Murai, M.
Osaka J. Math.
35 (1998), 835–854

BLOCKS OF FACTOR GROUPS AND HEIGHTS OF CHARACTERS

Masafumi MURAI

(Received May 9, 1997)

Introduction

Let G be a finite group and p a prime number. Let (K, R, k) be a p-modular system. We assume that K contains a primitive $|G|$-th root of unity and that k is algebraically closed. Let ν be the valuation of K normalized so that $\nu(p) = 1$. Let N be a normal subgroup of G and let V be an indecomposable oG-module such that V_N is indecomposable, where $o = R$ or k. As in [14], we say that a block B of G V-dominates a block $\overline{B}$ of G/N if there is an $o[G/N]$-module X in $\overline{B}$ such that $V \otimes \mathrm{Inf}X$ belongs to B, where $\mathrm{Inf}X$ denotes the inflation of X to G. In [14] we have shown that there is a natural relation between B and $\overline{B}$, if B V- dominates $\overline{B}$. In particular, if D is a defect group of B, then $\overline{B}$ has a defect group of the form QN/N with $D \cap N \leq Q \leq D$. Then, we shall show in Section 2 that Q chosen in this way is of a rather restricted nature. In fact, we see that $O_p(N_G(Q)) = Q$ and that Q is a Sylow intersection in G (Theorem 2.1). When, for example, V_N is irreducible, there exists a B-Brauer pair (Q, b_Q) (Theorem 2.8). As a consequence, we see there exist defect groups D and $\overline{D}$ respectively of B and $\overline{B}$ such that $Z(D)N/N \leq \overline{D} \leq DN/N$. Further, Q is then a "defect intersection". When V is the trivial module "V-domination" is nothing but the usual "domination", in which case we shall show even the existence of a weight (Q, S) belonging to B (in the sense of Alperin [2]) (Proposition 2.6).

In Section 1 we give an alternative proof of a result of Harris-Knörr [8].

In Section 3 we give an extendibility theorem for an irreducible character of a normal subgroup, the proof of which depends upon a result of Brauer on major subsections [4, (4C)] and a result of Knörr [11, Corollary 3.7 (i)].

As an application we study in Section 4 the following conjecture $(*)$ given by Robinson [17]. In [17] $(*)$ is proved under a conjecture related to Alperin's weight conjecture, cf. Theorem 5.1 in [17].

$(*)$ Let B be a block of a group G with defect group D. Then, for every irreducible character χ in B, $\mathrm{ht}\chi \leq \nu|D : Z(D)|$ and the equality holds only when D is abelian.

The conjecture $(*)$ is of course an extension of half of Brauer's height 0 conjecture and it is known to be true for p-blocks of p-solvable groups by the results of Fong [7] and Watanabe [18]. Indeed, Fong [7, (3C)] proves the inequality and Watanabe [18, Proposition] proves that the inequality is strict unless D is abelian.

Actually, we consider a "relative version" of $(*)$ as follows:

($\sharp$) Let N be a normal subgroup of G. For every irreducible character χ in a block of G with defect group D and every irreducible constituent ξ of χ_N, we have

$$\mathrm{ht}\chi - \mathrm{ht}\xi \leq \nu|DN : Z(D)N|$$

and the equality holds if and only if χ is afforded by a $Z(D)N$-projective RG-module.

If $N = 1$, ($\sharp$) boils down to $(*)$. (In fact, by Knörr's theorem [11], an irreducible character of G in a block with defect group D is afforded by a $Z(D)$-projective RG-module if and only if D is abelian, cf. Lemma 4.5 below.) Conversely, we show ($\sharp$) is true if $(*)$ is true for blocks of certain groups related with the factor group G/N (Theorem 4.3). Thus the assertions $(*)$ and ($\sharp$) turn out to be equivalent. Furthermore, based on Theorem 4.3, we give a reduction of $(*)$ to the case of quasi-simple groups (Theorem 4.6). As a special case we obtain that ($\sharp$) is true if G/N is p-solvable (Corollary 4.7), which extends the results of P. Fong and A. Watanabe mentioned above.

In this paper all oG-modules are assumed to be o-free of finite rank. For a block B of G, $d(B)$ is the defect of B. For an oG-module X in B, we define $\mathrm{ht}X$, the height of X, by $\mathrm{ht}X = \nu(\mathrm{rank}_o X) - \nu|G| + d(B)$. For an indecomposable module X, $\mathrm{vx}(X)$ denotes a vertex of X. For a group H, $Z(H)$ denotes the center of H.

Throughout this paper Knörr's papers [10, 11, 12] are of fundamental importance.

1. A result of Harris-Knörr

Let G be a group and let N be a normal subgroup of G. Let b be a block of N with defect group δ. Let b_1 be the Brauer correspondent of b in $N_N(\delta)$. Then Harris and Knörr [8] have proved

Theorem 1.1 (Harris-Knörr [8, Theorem]). *Block induction gives a defect-preserving bijection between the set of blocks of $N_G(\delta)$ covering b_1 and the set of blocks of G covering b.*

A module-theoretical proof of the above theorem is found in Alperin [1]. Here we give still another (module-theoretical) proof (under our assumption on the fields K and k).

Lemma 1.2. *Let L be a subgroup of G such that $N_N(\delta) \lhd L$. Then, for a block β of L such that β^G is defined, the following are equivalent:*

(i) *β covers some $N_G(\delta)$-conjugate of b_1.*

(ii) β^G *covers* b.

