho,\delta,\lambda}^m$, we define the pseudo-differential operator $p(X, D_x, X')$ by

$$p(X, D_x, X')u(x) = \frac{1}{(2\pi)^n} \iint e^{i(x-x')\cdot\xi} p(x, \xi, x') u(x') dx' d\xi$$

for any u in $\mathcal{S}$. In the present paper the integrations $\int$ are taken on $\mathbb{R}^n$. In particular if $p(x, \xi, x') \in S_{\rho,\delta,\lambda}^m$ is independent in x', that is, $p(x, \xi) \in S_{\rho,\delta,\lambda}^m$, the operator $p(X, D_x)$ is defined, as usual, by

$$p(X, D_x)u(x) = \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} p(x, \xi) \hat{u}(\xi) d\xi$$

where $\hat{u}(\xi)$ denotes the Fourier transform of $u(x)$, that is,

$$\hat{u}(\xi) = \int e^{-ix'\cdot\xi} u(x')dx'$$

For any functions $f(x)$ and $g(x)$ on $\mathbb{R}^n$, we define the inner product of $L^2(\mathbb{R}^n)$ by

$$(f, g) = \int f(x)\overline{g(x)}dx$$

and denote the usual L^2 norm of function $f(x)$ by

$$||f|| = \{\int |f(x)|^2 dx\}^{\frac{1}{2}}$$

For any real number s and $u \in \mathcal{S}$, we define the norm $||u||_{s,\lambda}$ by

$$\begin{aligned} ||u||_{s,\lambda} &= \{\int |\lambda(D_x)^s u(x)|^2 dx\}^{\frac{1}{2}} \\ &= \{\frac{1}{(2\pi)^n}\int |\lambda(\xi)^s \hat{u}(\xi)|^2 d\xi\}^{\frac{1}{2}} \end{aligned}$$

In particular if $s = 0$, the norm $|| \cdot ||_{s,\lambda}$ coincides with usual L^2 norm $|| \cdot ||$.

The space $H_{s,\lambda}$ is defined by the completion of the space $\mathcal{S}$ by the norm $|| \cdot ||_{s,\lambda}$. It is not difficult to see that the space $H_{s,\lambda}$ is a Hilbert space.

Let s and m be real numbers. For a symbol $p(x, \xi, x')$ in $S^m_{\rho,\delta,\lambda}$, we have

$$||p(X, D_x, X')u||_{s,\lambda} \leq C||u||_{s+m,\lambda}$$

for any u in $\mathcal{S}$ (see [5]).

2. Estimates of commutators

Let us consider commutators of pseudo-differential operators in $L^2(\mathbb{R}^n)$. The estimates is essential for the proof of sharp Gårding's inequality. However the estimates itself are interesting subject.

Let $0 \leq \delta < 1$ and $\lambda(\xi)$ be a basic weight function and $\phi(x)$ be an even function in $C_0^\infty(\mathbb{R}^n)$ satisfying $\int \phi(x)dx = 1$. For a function $b(x)$ on $\mathbb{R}^n$, we define

$$\tilde{b}(x, \xi) = \int \phi(z)b(x - \lambda(\xi)^{-\delta}z)dz$$

Then in [5], the following approximation theorem is shown.

Lemma 2.1. *If $b(x)$ is a bounded function, then $\tilde{b}(x,\xi)$ belongs to $S^0_{1,\delta,\lambda}$.*

(i) *If $b(x)$ is a function in $\mathcal{B}^1$, then $\tilde{b}_{(\alpha)}(x,\xi)$ belongs to $S^0_{1,\delta,\lambda}$ for $|\alpha| \le 1$ and we have*

$$||\{b(X) - \tilde{b}(X,D_x)\}u|| \le C||u||_{-\delta,\lambda}$$

for any u in $\mathcal{S}$.

(ii) *If $b(x)$ is a function in $\mathcal{B}^2$, then $\tilde{b}_{(\alpha)}(x,\xi)$ belongs to $S^0_{1,\delta,\lambda}$ for $|\alpha| \le 2$ and we have*

$$||\{b(X) - \tilde{b}(X,D_x)\}u|| \le C||u||_{-2\delta,\lambda}$$

for any u in $\mathcal{S}$.

From Lemma 2.1 we can prove the following lemma 2.2.

