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ON THE EULER CHARACTERISTIC OF THE ORBIT SPACE OF A PROPER Γ-COMPLEX

Dedicated to Professor Fuichi Uchida on his 60th birthday

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1. Introduction

Let Γ be a discrete groups. A Γ-CW-complex X is said to be Γ-*finite* if it satisfies the following two conditions:

1. For each cell σ of X, the isotropy subgroup Γ_σ of σ is of finite order.
2. The orbit space X/Γ is a finite complex.

In other words, X is Γ-finite if the action of Γ on X is *proper* and *cocompact.* For a Γ-finite Γ-CW-complex X, define the *equivariant Euler characteristic* $e(\Gamma, X)$ by

$$e(\Gamma, X) = \sum_{\sigma\in\mathcal{E}}(-1)^{\dim\sigma}\frac{1}{|\Gamma_\sigma|} \in \mathbb{Q},$$

where $\mathcal{E}$ is a set of representatives of Γ-orbits of cells of X and $|\Gamma_\sigma|$ is the order of Γ_σ. We agree $e(\Gamma, X) = 0$ when $X = \emptyset$. The equivariant Euler characteristic and its variants appear in various contexts of mathematics. See [3, 4, 9, 11] for instance. In particular, when X is a manifold, the orbit space X/Γ can be regarded as an orbifold and $e(\Gamma, X)$ is the *orbifold Euler characteristic* of X/Γ in the sense of [11].

In this paper, we prove the formula expressing the Euler characteristic of the orbit space of a Γ-finite Γ-CW-complex in terms of equivariant Euler characteristics. More precisely, let X be a Γ-finite Γ-CW-complex. For each $\gamma \in \Gamma$, the centralizer $C_\Gamma(\gamma)$ acts on the fixed point set X^γ. In this way X^γ is naturally a $C_\Gamma(\gamma)$-finite $C_\Gamma(\gamma)$-CW-complex and hence $e(C_\Gamma(\gamma), X^\gamma)$ is defined for each $\gamma \in \Gamma$. Our result is:

Theorem 1. *Let Γ be a discrete group, and X a Γ-finite Γ-CW-complex. Then*

$$\chi(X/\Gamma) = \sum_{\gamma\in\mathcal{F}(\Gamma)} e(C_\Gamma(\gamma), X^\gamma), \tag{1}$$

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where $\mathcal{F}(\Gamma)$ is a set of representatives of conjugacy classes of elements of finite order in Γ.

Note that there are finitely many γ's in $\mathcal{F}(\Gamma)$ with $X^\gamma \neq \emptyset$ and hence the summation in (1) makes sense.

In case a Γ-finite Γ-CW-complex X is a smooth Γ-manifold, then X^γ is a submanifold of X for each $\gamma \in \Gamma$ of finite order, so that the terms $e(C_\Gamma(\gamma), X^\gamma)$ in (1) can be regarded as orbifold Euler characteristics of orbifolds $X^\gamma/C_\Gamma(\gamma)$. Thus Theorem 1 gives the expression of the Euler characteristic of X/Γ in terms of orbifold Euler characteristics of $X^\gamma/C_\Gamma(\gamma)$.

When Γ is a finite group, a Γ-finite Γ-CW-complex is simply a finite Γ-CW-complex, and Theorem 1 implies

$$\chi(X/\Gamma) = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\chi(X^\gamma). \tag{2}$$

Thus Theorem 1 is a generalization of the well-known equation (2) for finite group actions. For the direct proof of the equation (2), see [6, p. 225].

If Γ is virtually torsion-free and X is a Γ-finite Γ-CW-complex such that X^γ is nonempty and $\mathbb{Q}$-acyclic for every element $\gamma \in \Gamma$ of finite order, then $e(C_\Gamma(\gamma), X^\gamma)$ coincides with the Euler characteristic $\chi(C_\Gamma(\gamma))$ of the group $C_\Gamma(\gamma)$ for every $\gamma \in \Gamma$ of finite order, and Theorem 1 reduces to the following formula due to K. S. Brown (cf. [5, p. 261]):

$$\sum_i (-1)^i \dim_{\mathbb{Q}} H_i(\Gamma, \mathbb{Q}) = \sum_{\gamma\in\mathcal{F}(\Gamma)} \chi(C_\Gamma(\gamma)).$$

The rest of this paper is organized as follows. In §2, we introduce Hattori-Stallings ranks of finitely generated projective $\mathbb{Q}\Gamma$-modules, where $\mathbb{Q}\Gamma$ denotes the rational group algebra of Γ.