Proof. Put $M = N_N(\delta)$. Let U be an indecomposable RG-module of height 0 in β^G. Then there is an indecomposable RL-module V of height 0 in β such that $V|U_L$ by [13, Corollary 1.7 (i)]. Let $b_1{}'$ be a block of M covered by β. Then there is an indecomposable RM-module W of height 0 in $b_1{}'$ such that $W|V_M$ by [13, Theorem 4.1] (see also [20, Proposition 2]). So there is an indecomposable RN-module X such that $X|U_N$ and that $W|X_M$. Let b' be the block of N containing X. Since $\mathrm{ht}W = 0$, $\mathrm{vx}(W)$ is a defect group of $b_1{}'$. Further we get

$$(1) \qquad \delta \lhd \mathrm{vx}(W) \leq \mathrm{vx}(X) \leq \delta',$$

where $\mathrm{vx}(X)$ is a vertex of X and δ' is a defect group of b'.

(i) $\Rightarrow$ (ii): In the above we may choose $b_1{}'$ so that $b_1{}' = b_1{}^x$ for some $x \in N_G(\delta)$. So $\mathrm{vx}(W) = \delta$. Hence X belongs to $(b_1{}^x)^N = (b_1{}^N)^x = b^x$ by the Nagao-Green theorem [14, Theorem 3.12]. Thus β^G covers b.

(ii) $\Rightarrow$ (i): We have $b' = b^x$ for some $x \in G$. So $\delta' = \delta^{xn}$ for some $n \in N$. Thus equality holds throughout in (1) and $\mathrm{vx}(W) = \delta = \delta^{xn}$. Hence X belongs to $(b_1{}')^N$ by the Nagao-Green theorem. So $(b_1{}')^N = b^x$. Put $y = (xn)^{-1} \in N_G(\delta)$. Then $((b_1{}')^y)^N = ((b_1{}')^N)^y = b^{xy} = b$, since $xy \in N$. On the other hand, since $b_1{}'$ has defect group δ, $(b_1{}')^y$ has defect group $\delta^y = \delta$. Thus $(b_1{}')^y = b_1$ by the First Main Theorem. Hence β covers $b_1{}' = b_1{}^{y^{-1}}$. This completes the proof. □

Proof of Theorem 1.1. Applying the First Main Theorem and Lemma 1.2 with $L = N_G(\delta)$, we get the result (cf. the proof of [8, Theorem]). □

2. Blocks of factor groups

Throughout this section we use the following notation:

Let N be a normal subgroup of a group G and let V be an indecomposable oG-module such that V_N is indecomposable, where $o = R$ or k. Let b be the block of N to which V_N belongs. (So b is G-invariant.) Let B be a block of G covering b. Let D be a defect group of B.

If $\overline{B}$ is a block of G/N which is V-dominated by B, then a defect group of $\overline{B}$ is contained in DN/N ([14, Theorem 1.4 (i)]). Since $DN/N \cong D/D \cap N$, we may choose a p-subgroup Q so that QN/N is a defect group of $\overline{B}$ and that $D \cap N \leq Q \leq D$. (We note that $D \cap N$ is a defect group of b by [10, Proposition 4.2].)

For a p-subgroup Q such that $D \cap N \leq Q \leq D$, we denote by $b(Q)$ a unique block of QN covering b. Since b is G-invariant, Q is a defect group of $b(Q)$ ([13, Lemma 4.13]). Further, Since $b(Q)$ is $N_G(QN)$-invariant, we see, by the Frattini argument, that $N_G(QN) = N_G(Q)N$. Let $b'(Q)$ be the Brauer correspondent of $b(Q)$ in $N_{QN}(Q) = QN_N(Q)$.

Theorem 2.1. *Let QN/N, $D \cap N \leq Q \leq D$, be a defect group of a block of G/N which is V-dominated by B. Then:*
(i) $O_p(N_G(Q)) = Q$.
(ii) *Q is a Sylow intersection in G.*

Proof. (i) By the First Main Theorem, $N_{G/N}(QN/N)$ has a block with defect group QN/N. In view of the natural isomorphism

$$N_{G/N}(QN/N) = N_G(Q)N/N \cong N_G(Q)/N_N(Q),$$

it follows that $N_G(Q)/N_N(Q)$ has a block with defect group $QN_N(Q)/N_N(Q)$. So $N_G(Q)/QN_N(Q)$ has a block of defect 0 and hence $O_p(N_G(Q)/QN_N(Q)) = 1$. Thus $Q \leq O_p(N_G(Q)) \leq O_p(QN_N(Q))$. On the other hand, since the block $b'(Q)$ has defect group Q, we get $O_p(QN_N(Q)) \leq Q$. Hence $O_p(N_G(Q)) = Q$.

(ii) As in the proof of (i), $N_G(Q)/N_N(Q)$ has a block with defect group $QN_N(Q)/N_N(Q)$. So $N_G(Q)/N_N(Q)$ has p-Sylow subgroups $L_i/N_N(Q)$, $i = 1, 2$, such that $L_1 \cap L_2 = QN_N(Q)$. Since $Q \cap N = Q \cap N_N(Q)$ is a defect group of a block of $N_N(Q)$ covered by $b'(Q)$, we can choose p-Sylow subgroups T_i, $i = 1, 2$, of $N_N(Q)$ such that $T_1 \cap T_2 = Q \cap N$. Choose p-Sylow subgroups S_i, $i = 1, 2$, of L_i such that $T_i \leq S_i$. Then

$$\begin{aligned} Q &\leq S_1 \cap S_2 \text{ (since } Q \text{ is a normal } p\text{-subgroup of } L_i,\ i = 1, 2) \\ &= S_1 \cap S_2 \cap QN_N(Q) \text{ (since } S_1 \cap S_2 \leq L_1 \cap L_2 = QN_N(Q)) \\ &= Q(S_1 \cap S_2 \cap N_N(Q)) \text{ (since } Q \leq S_1 \cap S_2) \\ &= Q(T_1 \cap T_2) \text{ (since } S_i \cap N_N(Q) = T_i,\ i = 1, 2) \\ &= Q(Q \cap N) = Q. \end{aligned}$$

Thus $S_1 \cap S_2 = Q$. Choose p-Sylow subgroups P_i, $i = 1, 2$, of G such that $S_i \leq P_i$. Then $P_1 \cap P_2 \cap N_G(Q) = S_1 \cap S_2 = Q$, since S_i, $i = 1, 2$, are p-Sylow subgroups of $N_G(Q)$. Thus we get $P_1 \cap P_2 = Q$. □

The following lemma is useful.