Lemma 2.2. (i) *If $b(x)$ is in $\mathcal{B}^1$, then we have*

$$||\{b(X) - \tilde{b}(X,D_x)\}u||_{\delta,\lambda} \le C||u||$$

for any u in $\mathcal{S}$.

(ii) *If $b(x)$ is a $\mathcal{B}^2$, then we have*

$$||\{b(X) - \tilde{b}(X,D_x)\}u||_{2\delta,\lambda} \le C||u||$$

for any u in $\mathcal{S}$.

Proof. We prove (i), and (ii) can be shown in a similar way.

For any u and v in $\mathcal{S}$, we have

$$(\{b(X) - \tilde{b}(X,D_x)\}u, v) = (u, \{\bar{b}(X) - \bar{b}(D_x, X')\}v)$$

where $\bar{b}(\xi, x') = \overline{\tilde{b}(x',\xi)}$. Then by using the asymptotic expansion formula of pseudo-differential operators (see, for example [4]), we have

$$\bar{b}(D_x, X') = \bar{\bar{b}}(X,D_x) + b_1(X,D_x)$$

where

$$b_1(x,\xi) = \sum_{j=1}^{N} \sum_{|\alpha|=j} \frac{1}{\alpha!}\tilde{b}^{(\alpha)}_{(\alpha)}(x,\xi) + R_N(x,\xi) \quad \text{and } R_N(x,\xi) \in S^{-N(1-\delta)}_{1,\delta,\lambda}$$

Since $b(x) \in \mathcal{B}^1$, we can see by Lemma 2.1 (i) that

$$\tilde{b}^{(\alpha)}_{(\alpha)}(x,\xi) \in S^{-(1-\delta)|\alpha|-\delta}_{1,\delta,\lambda} \quad \text{for } |\alpha| \neq 0$$

Hence taking N sufficiently large we can see

$$b_1(x,\xi) \in S^{-1}_{1,\delta,\lambda}$$

Using Lemma 2.1 (i) and the boundedness of pseudo-differential operators we have

$$\begin{aligned} ||\{\bar{b}(X) - \bar{b}(D_x, X')\}v|| &\leq C||\{b(X) - \tilde{b}(X, D_x)\}v|| + ||b_1(X, D_x)v|| \\ &\leq C||v||_{-\delta,\lambda} \end{aligned}$$

Therefore by Schwarz inequality and duality argument of the spaces $H_{s,\lambda}$, we have the estimate.

□

In order to show the main estimate in this section, the following theorem plays an essential role.

Theorem 2.3. *Let $b(x)$ be a function in $\mathcal{B}^2$, and let $0 \leq \delta < 1$. For a basic weight function $\lambda(\xi)$ we define a symbol $\tilde{b}(x,\xi)$ by*

$$\tilde{b}(x,\xi) = \int \phi(z) b(x - \lambda(\xi)^{-\delta} z) dz$$

where $\phi(x)$ is an even function in $\mathcal{S}$ with $\int \phi(x)dx = 1$. Then for any $s \in [0, 2\delta]$ we have

$$||\{b(X) - \tilde{b}(X, D_x)\}u||_{s,\lambda} \leq C||u||_{s-2\delta,\lambda}$$

for any u in $\mathcal{S}$.

Proof. For the proof, we use the three line theorem in complex analysis.

Let u and v be functions in $\mathcal{S}$ and we consider the complex function

$$f(z) = (\lambda(D_x)^{2\delta(1-z)}\{b(X) - \tilde{b}(X, D_x)\lambda(D_x)^{2\delta z}u,\ v)$$

Since u and v are in $\mathcal{S}$ and $\lambda(\xi) \geq 1$, it is clear that the function $f(z)$ is holomorphic in the complex $z = \sigma + i\tau$-plane $\mathbb{C}$. Since the symbol $\lambda(\xi)^z$ is in $S^{\mathrm{Re}z}_{1,0,\lambda}$, independent of x and $|\lambda(\xi)^{i\tau}| = 1$, we can see from the Lemma 2.1 and 2.2 that

$$\begin{aligned} |f(i\tau)| &\leq ||\lambda(D_x)^{2\delta}\{b(X) - \tilde{b}(X, D_x)\}\lambda(D_x)^{2i\delta\tau}u||\ ||v|| \leq C||u||\ ||v|| \\ |f(1+i\tau)| &\leq ||\{b(X) - \tilde{b}(X, D_x)\}\lambda(D_x)^{2\delta(1+i\tau)}u||\ ||v|| \leq C||u||\ ||v|| \end{aligned}$$