When X is a Γ-finite Γ-CW-complex, its cellular chain groups $C_i(X,\mathbb{Q})$ are finitely generated projective $\mathbb{Q}\Gamma$-modules. In §3, we will see that $e(C_\Gamma(\gamma), X^\gamma)$ can be expressed in terms of Hattori-Stallings ranks of cellular chain groups.

In §4, we will prove Theorem 1. The proof is done by the spectral sequence which converges to the homology of Γ with coefficients in the cellular chain complex $C_*(X, \mathbb{Q})$, together with properties of Hattori-Stallings ranks which will be discussed in §2 and §3.

In the final section §5, we will consider the two special cases where (i) Γ is a finite group or (ii) Γ is virtually torsion-free, and X^γ is nonempty and $\mathbb{Q}$-acyclic for every $\gamma \in \Gamma$ of finite order, both of which are mentioned above.

Throughout this paper, we emply the following conventions unless otherwise stated: Γ is a discrete group and $\mathbb{Q}\Gamma$ is its rational group algebra. A module over $\mathbb{Q}\Gamma$ is understood to be a left $\mathbb{Q}\Gamma$-module.

2. The Hattori-Stallings rank

The Hattori-Stallings rank was introduced by A. Hattori [8] and J. Stallings [10] and was studied by H. Bass in detail [1]. The reader should refer to [2] and [5, Chapter IX] for further detail of the Hattori-Stallings rank.

Let $\mathbb{Q}\Gamma$ be the rational group algebra of Γ. Define $[\mathbb{Q}\Gamma, \mathbb{Q}\Gamma]$ be the additive subgroup of $\mathbb{Q}\Gamma$ generated by $\alpha\beta - \beta\alpha$ $(\alpha, \beta \in \mathbb{Q}\Gamma)$. Set $T(\mathbb{Q}\Gamma) = \mathbb{Q}\Gamma/[\mathbb{Q}\Gamma, \mathbb{Q}\Gamma]$. It is easy to see that $T(\mathbb{Q}\Gamma)$ is isomorphic to the $\mathbb{Q}$-linear space spaned by the set of conjugacy classes of elements of Γ, i.e.,

$$T(\mathbb{Q}\Gamma) \cong \bigoplus_{\gamma \in \mathcal{C}(\Gamma)} \mathbb{Q} \cdot (\gamma), \tag{3}$$

where $\mathcal{C}(\Gamma)$ is a set of representatives of conjugacy classes of elements of Γ and (γ) is the conjugacy class of γ.

Let $\pi : \mathbb{Q}\Gamma \to T(\mathbb{Q}\Gamma)$ be the natural projection. Under the identification (3), π assigns $1 \cdot (\gamma) \in T(\mathbb{Q}\Gamma)$ to $\gamma \in \mathbb{Q}\Gamma$.

Let P be a finitely generated projective $\mathbb{Q}\Gamma$-module. Then P is a direct summand of a finitely generated free $\mathbb{Q}\Gamma$-module F. Choose such F and let $p : F \to P$ be the natural projection and $i : P \hookrightarrow F$ the inclusion. By fixing a basis of F, the composite $i \circ p$ can be identified with a square matrix M over $\mathbb{Q}\Gamma$.

DEFINITION. Under these assumptions, the *Hattori-Stallings rank* $r_\Gamma(P)$ of a finitely generated projective $\mathbb{Q}\Gamma$-module P is defined by

$$r_\Gamma(P) = \pi(\mathrm{tr} M) \in T(\mathbb{Q}\Gamma),$$

where $\mathrm{tr}M$ is the trace of M.

Note that $r_\Gamma(P)$ is well-defined, i.e., it is independent of various choices made. Under the identification (3), denote by $r_\Gamma(P)(\gamma)$ the coefficient of (γ) in $r_\Gamma(P)$.

We recall some properties of the Hattori-Stallings rank, which will be used later.

Proposition 2 (cf. [1, §2 (2.5)]). *If P_1 and P_2 are finitely generated projective $\mathbb{Q}\Gamma$-modules, then*

$$r_\Gamma(P_1 \oplus P_2) = r_\Gamma(P_1) + r_\Gamma(P_2).$$

Proposition 3 (cf. [1, §6 (6.3)]). *Let Γ' be a subgroup of finite index of Γ, P a finitely generated projective $\mathbb{Q}\Gamma$-module. Regarding P as a finitely generated projective $\mathbb{Q}\Gamma'$-modules by the restriction of scalars, one has*

$$r_{\Gamma'}(P)(\gamma) = (C_\Gamma(\gamma), C_{\Gamma'}(\gamma)) \cdot r_\Gamma(P)(\gamma)$$

for all $\gamma \in \Gamma'$, *where* $(C_\Gamma(\gamma), C_{\Gamma'}(\gamma))$ *is the index of* $C_{\Gamma'}(\gamma)$ *in* $C_\Gamma(\gamma)$.