Lemma 2.2. *Let H be a subgroup of G with $H \geq N$. Let U be an oH-module such that U_N is indecomposable. Let Q be a p-subgroup with $QN \lhd H$. Let W be a projective indecomposable $o[H/QN]$-module. Then, $U \otimes \mathrm{Inf}W$ is indecomposable, and for a p-subgroup S of H, S is a vertex of U_{QN} if and only if S is a vertex of $U \otimes \mathrm{Inf}W$. Further, $SN = QN$ for such S.*

Proof. If $o = R$, let πR be the maximal ideal of R. If $o = k$, let $\pi = 0$. As is well-known, $W/\pi W$ is indecomposable, so $U \otimes \mathrm{Inf}W$ is indecomposable by [14, Lemma 1.1 (i)]. Clearly $\mathrm{Inf}W$ is QN-projective, so we have

(1) $U \otimes \mathrm{Inf}W$ is QN-projective.
Also we have
(2) $(U \otimes \mathrm{Inf}W)_{QN} \cong (\mathrm{rank}_o W)U_{QN}$.
If S is a vertex of U_{QN}, then (1) and (2) imply that S is a vertex of $U \otimes \mathrm{Inf}W$. Further, $U_{QN} \cong (U_{SN})^{QN}$ by Green's indecomposability theorem. So $SN = QN$. Conversely, let S be a vertex of $U \otimes \mathrm{Inf}W$. Then, since $QN \triangleleft H$, (1) implies $S \leq QN$. Then (2) implies S is a vertex of U_{QN}. This completes the proof. □

For a p-subgroup Q such that $D \cap N \leq Q \leq D$, let $b(Q)$ and $b'(Q)$ be as before. We denote by $BL(N_G(Q)N|b(Q))$ and $BL(N_G(Q)|b'(Q))$ the set of blocks of $N_G(Q)N$ covering $b(Q)$ and the set of blocks of $N_G(Q)$ covering $b'(Q)$, respectively. For a subgroup H of G, let

$$BL(H, B) = \{\beta \mid \beta \text{ is a block of } H \text{ such that } \beta^G = B\}.$$

Lemma 2.3. *Block induction gives a defect-preserving bijection between $BL(N_G(Q), B)$ and $BL(N_G(Q)N, B)$.*

Proof. Let $\beta \in BL(N_G(Q)N, B)$. Then, since $B = \beta^G$ covers b, we see, by [14, Lemma 1.3], β covers b and hence $b(Q)$. So $BL(N_G(Q)N, B) \subseteq BL(N_G(Q)N \mid b(Q))$. Let $\beta' \in BL(N_G(Q), B)$. Then, since $(\beta'^{N_G(Q)N})^G = B$, $\beta'^{N_G(Q)N}$ covers $b(Q)$ by the same reason, so β' covers $b'(Q)$ by Lemma 1.2. Thus $BL(N_G(Q), B) \subseteq BL(N_G(Q)|b'(Q))$. Hence the result follows from Theorem 1.1 (with $(N_G(Q)N, QN, b(Q))$ in place of (G, N, b)) and the transitivity of block induction. □

REMARK. For any block β of $N_G(Q)N$ covering b, β^G is defined. In fact, since β covers $b(Q)$, β has a defect group P with $P \geq Q$. Since $C_G(P) \leq C_G(Q) \leq N_G(Q)N$, β^G is defined.

Proposition 2.4. *Let Q be a p-subgroup of G such that $D \cap N \leq Q \leq D$. Let $\overline{\beta}$ be a block of $N_{G/N}(QN/N) = N_G(Q)N/N$. Then the following are equivalent:*
(i) *$\overline{\beta}^{G/N}$ is V-dominated by B.*
(ii) *$\overline{\beta}$ is $V_{N_G(Q)N}$-dominated by some $\beta \in BL(N_G(Q)N, B)$.*
(iii) *$\overline{\beta}$ is $V_{N_G(Q)N}$-dominated by $\beta'^{N_G(Q)N}$ for some $\beta' \in BL(N_G(Q), B)$.*

Proof. (i) ⇔ (ii): Put $H = N_G(Q)N$. We can choose a projective indecomposable $o[H/QN]$-module W which lies in $\overline{\beta}$ as an H/N-module. Then W has vertex QN/N. Let U be the Green correspondent of W with respect to $(G/N, H/N, QN/N)$. So U lies in $\overline{\beta}^{G/N}$ by the Nagao-Green theorem [16, Theorem

5.3.12]. Clearly $V_H \otimes \mathrm{Inf}W|(V \otimes \mathrm{Inf}U)_H$. By Lemma 2.2 with V_H in place of U, $V_H \otimes \mathrm{Inf}W$ is indecomposable, so there is an indecomposable summand X of $V \otimes \mathrm{Inf}U$ such that $V_H \otimes \mathrm{Inf}W|X_H$. Let S be a vertex of $V_H \otimes \mathrm{Inf}W$. Then by Lemma 2.2, we obtain $SN = QN$. So $C_G(S) \leq N_G(S) \leq N_G(QN) = H$. Thus, if β is the block of H containing $V_H \otimes \mathrm{Inf}W$, then X lies in β^G by the Nagao-Green theorem. So $V \otimes \mathrm{Inf}U$ belongs to β^G by [14, Theorem 1.2]. Thus, (i) is equivalent to (ii) (by [14, Theorem 1.2 (ii)]).

(ii) $\Leftrightarrow$ (iii): This follows from Lemma 2.3. This completes the proof. □

Now we can refine [14, Theorem 1.4 (ii)].