Hence from the three line theorem (see [7]), we have

$$|f(\sigma)| \leq C||u||\,||v||$$

for $0 < \sigma < 1$. Taking $\sigma = 1 - \frac{s}{2\delta}$, we have

$$|(\lambda(D_x)^s\{b(X) - \tilde{b}(X, D_x)\}\lambda(D_x)^{-s+2\delta}u,\ v)| \leq C||u||\,||v||$$

for any u and v in $\mathcal{S}$. Hence using the usual duality argument, we can get the inequality. □

Theorem 2.3 implies the following estimate.

Theorem 2.4. *Let $b(x)$ be a function in $\mathcal{B}^2$ and let $a(\xi)$ in $S^s_{1,0,\lambda}$ with $0 < s < 1$. Then we have*

$$||[a(D_x), b(X)]u||_{0,\lambda} \leq C||u||_{s-1,\lambda} \tag{2.1}$$

for any u in $\mathcal{S}$.

Proof. We take an even function $\phi(x)$ in $C_0^\infty(\mathbb{R}^n)$ such that $\int \phi(x)dx = 1$. For $b(x)$ we define a symbol $\tilde{b}(x, \xi)$ by

$$\tilde{b}(x, \xi) = \int \phi(z)b(x - \lambda(\xi)^{-\delta}z)dz$$

for $s < \delta = \frac{1+s}{2} (< 1)$. We write

$$\begin{aligned}[a(D_x), b(X)]u = a(D_x)&\{b(X) - \tilde{b}(X, D_x)\}u(x)\\ &+ a(D_x)\tilde{b}(X, D_x)u(x) - \tilde{b}(X, D_x)a(D_x)u(x)\\ &+ \{b(X) - \tilde{b}(X, D_x)\}a(D_x)u(x)\end{aligned}$$

Since $a(\xi) \in S^s_{1,0,\lambda}$, we can see that the first term can be estimated by

$$||a(D_x)\{b(X) - \tilde{b}(X, D_x)\}u(x)||_{0,\lambda} \leq C||\lambda(D_x)^s\{b(X) - \tilde{b}(X, D_x)\}u(x)||_{0,\lambda}$$

and therefore by Theorem 2.3 we have

$$\begin{aligned}||a(D_x)\{b(X) - \tilde{b}(X, D_x)\}u(x)||_{0,\lambda} &\leq C||u||_{s-2\delta,\lambda}\\ &\leq C||u||_{s-1,\lambda}\end{aligned} \tag{2.2}$$

The third term is estimated by Lemma 2.1(ii) and we have

$$\begin{aligned} \|\{b(X) - \tilde{b}(X, D_x)\}a(D_x)u(x)\| &\leq C\|a(D_x)u\|_{-2\delta,\lambda} \\ &\leq C\|a(D_x)u\|_{s-1,\lambda} \end{aligned} \tag{2.3}$$

The second term is estimated by the usual asymptotic expansion formula for pseudo-differential operators (see [4]), that is, we have

$$a(D_x)\tilde{b}(X, D_x) = b_L(X, D_x)$$

and

$$b_L(x, \xi) \sim \tilde{b}(x, \xi)a(\xi) + \sum_{j=1}^{\infty} b_j(x, \xi)$$

$$b_j(x, \xi) = \sum_{|\alpha|=j} \frac{1}{a!}\tilde{b}_{(\alpha)}(x, \xi)a^{(\alpha)}(\xi)$$

Since the symbols $\tilde{b}_{(\alpha)}(x, \xi)$ belong to $S^0_{1,\delta,\lambda}$ for $|\alpha| \leq 2$, we can see that $b_1(x, \xi) \in S^{s-1}_{1,\delta,\lambda}$ and $b_j(x, \xi) \in S^{s-j+(j-2)\delta}_{1,\delta,\lambda}$ for $j \geq 2$. Hence we can write

$$a(D_x)\tilde{b}(X, D_x)u - \tilde{b}(X, D_x)a(D_x)u = B_1(X, D_x)u$$

where $B_1(x, \xi)$ belongs to $S^{s-1}_{1,\delta,\lambda}$ Therefore we have

$$\begin{aligned} \|[a(D_x)\tilde{b}(X, D_x) - \tilde{b}(X, D_x)a(D_x)]u(x)\| &\leq \|B_1(X, D_x)u\| \\ &\leq C\|u(x)\|_{s-1,\lambda} \end{aligned} \tag{2.4}$$

From the estimates (2.2), (2.3) and (2.4), we have the estimate (2.1).