Let $f : \Gamma_1 \to \Gamma_2$ be a group homomorphism. Let P be a finitely generated projective $\mathbb{Q}\Gamma_1$-module. Then $\mathbb{Q}\Gamma_2 \otimes_{\mathbb{Q}\Gamma_1} P$ is a finitely generated projective $\mathbb{Q}\Gamma_2$-modules, where $\mathbb{Q}\Gamma_2$ is regarded as a right $\mathbb{Q}\Gamma_1$-module via f. Let $T(f) : T(\mathbb{Q}\Gamma_1) \to T(\mathbb{Q}\Gamma_2)$ be the homomorphism induced by f.

Proposition 4 (cf. [1, §2 (2.9)]). *Under these assumptions, one has*

$$r_{\Gamma_2}(\mathbb{Q}\Gamma_2 \otimes_{\mathbb{Q}\Gamma_1} P) = T(f)(r_{\Gamma_1}(P)).$$

In case Γ is a finite group, Hattori-Stallings ranks can be determined by the character:

Proposition 5 (cf. [1, §5 (5.8)]). *Let* Γ *be a finite group. Let* V *be a* $\mathbb{Q}\Gamma$*-module which is finite dimensional over* $\mathbb{Q}$. *Then* V *is finitely generated and projective, and one has*

$$r_\Gamma(V)(\gamma) = \frac{\chi(\gamma^{-1})}{|C_\Gamma(\gamma)|},$$

where $\chi : \Gamma \to \mathbb{Q}$ *is the character of* V.

3. Hattori-Stallings ranks and equivariant Euler characteristics

Now we consider the equivariant Euler characteristic $e(\Gamma, X)$. First we invoke the following elementary lemma, which may be well-known.

Lemma 6. *Let* X *be a* Γ*-finite* Γ*-CW-complex. Then its cellular chain group* $C_i(X, \mathbb{Q})$ *is a finitely generated projective* $\mathbb{Q}\Gamma$*-module.*

Proof. $C_i(X, \mathbb{Q})$ has a direct sum decomposition as a $\mathbb{Q}\Gamma$-module:

$$C_i(X, \mathbb{Q}) \cong \bigoplus_{\sigma} \mathbb{Q}\Gamma \otimes_{\mathbb{Q}\Gamma_\sigma} \mathbb{Q}, \tag{4}$$

where σ ranges representatives of Γ-orbits of i-cells of X, Γ_σ is the isotropy subgroup of σ, $\mathbb{Q}$ is regarded as a left $\mathbb{Q}\Gamma_\sigma$-module with the trivial action of Γ_σ, and $\mathbb{Q}\Gamma$ is regarded naturally as a right $\mathbb{Q}\Gamma_\sigma$-module. Since X is Γ-finite, each Γ_σ is a finite subgroup of Γ, which implies that $\mathbb{Q}$ is always finitely generated projective $\mathbb{Q}\Gamma_\sigma$-module (cf. Proposition 5). Thus $\mathbb{Q}\Gamma \otimes_{\mathbb{Q}\Gamma_\sigma} \mathbb{Q}$ is a finitely generated projective $\mathbb{Q}\Gamma$-module. As the number of Γ-orbits of cells of X is finite, so is the number of direct summands in (4), which yields the lemma. □

By Lemma 6, the Hattori-Stallings rank of $C_i(X, \mathbb{Q})$ can be defined.

Lemma 7. *Let X be a Γ-CW-complex. Then*

$$r_\Gamma(C_i(X,\mathbb{Q}))(1) = \sum_\sigma \frac{1}{|\Gamma_\sigma|},$$

where σ ranges representatives of Γ-orbits of i-cells of X.