Corollary 2.5. *There exists a block of G/N with defect group DN/N which is V-dominated by B. Furthermore, the number of blocks of G/N with defect group DN/N which are V-dominated by B equals the number of blocks of $N_G(D)N/N$ with defect group DN/N which are $V_{N_G(D)N}$-dominated by $\tilde{B}^{N_G(D)N}$, where $\tilde{B}$ is the Brauer correspondent of B in $N_G(D)$.*

Proof. Put $H = N_G(D)N$. By the First Main Theorem, there is a bijection between the set of blocks of G/N with defect group DN/N and the set of blocks of H/N with defect group DN/N. By Proposition 2.4, it suffices to show

(1) $BL(N_G(D), B) = \{\tilde{B}\}$.

(2) $\tilde{B}^H$ V_H-dominates a block $\overline{\beta}$ of H/N, and for any such $\overline{\beta}$, $\overline{\beta}$ has defect group DN/N.

(1) follows from the First Main Theorem. To prove (2), put $\beta = \tilde{B}^H$. Then, since $\beta^G = B$ covers b, β covers b by [14, Lemma 1.3]. So, by [14, Theorem 1.2 (i)], β V_H-dominates a block $\overline{\beta}$ of H/N. Let Q_1 be a defect group of $\overline{\beta}$. Since D is a defect group of β, we get $Q_1 \leq_{H/N} DN/N$ by [14, Theorem 1.4 (i)]. On the other hand, $Q_1 \geq_{H/N} DN/N$, since DN/N is normal in H/N. So $Q_1 =_{H/N} DN/N$. Thus (2) is proved. □

In the case of usual domination, we have the following:

Proposition 2.6. *Let QN/N, $D \cap N \leq Q \leq D$, be a defect group of a block of G/N which is dominated by B. Then there is a weight (Q, S) belonging to B.*

Proof. Let $\overline{B}$ be a block of G/N with defect group QN/N which is dominated by B. Let $\overline{\beta}$ be the Brauer correspondent of $\overline{B}$ in $N_G(Q)N/N$. Let β be a unique block of $N_G(Q)N$ dominating $\overline{\beta}$. We have $\beta^G = B$ by Proposition 2.4. Let W be an irreducible $k[N_G(Q)N/N]$-module in $\overline{\beta}$. Then W has vertex QN/N, so if $\mathrm{Inf}W$ is the inflation to $N_G(Q)N$ of W, then $\mathrm{Inf}W$ has vertex Q (note that Q is a p-Sylow subgroup of QN). Put $S = (\mathrm{Inf}W)_{N_G(Q)}$. Then S is irreducible and has vertex Q. Let

β' be the block of $N_G(Q)$ containing S. Then, by using the Green correspondence and the Nagao-Green theorem, we see that $\beta'^{N_G(Q)N} = \beta$. So $\beta'^G = B$. Hence (Q, S) is a weight belonging to B. This completes the proof. □

In the rest of this section we consider mainly the case when V_N is an irreducible oN-module. In this case as well, defect groups of the blocks of G/N which are V-dominated by B are rather restricted, though the condition we give below is not so strong as Proposition 2.6. We prepare the following lemma, which complements 1.21 Remark in Knörr [12]. For the definition of virtually irreducible modules (lattices) and basic properties of them, see Knörr [12].

Lemma 2.7. *Let W be an irreducible $o[G/N]$-module.*

(i) *If $o = R$ and V_N is virtually irreducible RN-module, then $V \otimes \mathrm{Inf}W$ is virtually irreducible.*

(ii) *If $o = k$ and $\mathrm{End}_{kN}(V_N) = k$, then $\mathrm{End}_{kG}(V \otimes \mathrm{Inf}W) = k$.*

Proof. (i) Let $\phi \in \mathrm{End}_{RG}(V \otimes \mathrm{Inf}W)$. Let $\{w_i\}$ be an R-basis of W. We may write

$$(v \otimes w_i)\phi = \sum_j v\phi_{ij} \otimes w_j, \quad v \in V,$$

where ϕ_{ij} are uniquely determined elements of $\mathrm{End}_{RN}(V_N)$. Put $E = \mathrm{End}_{RN}(V_N)$ and $n = \mathrm{rank}_R W$. Let $\phi F \in \mathrm{Mat}_n(E)$ be the matrix whose (i, j)-entry is ϕ_{ij}. Clearly F is an R-algebra monomorphism from $\mathrm{End}_{RG}(V \otimes \mathrm{Inf}W)$ to $\mathrm{Mat}_n(E)$. Put

$$w_i g = \sum_j a_{ij}(g) w_j, \quad a_{ij}(g) \in R, \text{ for every } g \in G.$$

Then we get

$$\sum_s a_{is}(g)\phi_{sj} = \sum_s {\phi_{is}}^g a_{sj}(g),$$

where ${\phi_{is}}^g$ is defined by the rule: $v{\phi_{is}}^g = vg^{-1}\phi_{is}g$, $v \in V$. Taking the traces of both sides, we get

$$\sum_s a_{is}(g)\mathrm{tr}(\phi_{sj}) = \sum_s \mathrm{tr}(\phi_{is}) a_{sj}(g).$$

This shows that the R-endomorphism Φ of W defined by

$$w_i\Phi = \sum_j \mathrm{tr}(\phi_{ij}) w_j$$

is an RG-endomorphism of W. So by assumption on W,
(1) $\operatorname{tr}(\phi_{ii}) = \operatorname{tr}(\phi_{11})$ for all i, and $\operatorname{tr}(\phi_{ij}) = 0$ if $i \neq j$.
Thus

$$\operatorname{tr}(\phi) = \sum_i \operatorname{tr}(\phi_{ii}) = (\operatorname{rank}_R W)\operatorname{tr}(\phi_{11}).$$