□

In particular we have

Corollary 2.5. *Let $b(x)$ be a function in $\mathcal{B}^2$ and let $a(\xi)$ be in $S^{\frac{1}{2}}_{1,0,\lambda}$. Then we have*

$$\|[a(D_x), b(X)]u\|_{0,\lambda} \leq C\|u\|_{-\frac{1}{2},\lambda}$$

for any u in $\mathcal{S}$.

REMARK 1. We note that if the basic weight function $\lambda(\xi) =<\xi>= (1 + |\xi|^2)^{\frac{1}{2}}$, we can get more general results for $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ than the ones in this section. Because in case of $<\xi>$, we can use the kernel representations of the operators(see [5]). We can see sharper results in [1] than in [5] in the case $\lambda(\xi) =<\xi>$.

3. A sharp form of Gårding's inequality

Let us begin with the following inequality, which we can say a sharp form of Gårding's inequality.

Theorem 3.1. *Let $\delta < 1$. We assume that a symbol $p(x,\xi)$ in $S^m_{1,\delta,\lambda}$ satisfies that $p_{(\beta)}(x,\xi)$ are in $S^m_{1,\delta,\lambda}$ for $|\beta| \le 2$ and*

$$\mathrm{Re}\, p(x,\xi) \ge 0$$

for some constant c_0. Then we have

$$\mathrm{Re}\ (p(X,D_x)u,\ u) \ge -C\|u\|^2_{\frac{m-1}{2},\lambda}$$

for any u in $\mathcal{S}$.

For the proof of this theorem 3.1 we use the following lemma

Lemma 3.2. (see [5]) *Let τ be a real number. Let $\psi(x)$ be a infinitely smooth function on $\mathbb{R}^n$, and let $\lambda(\xi)$ bea basic weight function. Then for any α we have*

$$\partial_\xi^\alpha\{\psi(\lambda(\xi)^\tau x)\} = \sum_{|\alpha'|\le|\alpha|} \phi_{\alpha',\alpha}(\xi)\{\lambda(\xi)^\tau x\}^{\alpha'}\psi^{(\alpha')}(\lambda(\xi)^\tau x)$$

where $\phi_{\alpha',\alpha}(\xi)$ belong to $S^{-|\alpha|}_{\lambda,1,0}$ for all α' with $|a'| \le |\alpha|$.

Proof of Theorem 3.1. First we note that we can assume that the symbol $p(x,\xi)$ be real-valued. In fact, if a real-valued symbol $r(x,\xi) \in S^m_{1,\delta,\lambda}$ satisfies the same assumption of $p(x,\xi)$ in Theorem 3.1, then we have

$$\mathrm{Re}(ir(X,D_x)u,\ u) = \frac{1}{2}\mathrm{Im}(\{r(X,D_x) - r^*(X,D_x)\}u,\ u)$$

Since the symbol $r(x,\xi)$ is real-valued, by using the expansion formula for the symbol of the formal adjoint operator $r^*(X,D_x)$ we have

$$r^*(x,\xi) \sim r(x,\xi) + \sum_{j=1}^{\infty}\sum_{|\alpha|=j}\frac{1}{\alpha!}r^{(\alpha)}_{(\alpha)}(x,\xi)$$

Therefore using the assumption we have

$$r(x,\xi) - r^*(x,\xi) = R(x,\xi) \in S^{m-1}_{1,\delta,\lambda}$$

From this relation we have

$$\begin{aligned}|\mathrm{Re}(ir(X,D_x)u,\ u)| &= \frac{1}{2}|\mathrm{Im}(\{r(X,D_x)-r^*(X,D_x)\}u,\ u)| \\ &= \frac{1}{2}|(R(X,D_x)u,\ u)| \\ &\le C||u||^2_{\frac{m-1}{2},\lambda}\end{aligned}$$

Now we assume that the symbol $p(x,\xi)$ is real and non-negative. We take an even and real-valued function $\psi(x)$ in $\mathcal{S}$ such that $\int \psi(x)^2 dx = 1$ and we put

$$p_G(x,\xi,x') = \int \psi(\lambda(\xi)^{\frac{1}{2}}(x-z))\psi(\lambda(\xi)^{\frac{1}{2}}(x'-z))p(z,\xi)dz\lambda(\xi)^{\frac{n}{2}}$$