Proof. We have

$$\begin{aligned} r_\Gamma(C_i(X,\mathbb{Q}))(1) &= r_\Gamma\left(\bigoplus_\sigma \mathbb{Q}\Gamma \otimes_{\mathbb{Q}\Gamma_\sigma} \mathbb{Q}\right)(1) && \text{by (4)} \\ &= \sum_\sigma r_\Gamma(\mathbb{Q}\Gamma \otimes_{\mathbb{Q}\Gamma_\sigma} \mathbb{Q})(1) && \text{by Proposition 2} \\ &= \sum_\sigma T(i)(r_{\Gamma_\sigma}(\mathbb{Q}))(1) && \text{by Proposition 4,} \end{aligned}$$

where σ ranges representatives of Γ-orbits of i-cells of X and $T(i) : T(\mathbb{Q}\Gamma_\sigma) \to T(\mathbb{Q}\Gamma)$ is the map induced by the inclusion $i : \Gamma_\sigma \hookrightarrow \Gamma$. From Proposition 5 we conclude

$$T(i)(r_{\Gamma_\sigma}(\mathbb{Q}))(1) = r_{\Gamma_\sigma}(\mathbb{Q})(1) = \frac{1}{|\Gamma_\sigma|},$$

proving the lemma. □

By virtue of Lemma 7, we have

(5) $$e(\Gamma, X) = \sum_i (-1)^i r_\Gamma(C_i(X,\mathbb{Q}))(1).$$

Together with the result of K. S. Brown [4], we obtain the relation between the Hattori-Stallings rank of $C_i(X,\mathbb{Q})$ and $e(C_\Gamma(\gamma), X^\gamma)$ as follows:

Proposition 8. *Let X be a Γ-finite Γ-CW-complex. Then*

(6) $$e(C_\Gamma(\gamma), X^\gamma) = \sum_i (-1)^i r_\Gamma(C_i(X,\mathbb{Q}))(\gamma)$$

for every $\gamma \in \Gamma$.

Proof. A direct consequence of the equality (5) and [4, Theorem 3.1 (iii)]. □

4. Proof of Theorem 1

Let X be a Γ-finite Γ-CW-complex. Let $H_*(\Gamma, C_*(X,\mathbb{Q}))$ be the homology of Γ with coefficients in the cellular chain complex $C_*(X,\mathbb{Q})$, which is isomorphic to the Borel homology (equivariant homology) $H_*^{\Gamma}(X,\mathbb{Q})$ (cf. [5, Chapter VII]). Since the isotropy subgroup of every cell of X is finite, the Borel homology of X is isomorphic to the rational homology of the orbit space:

$$H_*(\Gamma, C_*(X,\mathbb{Q})) \cong H_*^{\Gamma}(X,\mathbb{Q}) \cong H_*(X/\Gamma,\mathbb{Q}). \tag{7}$$

Lemma 9. *Let X be a Γ-finite Γ-CW-complex. Then*

$$\sum_i (-1)^i \dim_{\mathbb{Q}} H_i(\Gamma, C_*(X,\mathbb{Q})) = \sum_i (-1)^i \dim_{\mathbb{Q}} \mathbb{Q} \otimes_{\mathbb{Q}\Gamma} C_i(X,\mathbb{Q}).$$

Proof. Consider the spectral sequence

$$E^1_{i,j} = H_j(\Gamma, C_i(X,\mathbb{Q})) \Rightarrow H_{i+j}(\Gamma, C_*(X,\mathbb{Q}))$$

(cf. [5, §VII.5 and §VII.7]). Since $C_i(X,\mathbb{Q})$ is a projective $\mathbb{Q}\Gamma$-module for all i, we have

$$E^1_{i,j} \cong \begin{cases} \mathbb{Q} \otimes_{\mathbb{Q}\Gamma} C_i(X,\mathbb{Q}) & j = 0 \\ 0 & j \neq 0. \end{cases}$$

As $\dim_{\mathbb{Q}} \mathbb{Q} \otimes_{\mathbb{Q}\Gamma} C_i(X,\mathbb{Q}) < \infty$ for all i, we obtain the desired equation. □

Now we prove Theorem 1. By Proposition 4 (take Γ_2 to be the trivial subgroup),

$$\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Q}\Gamma} C_i(X,\mathbb{Q})) = \sum_{\gamma \in \mathcal{C}(\Gamma)} r_{\Gamma}(C_i(X,\mathbb{Q}))(\gamma). \tag{8}$$