So

$$\nu(\operatorname{tr}(\phi)) = \nu(\operatorname{rank}_R W) + \nu(\operatorname{tr}(\phi_{11})) \geq \nu(\operatorname{rank}_R(V \otimes \operatorname{Inf} W)),$$

since V_N is virtually irreducible. It remains to show that if the equality holds here then ϕ is invertible. Assume the equality holds. Since V_N is virtually irreducible, (1) yields that ϕ_{ii} are invertible for all i and that $\phi_{ij} \in J(E)$ if $i \neq j$, where $J(E)$ is the radical of E. Let

$$\alpha : \operatorname{Mat}_n(E) \to \operatorname{Mat}_n(E)/J(\operatorname{Mat}_n(E)) \ (\cong \operatorname{Mat}_n(E/J(E)))$$

be the natural map. Then, by the above, $\phi F \alpha$ is invertible. So ϕF is invertible and then ϕ is invertible. This completes the proof.

(ii) cf. the proof of 1.21 Remark in Knörr [12]. □

We say (Q, b_Q) is a B-Brauer pair if b_Q is a block of $QC_G(Q)$ with defect group Q and $(b_Q)^G = B$. We refer to Brauer [5] for the basic facts about Brauer pairs.

Theorem 2.8. *Let QN/N, $D \cap N \leq Q \leq D$, be a defect group of a block $\overline{B}$ of G/N which is V-dominated by B. Assume either of the following:*
(a) *$o = R$ and V is an RG-module such that V_N is virtually irreducible.*
(b) *$o = k$ and V is a kG-module such that $\operatorname{End}_{kN}(V_N) = k$.*
Then
(i) *There is a B-Brauer pair (Q, b_Q).*
(ii) *For some defect group D_1 of B, we have $Z(D_1)N/N \leq QN/N \leq D_1N/N$.*
In particular if D is abelian, then every block of G/N V-dominated by B has DN/N as a defect group.
(iii) *There exist defect groups D_1 and D_2 of B such that $Q = D_1 \cap D_2$, that is, Q is a "defect intersection".*

Proof. Put $H = N_G(Q)N$.

(i) Let $\overline{\beta}$ be the Brauer correspondent of $\overline{B}$ in H/N and let β be a unique block of H which V_H-dominates $\overline{\beta}$.

Let W be an irreducible $o[H/N]$-module in $\overline{\beta}$ with $\operatorname{Ker} W \geq QN/N$. Let S be a vertex of $V_H \otimes \operatorname{Inf} W$. By Lemma 2.2, $SN = QN$. We claim that in both cases there exists a β-Brauer pair (S, b_S).

CASE (a). By Lemma 2.7, $V_H \otimes \mathrm{Inf}W$ is a virtually irreducible RH-module in β. So, by Knörr's theorem [11, Corollary 3.7 (i)] (or [12, Corollary 4.11]), there is a β-Brauer pair (S, b_S).

CASE (b). By Lemma 2.7 (ii), $\mathrm{End}_{kH}(V_H \otimes \mathrm{Inf}W) = k$. So, by Knörr [11, Theorem 3.3], there is a β-Brauer pair (S, b_S).

Thus the claim is proved. Now there is a primitive β-Brauer pair (P, b_P) such that $(S, b_S) \subseteq (P, b_P)$. Then, since $S \leq P \cap SN \leq P$, there is a β-Brauer pair $(P \cap SN, b_{P \cap SN})$. On the other hand, since P is a defect group of β and β covers $b(Q)$, $P \cap QN = P \cap SN$ is a defect group of $b(Q)$. Thus $P \cap SN$ is QN-conjugate to Q. Thus there is a β-Brauer pair (Q, b_Q). Then b_Q is a block of $QC_H(Q) = QC_G(Q)$ with defect group Q and $(b_Q)^G = ((b_Q)^H)^G = \beta^G = B$. Thus (i) is proved.

(ii) This follows from (i) and the Brauer-Olsson theorem [5, (4K)].

(iii) Let β be as in the proof of (i). From the proof of (i), we see there is a β-Brauer pair (Q, b_Q). Put $(b_Q)^{N_G(Q)} = \beta'$. From the proof of Theorem 2.1 (ii), we see there are p-Sylow subgroups S_i, $i = 1$, 2, of $N_G(Q)$ with $S_1 \cap S_2 = Q$. Let U_i, $i = 1$, 2, be defect groups of β' such that $S_i \geq U_i$. Then $Q = S_1 \cap S_2 \geq U_1 \cap U_2 \geq Q$, so $U_1 \cap U_2 = Q$. Now, as in the proof of (i), we have $\beta'^G = B$. Then we see that there is a defect group D_1 of B such that $U := N_{D_1}(Q)$ is a defect group of β', cf. [16, Theorem 5.5.21]. Thus there are x, $y \in N_G(Q)$ such that $U_1 = U^x$ and $U_2 = U^y$. Then $N_G(Q) \cap {D_1}^x \cap {D_1}^y = U^x \cap U^y = U_1 \cap U_2 = Q$, and so ${D_1}^x \cap {D_1}^y = Q$. This completes the proof. □

REMARK. When V is the trivial module, "B V-dominates $\overline{B}$" coincides with "B dominates $\overline{B}$" (or "B contains $\overline{B}$"). In this case, the last assertion of Theorem 2.8 (ii) is proved in Berger and Knörr [3, Step 2 of the proof of Theorem].

3. Extension of a character of a normal subgroup

Throughout this section, we use the following notation: Let N be a normal subgroup of a group G. Let b be a block of N. Let B be a block of G covering b. Let D be a defect group of the Fong-Reynolds correspondent of B in the inertial group of b in G. Put $\delta = D \cap N$. So δ is a defect group of b.

If Y is a subgroup of a group X and β is a block of Y, then for a character χ of X, we denote by χ_β the β-component of χ_Y and call it the β-component of χ.