Then using the lemma 3.2 we can see that $p_G(x,\xi,x')$ is in $S^m_{1,\frac{1}{2},\lambda}$ and changing the order of integrations we have

$$\mathrm{Re}(p_G(X,D_x,X')w,\ w) \ge 0$$

for any $w \in \mathcal{S}$(see [6]). Moreover by using the formula of simplified symbols in [4] for the operator with double symbols we can see that the operator $p_G(X,D_x,X')$ can be written asymptotically as

$$p_G(X,D_x,X') \sim \sum_{j=0}^{\infty} p_j(X,D_x)$$

where $p_j(x,\xi)$ is in $S^{m-\frac{j}{2}}_{1,\frac{1}{2},\lambda}$ for any j and has the form

$$p_j(x,\xi) = \sum_{|\alpha|=j} \frac{1}{\alpha!} p^{(\alpha)}_{G,(\alpha)}(x,\xi,x)$$

In particular, $p_0(x,\xi)$ can be written as

$$\begin{aligned}p_0(x,\xi) &= p_G(x,\xi,x) \\ &= \lambda(\xi)^{\frac{n}{2}} \int \psi(\lambda(\xi)^{\frac{1}{2}}(x-z))^2 p(z,\xi)dz \\ &= \lambda(\xi)^{\frac{n}{2}} \int \psi(\lambda(\xi)^{\frac{1}{2}}z)^2 p(x-z,\xi)dz \\ &= \int \psi(z)^2\ p(x-\lambda(\xi)^{-\frac{1}{2}}z,\xi)dz\end{aligned}$$

By using the Taylor expansion for the second expression, we have

$$p(x-z,\xi) = p(x,\xi) + \sum_{|\beta|=1} ip_{(\beta)}(x,\xi)z^{\beta} + R_2(z,x,\xi)$$

where the remainder term $R_2(z,x,\xi)$ is

$$R_2(z,x,\xi) = \sum_{|\beta|=2} \frac{-2}{\beta!} \int_0^1 (1-t) z^{\beta} p_{(\beta)}(x-tz,\xi)dt$$

Since $\psi(z)$ is an even function we see that

$$\int \psi(\lambda(\xi)^{\frac{1}{2}} z)^2 z^{\beta} dz = 0$$

for $|\beta| = 1$. Therefore we have

$$\begin{aligned} p_0(x,\xi) &= \lambda(\xi)^{\frac{n}{2}} \int \psi(\lambda(\xi)^{\frac{1}{2}} z)^2 p(x,\xi) dz + \lambda(\xi)^{\frac{n}{2}} \int \psi(\lambda(\xi)^{\frac{1}{2}} z)^2 R_2(z,x,\xi)\} \, dz \\ &= p(x,\xi) + \lambda(\xi)^{\frac{n}{2}} \sum_{|\beta|=2} \frac{-2}{\beta!} \int_0^1 (1-t) \int z^{\beta} \psi(\lambda(\xi)^{\frac{1}{2}} z)^2 p_{(\beta)}(x-tz,\xi) \, dz \\ &= p(x,\xi) + r_2(x,\xi) \end{aligned}$$

From the assumption of the symbol $p(x,\xi)$, the symbols $p_{(\beta)}(x-tz,\xi)$ belong to $S^m_{\lambda,1,\delta}$ for $|\beta| = 2$. Hence using Lemma 3.2, we can see that

$$r_2(x,\xi) = \lambda(\xi)^{\frac{n}{2}} \sum_{|\beta|=2} \frac{-2}{\beta!} \int_0^1 (1-t) \cdot \int z^{\beta} \psi(\lambda(\xi)^{\frac{1}{2}} z)^2 p_{(\beta)}(x-tz,\xi) \, dz$$

is in $S^{m-1}_{\lambda,1,\delta}$. Thus we can write

$$p_0(x,\xi) = p(x,\xi) + r_2(x,\xi)$$

with symbol $r_2(x,\xi)$ in $S^{m-1}_{\lambda,1,\delta}$.