Hence

$$\begin{aligned}
\chi(X/\Gamma) &= \sum_i (-1)^i \dim_{\mathbb{Q}} H_i(\Gamma, C_*(X,\mathbb{Q})) && \text{by (7)} \\
&= \sum_i (-1)^i \dim_{\mathbb{Q}} \mathbb{Q} \otimes_{\mathbb{Q}\Gamma} C_i(X,\mathbb{Q}) && \text{by Lemma 9} \\
&= \sum_i (-1)^i \left(\sum_{\gamma \in \mathcal{C}(\Gamma)} r_{\Gamma}(C_i(X,\mathbb{Q}))(\gamma) \right) && \text{by (8)}
\end{aligned}$$

$$= \sum_{\gamma \in \mathcal{C}(\Gamma)} \left(\sum_i (-1)^i r_\Gamma(C_i(X, \mathbb{Q}))(\gamma) \right)$$
$$= \sum_{\gamma \in \mathcal{C}(\Gamma)} e(C_\Gamma(\gamma), X^\gamma) \qquad \text{by Proposition 8.}$$

For an element γ of infinite order, we have $X^\gamma = \emptyset$ and hence $e(C_\Gamma(\gamma), X^\gamma) = 0$, which proves Theorem 1.

5. Remarks

5.1. Finite group actions

Suppose that Γ is a finite group. Let X be a finite Γ-complex. By Proposition 5, we have

$$e(C_\Gamma(\gamma), X^\gamma) = \sum_i (-1)^i r_{C_\Gamma(\gamma)}(C_i(X^\gamma, \mathbb{Q}))(1) = \frac{\chi(X^\gamma)}{|C_\Gamma(\gamma)|}.$$

By Theorem 1, we have

$$\chi(X/\Gamma) = \sum_{\gamma \in \mathcal{C}(\Gamma)} \frac{\chi(X^\gamma)}{|C_\Gamma(\gamma)|} = \frac{1}{|\Gamma|} \sum_{\gamma \in \mathcal{C}(\Gamma)} \frac{|\Gamma|}{|C_\Gamma(\gamma)|} \chi(X^\gamma).$$

Since $|\Gamma|/|C_\Gamma(\gamma)|$ is the cardinality of the conjugacy class (γ), we obtain

$$\chi(X/\Gamma) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi(X^\gamma).$$

Hence Theorem 1 implies the well-known equality for finite group actions.

5.2. Euler characteristics of groups

Suppose that Γ is a group of finite homological type, then one can define its Euler characteristic $\chi(\Gamma)$ in the sense of C. T. C. Wall [12]. See [5, Chapter IX] for relevant definitons. Suppose in addition the centralizer $C_\Gamma(\gamma)$ is of finite homological type for every $\gamma \in \Gamma$ of finite order. Under these assumptions, K. S. Brown obtained the following formula:

$$\tilde{\chi}(\Gamma) = \sum_{\gamma \in \mathcal{F}(\Gamma)} \chi(C_\Gamma(\gamma)), \tag{9}$$

where $\tilde{\chi}(\Gamma)$ is the *naive Euler characteristic* of Γ defined by $\tilde{\chi}(\Gamma) = \sum_i (-1)^i \dim_{\mathbb{Q}} H_i(\Gamma, \mathbb{Q})$ (cf. [5, p. 261]). This formula was used by J. Harer and D. Zagier in the computation of the Euler characteristic of the moduli space of curves [7].

We will give a relation between the equation (9) and Theorem 1. Let Γ be a discrete group and X a Γ-finite Γ-CW complex such that X^γ is nonempty and $\mathbb{Q}$-acyclic for every $\gamma \in \Gamma$ of finite order. If Γ is virtually torsion-free, then $C_\Gamma(\gamma)$ is of finite homological type for every $\gamma \in \Gamma$ of finite order (including $\gamma = 1$), and $\chi(C_\Gamma(\gamma))$ coincides with $e(C_\Gamma(\gamma), X^\gamma)$ (cf. [4, pp. 111-112]). In this case the equation (1) in Theorem 1 reduces to the equation (9), since

$$H_*(\Gamma, \mathbb{Q}) \cong H_*^\Gamma(X, \mathbb{Q}) \cong H_*(X/\Gamma, \mathbb{Q}).$$

However, we claim the equation (9) for this special case can be deduced without the use of the spectral sequence appeared in the proof of Lemma 9. To see this, observe $\varepsilon : C_*(X, \mathbb{Q}) \to \mathbb{Q}$ is a projective resolution of $\mathbb{Q}$ over $\mathbb{Q}\Gamma$, where $\mathbb{Q}$ is regarded as a $\mathbb{Q}\Gamma$-module with the trivial Γ-action and ε is the augmentation. Hence

$$\tilde{\chi}(\Gamma) = \sum_i (-1)^i \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Q}\Gamma} C_i(X, \mathbb{Q})).$$

Now the claim follows from this together with (8) and Proposition 8.

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