The following theorem plays an important role in Section 4.

Theorem 3.1. *Let the notation be as above. For any D-invariant irreducible character ξ in b, there exists a D-invariant extension of ξ to $Z(D)N$.*

For the proof we prepare a lemma, which extends [13, Proposition 4.15 (i) (in case (1))].

Lemma 3.2. *Let A be an abelian subgroup of $C_D(\delta)$. Then every irreducible character in b extends to AN.*

Proof. Put $L = AN$. Let ξ be an irreducible character in b. Let ζ be an irreducible character of L lying over ξ. Since L/N is a p-group, there exist a subgroup H and a character η of H with the following properties: $N \leq H \leq L$, $\eta_N = \xi$ and $\eta^L = \zeta$, cf. Isaacs [9, Theorem 6.22]. Let V be an RH-module affording η. If $\hat{b}$ is a block of L to which ζ belongs, then $A\delta$ is a defect group of $\hat{b}$, cf. [13, Lemma 4.13]. Then, since V^L affords ζ and V^L is H-projective, we get $Z(A\delta) \leq {}_LH$ by [11, Corollary 3.7 (i)] (or [12, Corollary 4.11]). Clearly $A \leq Z(A\delta)$, so $A \leq {}_LH$ and $L = H$. Thus ζ is an extension of ξ to L. □

Proof of Theorem 3.1. Put $L = Z(D)N$. Since L is a normal subgroup of DN, the assertion makes sense. Applying Lemma 3.2 with $A = Z(D)$, we see that there exists an extension ζ of ξ to L. Fix any element x of D. Since $\xi^x = \xi$, ζ^x is also an extension of ξ to L. So there is a unique irreducible (linear) character $\lambda = \lambda_x$ of L/N such that $\zeta^x = \zeta\lambda$.

Let u be any element of $Z(D)$. If B' is a unique block of DN covering b, then D is a defect group of B', cf. [13, Lemma 4.13]. So there is a block b' of $C_{DN}(u)$ such that $b'^{DN} = B'$ and that D is a defect group of b'. (In fact, it suffices to choose the block of $C_{DN}(u)$ induced by a root of B' in $DC_{DN}(D)$.) Now $C_L(u) \lhd C_{DN}(u)$ and $C_{DN}(u) = DC_L(u)$. So b' covers a unique (D-invariant) block, say b_1, of $C_L(u)$, and b_1 has $D \cap C_L(u) = Z(D)\delta$ as a defect group. Let B_1 be a unique block of L covering b. Then clearly B_1 is D-invariant and, by [13, Lemma 4.13], $Z(D)\delta$ is a defect group of B_1. Since $b'^{DN} = B'$, and b_1 and B_1 are D-invariant, it readily follows that ${b_1}^L = B_1$.

Now we consider the b_1-component of $\zeta^x = \zeta\lambda$. Let e be the block idempotent of $RC_L(u)$ corresponding to b_1. Then for $h \in C_L(u)$,

$$(\zeta^x)_{b_1}(h) = \zeta^x(he) = \zeta(h^{x^{-1}}e) = (\zeta_{b_1})^x(h),$$

since $e^x = e$. So $(\zeta^x)_{b_1} = (\zeta_{b_1})^x$. On the other hand, put $e = \sum_y a_y y$, where $a_y \in R$ and y ranges over the p'-elements of $C_L(u)$. Then for $h \in C_L(u)$,

$$(\zeta\lambda)_{b_1}(h) = \sum_y a_y \zeta(hy)\lambda(h) = (\zeta_{b_1}\lambda)(h).$$

Thus $(\zeta\lambda)_{b_1} = \zeta_{b_1}\lambda$ and we have shown $(\zeta_{b_1})^x = \zeta_{b_1}\lambda$. Evaluating at u, we get $\zeta_{b_1}(u) = \zeta_{b_1}(u)\lambda(u)$. Since $Z(D)\delta$ is a common defect group of b_1 and B_1, $\zeta_{b_1}(u) \neq 0$ by Brauer [4, (4C)]. Thus we get $\lambda(u) = 1$. Since u is an arbitrary element of $Z(D)$, this shows that λ is the trivial character. So ζ is $\langle x \rangle$-invariant and, since $x \in D$ is arbitrary, we get that ζ is D-invariant. This completes the proof. □

REMARK. For alternative proofs of Brauer [4, (4C)], see [6, Proposition 3.4.1], [15, Corollary 1.10, Corollary 2.6], [19, Lemma].

4. Robinson's conjecture

We recall from Introduction Robinson's conjecture:

(∗) Let B be a block of a group G with defect group D. Then, for every irreducible character χ in B, ht $\chi \leq \nu|D : Z(D)|$ and the equality holds only when D is abelian.

We shall give a "relative version" of the conjecture (∗) and reduce (∗) to the case of quasi-simple groups. In this section we assume that the field K contains a primitive $|G|^3$-th root of unity.

In the following Lemmas 4.1 and 4.2, we use the following notation: N is a normal subgroup of a group G, B is a block of G, χ is an irreducible character in B, and ξ is an irreducible constituent of χ_N. Let $T_G(\xi)$ be the inertial group of ξ in G. Let $\mathrm{Irr}(T_G(\xi)|\xi)$ be the set of irreducible characters of $T_G(\xi)$ lying over ξ.