Similarly for $|\alpha| = 1$, since

$$p^{(\alpha)}_{G,(\alpha)}(x,\xi,x) = \partial^{\alpha}_{\xi}\{\lambda(\xi)^{\frac{n+1}{2}} \int \psi(\lambda(\xi)^{\frac{1}{2}} z) \psi^{(\alpha)}(\lambda(\xi)^{\frac{1}{2}} z) p(x-z,\xi) dz\}$$

we can see that

$$p^{(\alpha)}_{G,(\alpha)}(x,\xi,x) = \partial^{\alpha}_{\xi}\{\lambda(\xi)^{\frac{n+1}{2}} \int \psi(\lambda(\xi)^{\frac{1}{2}} z) \psi^{(\alpha)}(\lambda(\xi)^{\frac{1}{2}} z) \{p(x,\xi) + R_1(z,x,\xi)\} dz\}$$

where the remainder term $R_1(z,x,\xi)$ is

$$R_1(z,x,\xi) = \sum_{|\beta|=1} i\int_0^1 (1-t)z^\beta p_{(\beta)}(x-tz,\xi)dt$$

Since $\int \psi(z)\psi^{(\alpha)}(z)dz = 0$ for $|\alpha| = 1$, we can see that

$$\begin{aligned} p^{(\alpha)}_{G,(\alpha)}(x,\xi,x) =& \partial_\xi^\alpha\{\lambda(\xi)^{\frac{n+1}{2}} \int \psi(\lambda(\xi)^{\frac{1}{2}}z)\psi^{(\alpha)}(\lambda(\xi)^{\frac{1}{2}}z)R_1(z,x,\xi)\, dz\} \\ =& \sum_{|\beta|=1} i\int_0^1 (1-t)dt \\ &\times \partial_\xi^\alpha\{\lambda(\xi)^{\frac{n+1}{2}} \int \psi(\lambda(\xi)^{\frac{1}{2}}z)\psi^{(\alpha)}(\lambda(\xi)^{\frac{1}{2}}z)z^\beta p_{(\beta)}(x-tz,\xi)\, dz\} \end{aligned}$$

for $|\alpha| = 1$. In a similar way to the estimate of $R_2(z,x,\xi)$, we can see from Lemma 3.2 and the assumption of the symbol $p(x,\xi)$ that $p_{1,\alpha}(x,\xi) = p^{(\alpha)}_{G,(\alpha)}(x,\xi,x)$ belongs to $S^{m-1}_{\lambda,1,\delta}$ for $|\alpha| = 1$. Therefore we can see that

$$p_1(x,\xi) \in S^{m-1}_{\lambda,1,\delta}$$

Thus we can write

$$p_G(X,D_x,X') = p(X,D_x) + R(X,D_x) + Q(X,D_x)$$

where $Q(x,\xi) \in S^{m-1}_{\lambda,1,\delta}$ and $R(x,\xi) \in S^{m-1}_{\lambda,1,\frac{1}{2}}$. Now from the L^2-boundedness theorems and the algebra of pseudo-differential operators with symbols in $\cup_{m\in\mathbb{R}} S^m_{\lambda,1,\delta}$ for any δ with $0 \le \delta < 1$, we can see that

$$\begin{aligned} |(Q(X,D_x)u,\ u)| &\le ||Q(X,D_x)u||_{\frac{-m+1}{2},\lambda}||u||_{\frac{m-1}{2},\lambda} \le C||u||_{\frac{m-1}{2},\lambda}{}^2 \\ |(R(X,D_x)u,\ u)| &\le ||R(X,D_x)u||_{\frac{-m+1}{2},\lambda}||u||_{\frac{m-1}{2},\lambda} \le C||u||_{\frac{m-1}{2},\lambda}{}^2 \end{aligned}$$

Therefore we have

$$\begin{aligned} &\mathrm{Re}(p(X,D_x)u,\ u) \\ &\quad = \mathrm{Re}(p_G(X,D_x,X')u,\ u) - \mathrm{Re}(R(X,D_x)u,\ u) - \mathrm{Re}(Q(X,D_x)u,\ u) \\ &\quad \ge -|(Q(X,D_x)u,\ u)| - |(R(X,D_x)u,\ u)| \\ &\quad \ge -C||u||_{\frac{m-1}{2},\lambda}{}^2 \end{aligned}$$

□

If a function $\lambda(\xi)$ is a basic weight function, then we can see that for $0 < \rho \le 1$ the fractional power $\lambda(\xi)^\rho$ is also a basic weight function. Using this fact and Theorem 3.1 we have