Lemma 4.1. *Let $\tilde{\chi} \in \mathrm{Irr}(T_G(\xi)|\xi)$ be such that $\tilde{\chi}^G = \chi$. Let $\tilde{B}$ be the block of $T_G(\xi)$ to which $\tilde{\chi}$ belongs. Let b be the block of N to which ξ belongs and assume that b is G-invariant. Let $\tilde{D}$ be a defect group of $\tilde{B}$. Then for every defect group D of B with $D \geq \tilde{D}$, we have $C_D(\tilde{D}) \leq \tilde{D}$. In particular, $Z(D) \leq Z(\tilde{D})$.*

Proof. Put $S_G(b) = \cap T_G(\eta)$, where η ranges over the irreducible characters in b. Since b is G-invariant, we see that $S_G(b) \lhd G$. Then, by Knörr [10], there is a block B_1 of $S_G(b)$ with defect group $\tilde{D} \cap S_G(b)$ which is covered by $\tilde{B}$. Since B also covers B_1, $D \cap S_G(b)$ is G-conjugate to $\tilde{D} \cap S_G(b)$. So, since $\tilde{D} \leq D$, we have $\tilde{D} \cap S_G(b) = D \cap S_G(b)$. On the other hand, $\tilde{D} \cap N = D \cap N$ is a defect group of b. Then, by [13, Lemma 4.14 (ii)], $C_D(\tilde{D}) \leq C_D(\tilde{D} \cap N) = C_D(D \cap N) \leq S_G(b)$. So $C_D(\tilde{D}) \leq S_G(b) \cap D = S_G(b) \cap \tilde{D} \leq \tilde{D}$. This completes the proof. □

Recently Watanabe [20] obtained simpler proofs of some results of [13] and [14]. Applying her method, we obtain the following.

Lemma 4.2. *Let the notation be as above and let D be a defect group of B. If χ is afforded by a $Z(D)N$-projective RG-module, then*

$$\mathrm{ht}\chi - \mathrm{ht}\xi \geq \nu|DN : Z(D)N|.$$

Proof. Let U be a $Z(D)N$-projective RG-module affording χ. Let Q be a vertex of U with $Q \leq D$. Then

$$\nu(\mathrm{rank}_R U) \geq \nu|G : QN| + \nu(\mathrm{rank}_R V),$$

where V is some indecomposable summand of U_N, cf. the proof of Proposition 2 in [20]. Then, since $\mathrm{rank}_R V$ is a multiple of $\xi(1)$, we get

$$\mathrm{ht}\chi - \mathrm{ht}\xi \geq \nu|DN : QN|.$$

By Knörr [11], $Q \geq_G Z(D)$. So, since $Q \leq_G Z(D)N$, we get $QN =_G Z(D)N$. Thus the result follows. □

The following is a "relative version" of Robinson's conjecture.

Theorem 4.3. *Let N be a normal subgroup of a group G with the following property*:

($*$) *is true for every block of every central extension of H/N for every subgroup H with $N \leq H \leq G$.*

Let B be a block of G with defect group D. Let χ be an irreducible character in B and let ξ be an irreducible constituent of χ_N. Then

$$\mathrm{ht}\chi - \mathrm{ht}\xi \leq \nu|DN : Z(D)N|$$

and the equality holds if and only if χ is afforded by a $Z(D)N$-projective RG-module.

Proof. First we note that in the statement of Theorem 4.3 the choice of D is an immaterial thing.

The proof is done by induction on $|G/N|$, the assertion being trivially true if $G = N$. It suffices to prove the inequality and the "only if" part. In fact, then the "if" part follows from the inequality and Lemma 4.2.

Let b be the block of N to which ξ belongs. By the Fong-Reynolds theorem and the induction hypothesis, we may assume that b is G-invariant. We divide the proof into several steps.

STEP 1. We may assume ξ is G-invariant.

Proof. Let $\tilde{\chi} \in \mathrm{Irr}(T_G(\xi)|\xi)$ be such that $\tilde{\chi}^G = \chi$. Let $\tilde{B}$ be the block of $T_G(\xi)$ to which $\tilde{\chi}$ belongs. We have

$$\mathrm{ht}\chi = \mathrm{ht}\tilde{\chi} + d(B) - d(\tilde{B}). \tag{1.a}$$

Let $\tilde{D}$ be a defect group of $\tilde{B}$. Since $\tilde{B}^G = B$, $\tilde{D} \leq D^g$ for some $g \in G$. So we may assume $\tilde{D} \leq D$ without loss of generality. If $T_G(\xi) < G$, then, by induction,

$$\mathrm{ht}\tilde{\chi} - \mathrm{ht}\xi \leq \nu|\tilde{D}N : Z(\tilde{D})N|. \tag{1.b}$$

Since b is G-invariant, we have $Z(\tilde{D}) \geq Z(D)$ by Lemma 4.1. From (1.a) and (1.b) we get

$$\begin{aligned} \mathrm{ht}\chi - \mathrm{ht}\xi &\leq d(B) - d(\tilde{B}) + \nu|\tilde{D}N : Z(\tilde{D})N| \\ &= d(B) + \nu|N : \tilde{D} \cap N| - \nu|Z(\tilde{D})N| \\ &= d(B) + \nu|N : D \cap N| - \nu|Z(\tilde{D})N| \\ &\quad (\text{since } \tilde{D} \cap N = D \cap N) \\ &\leq \nu|DN : Z(D)N| \quad (\text{since } Z(\tilde{D})N \geq Z(D)N). \end{aligned}$$

Thus

$$\text{(1.c)} \qquad \mathrm{ht}\chi - \mathrm{ht}\xi \leq \nu|DN : Z(D)N|.$$

If the equality holds in (1.c), then the equality holds throughout. So $Z(\tilde{D})N = Z(D)N$. Also, since the equality holds in (1.b), we see by induction that $\tilde{\chi}$ is afforded by a $Z(\tilde{D})N$-projective $RT_G(\xi)$-module V. Then V^G is a $Z(D)N$-projective RG-module affording χ. Thus we may assume $G = T_G(\xi)$.

The following step extends Step 5 of the proof of Theorem in [3] or (♯) in the proof of Theorem 6.1 in [13].