Corollary 3.3. *Let $0 \leq \delta < \rho \leq 1$. We assume that a symbol $p(x,\xi)$ in $S^m_{\rho,\delta,\lambda}$ satisfies that $p_{(\beta)}(x,\xi)$ are in $S^m_{\rho,\delta,\lambda}$ for $|\beta| \leq 2$ and*

$$\mathrm{Re}\, p(x,\xi) \geq 0$$

Then we have

$$\mathrm{Re}\,(p(X,D_x)u,\ u) \geq -C||u||^2_{\frac{m-\rho}{2},\lambda}$$

for any u in $\mathcal{S}$.

Proof. Since $\lambda(\xi)^\rho$ is a basic weight function, we see that

$$S^m_{\rho,\delta,\lambda} = S^{\frac{m}{\rho}}_{1,\frac{\delta}{\rho},\lambda^\rho}$$

Hence we can see that the symbol $p(x,\xi)$ in Corollary satisfies the assumptions of the one in Theorem 3.1 as the class of symbols in $S^{\frac{m}{\rho}}_{1,\frac{\delta}{\rho},\lambda^\rho}$. Therefore we see that

$$\begin{aligned}\mathrm{Re}\,(p(X,D_x)u,\ u) &\geq -C||u||^2_{(\frac{m}{\rho}-1)/2,\lambda^\rho}\\ &= -C||u||^2_{\frac{m-\rho}{2},\lambda}\end{aligned}$$

□

REMARK 2. Let $0 < \rho < 1$. Then we can show a similar sharp form of Garding's inequality to Theorem 3.1, under the assumption that the symbol $p(x,\xi)$ belongs to $S^m_{\rho,\rho,\lambda}$ and $p_{(\beta)}(x,\xi)$ belongs to $S^m_{\rho,\rho,\lambda}$ for any β with $|\beta| \leq 2$, by using the similar approximation $p_G(x,\xi,x')$ defined by

$$p_G(x,\xi,x') = \int \psi(\lambda(\xi)^{\frac{\rho}{2}}(x-z))\psi(\lambda(\xi)^{\frac{\rho}{2}}(x'-z))p(z,\xi)dz\lambda(\xi)^{\frac{n\rho}{2}}$$

and L^2-boundedness theorem(Theorem of Calderon and Vaillancourt, see [4]) of operators with symbols in $S^0_{\rho,\rho,\lambda}$.

Now using the commutator estimates in section 2 we can show the following sharp form of Garding's inequality.

Theorem 3.4. *Let $a_j(\xi)$ and $c_j(\xi)$ be in $S^m_{1,0,\lambda}$ and let $b_j(x)$ be $\mathcal{B}^2$ functions for $j = 1,\dots,N$. We assume that*

$$\mathrm{Re}\sum_{j=1}^{N} a_j(\xi)b_j(x)c_j(\xi) \geq 0$$

Then there exists a positive constant C such that

$$\mathrm{Re}\sum_{j=1}^{m}(b_j(X)a_j(D_x)u,\ c_j(D_x)u) \geq -C||u||^2_{m-\frac{1}{2},\lambda}$$

for any $u \in \mathcal{S}$.

Proof. We set $\tilde{a_j}(\xi) = \lambda(\xi)^{-m+\frac{1}{2}}a_j(\xi)$ and $\tilde{c_j}(\xi) = \lambda(\xi)^{-m+\frac{1}{2}}c_j(\xi)$. From the assumption we see that $\tilde{a}_j(\xi)$ and $\tilde{c}_j(\xi)$ are in $S^{\frac{1}{2}}_{1,0,\lambda}$. So writing

$$\sum_{j=1}^{N}(b_j(X)a_j(D_x)u,\ c_j(D_x)u) = \sum_{j=1}^{N}(b_j(X)\tilde{a_j}(D_x)\lambda(D_x)^{m-\frac{1}{2}}u,\ \tilde{c_j}(D_x)\lambda(D_x)^{m-\frac{1}{2}}u)$$

we put

$$\begin{aligned}\sum_{j=1}^{N}(b_j(X)a_j(D_x)u,\ c_j(D_x)u) &= \sum_{j=1}^{N}(b_j(X)\tilde{c_j}(D_x)\tilde{a_j}(D_x)v,\ v)\\ &\quad+\sum_{j=1}^{N}([\tilde{c_j}(D_x),b_j(X)]\tilde{a_j}(D_x)v,\ v)\\ &= I + II\end{aligned} \tag{3.1}$$

where $v = \lambda(D_x)^{m-\frac{1}{2}}u$. Since $\tilde{a}_j(\xi)$ and $\tilde{c}_j(\xi)$ are in $S^{\frac{1}{2}}_{1,0,\lambda}$, using the commutator estimate in Corollary 2.5 we can see that

$$||[\tilde{c_j}(D_x),b_j(X)]\tilde{a_j}(D_x)v|| \leq C||v|| = C||u||_{m-\frac{1}{2},\lambda}$$

Hence the second term II of (3.1) can be estimated by

$$|II| \leq \sum_{j=1}^{N}|([\tilde{c_j}(D_x),b_j(X)]\tilde{a_j}(D_x)v,v)| \leq C||u||_{m-\frac{1}{2},\lambda}{}^2$$

Now we consider the operator

$$p(X,D_x) = \sum_{j=1}^{N}b_j(X)\tilde{c_j}(D_x)\tilde{a_j}(D_x)$$

with symbol

$$p(x,\xi) = \sum_{j=1}^{N}b_j(x)\tilde{c_j}(\xi)\tilde{a_j}(\xi)$$

For the symbol $p(x,\xi)$ we define a new symbol $\tilde{p}(x,\xi)$ by

$$\begin{aligned}\tilde{p}(x,\xi) &= \int \phi(y)p(x-\lambda(\xi)^{-\frac{1}{2}}y,\xi)dy \\ &= \int \phi(\lambda(\xi)^{\frac{1}{2}}(x-y))p(y,\xi)dy\lambda(\xi)^{\frac{n}{2}}\end{aligned}$$

where $\phi(x)$ is a non-negative function in $C_0^\infty(\mathbb{R}^n)$ with $\int \phi(x)dx = 1$. Then by Lemma 2.1 (ii) we can see that the symbol $\tilde{p}(x,\xi)$ belongs to $S^1_{1,\frac{1}{2},\lambda}$, $\tilde{p}_{(\beta)}(x,\xi)$ belongs to $S^1_{1,\frac{1}{2},\lambda}$ for any $|\beta| \le 2$ and satisfies

$$||\{p(X,D_x)) - \tilde{p}(X,D_x)\}v|| \le \sum_{j=1}^N ||\{b_j(X) - \tilde{b_j}(X,D_x)\}\tilde{c_j}(D_x)\tilde{a_j}(D_x)v||$$

where

$$\tilde{b_j}(x,\xi) = \int \phi(y)b_j(x-\lambda(\xi)^{-\frac{1}{2}}y)dy$$

Then by Lemma 2.1 (ii) we have

$$||\{b_j(X) - \tilde{b_j}(X,D_x)\}w|| \le C||w||_{-1,\lambda}$$

for any $w \in \mathcal{S}$ and therefore we have

$$\begin{aligned}||\{p(X,D_x)) - \tilde{p}(X,D_x)\}v|| &\le C\sum_{j=1}^N ||\tilde{c}_j(D_x)\tilde{a}_j(D_x)v||_{-1,\lambda} \\ &\le C||v|| \qquad (3.2)\end{aligned}$$

Moreover $\tilde{p}(x,\xi)$ satisfies

$$\text{Re}\,\tilde{p}(x,\xi) \ge 0$$

Thus the symbol $\tilde{p}(x,\xi)$ satisfies the assumption in the Theorem 3.1 with $m = 1$ and $\delta = \frac{1}{2}$. Therefore we have

$$\text{Re}\,(\tilde{p}(X,D_x)v,\ v) \ge -C||u||_{\frac{1}{2},\lambda}{}^2$$

From (3.1) we see

$$\begin{aligned}\text{Re}\sum_{j=1}^N(b_j(X)a_j(D_x)u,c_j(D_x)u) &= \text{Re}\{I+II\} \\ &= \text{Re}(\tilde{p}(X,D_x)v,v) + \text{Re}(\{p(X,D_x) - \text{Re}(\tilde{p}(X,D_x)\}v,v) + II \\ &\ge -||\{p(X,D_x) - \text{Re}(\tilde{p}(X,D_x)\}v||\cdot||v|| - |II| \\ &\ge -C||v||^2\end{aligned}$$

Hence we have the theorem.

□

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