STEP 2. There exists a central extension of G,

$$1 \to Z \to \hat{G} \xrightarrow{f} G \to 1$$

with the following properties:
(2.a) $f^{-1}(N) = Z \times N_1$ for a normal subgroup N_1 of $\hat{G}$.
(2.b) ξ extends to $\hat{G}$.(Here we identify N with N_1 by (2.a).)
(2.c) Z is a finite cyclic group.
(2.d) There is a subgroup L of $\hat{G}$ such that $f^{-1}(Z(D)N) = Z \times L$ and that L is normal in $f^{-1}(DN)$.
(2.e) K is a splitting field for every subgroup of $\hat{G}$.

Proof. By Theorem 3.1, there is a D-invariant extension ζ of ξ to $Z(D)N$. Let $\rho : Z(D)N \to GL(\xi(1), K)$ be a representation affording ζ. Let T be a transversal of N in G with $1 \in T$. Since ξ and ζ are G-invariant and D-invariant, respectively, we can choose by standard arguments $\tilde{\rho}(t) \in GL(\xi(1), F)$ such that:

$$\begin{aligned} &\tilde{\rho}(t)\rho(n)\tilde{\rho}(t)^{-1} = \rho(tnt^{-1}), n \in N, \text{ for } t \in T - DN, \\ &\tilde{\rho}(t)\rho(x)\tilde{\rho}(t)^{-1} = \rho(txt^{-1}), x \in Z(D)N, \text{ for } t \in T \cap (DN - Z(D)N), \\ &\det \tilde{\rho}(t) = 1, \text{ for } t \in T - Z(D)N, \end{aligned}$$

where F is a suitable extension of K. For $t \in T \cap Z(D)N$, put $\tilde{\rho}(t) = \rho(t)$. For $g \in G$, write $g = tn$, $t \in T$, $n \in N$ and put $\tilde{\rho}(g) = \tilde{\rho}(t)\rho(n)$. Then

$$\text{(2.f)} \qquad \tilde{\rho}(g)\rho(n)\tilde{\rho}(g)^{-1} = \rho(gng^{-1}), \quad g \in G, n \in N, \text{and}$$

$$\text{(2.g)} \qquad \tilde{\rho}(x) = \rho(x), \quad x \in Z(D)N.$$

Further,

$$\text{(2.h)} \qquad \tilde{\rho}(g)\rho(x)\tilde{\rho}(g)^{-1} = \rho(gxg^{-1}), \quad g \in DN, x \in Z(D)N.$$

Let F^* be the multiplicative group of F. By (2.f) and (2.g), there is a factor set $\alpha : G \times G \to F^*$ satisfying the following:

$$\text{(2.i)} \qquad \tilde{\rho}(g)\tilde{\rho}(h) = \alpha(g,h)\tilde{\rho}(gh), \quad g, h \in G, \text{and}$$

$$\text{(2.j)} \qquad \alpha(x,y) = 1, \quad x, y \in Z(D)N.$$

Then, taking determinants in (2.i), we get $\alpha(g,h)^r = 1$, g, $h \in G$, where $r = |Z(D)N|\xi(1)$.

Now let Z be the cyclic subgroup of order r of K^*. (Since r divides $|G|^2$ and K contains a primitive $|G|^3$-th root of unity, Z exists.) Let

$$1 \to Z \to \hat{G} \xrightarrow{f} G \to 1$$

be the central extension of G corresponding to the factor set α. So $\hat{G} = Z \times G$ as a set and the multiplication in it is defined by

$$(z,g)(w,h) = (zw\alpha(g,h), gh), \;\; z, w \in Z, \;\; g, h \in G.$$

We show that this central extension is a required one. To prove (2.d), put $L = \{(1,x) | x \in Z(D)N\}$. By (2,j), L is a subgroup of $f^{-1}(Z(D)N)$ and $f^{-1}(Z(D)N) = Z \times L$. To show that L is normal in $f^{-1}(DN)$, it suffices to prove $(z,g)(1,x) = (1, gxg^{-1})(z,g)$, $z \in Z$, $g \in DN$, $x \in Z(D)N$; namely $\alpha(g,x) = \alpha(gxg^{-1}, g)$. Now

$$\begin{aligned}
\alpha(gxg^{-1}, g)I &= \tilde{\rho}(gxg^{-1})\tilde{\rho}(g)\tilde{\rho}(gx)^{-1} \text{ (by (2.i))} \\
&= \tilde{\rho}(g)\rho(x)\tilde{\rho}(g)^{-1}\tilde{\rho}(g)\tilde{\rho}(gx)^{-1} \text{ (by (2.g) and (2.h))} \\
&= \tilde{\rho}(g)\rho(x)\tilde{\rho}(gx)^{-1} \\
&= \alpha(g,x)I \text{ (by (2.g) and (2.i))},
\end{aligned}$$

where I is the identity matrix of degree $\xi(1)$. Thus (2.d) follows. To show (2.a) and (2.b), put $N_1 = \{(1,n) | n \in N\}$. Then we have $f^{-1}(N) = Z \times N_1$ by (2.j). Similar computation as in the above shows that N_1 is a normal subgroup of $\hat{G}$. If we let $\hat{\rho}((z,g)) = z\tilde{\rho}(g)$, $z \in Z$, $g \in G$, then $\hat{\rho}$ is a representation of $\hat{G}$ and, since

$\hat{\rho}((1,n)) = \rho(n)$ for $n \in N$, $\hat{\rho}$ affords an extension of ξ to $\hat{G}$. Since $|\hat{G}| = r|G|$ divides $|G|^3$ and K contains a primitive $|G|^3$-th root of unity, (2.e) follows. This completes the proof.

We fix a central extension $\hat{G}$ of G as above. Let $\hat{\chi}$ be the inflation of χ to $\hat{G}$. Let $\hat{B}$ be the block of $\hat{G}$ to which $\hat{\chi}$ belongs and let $\hat{D}$ be a defect group of $\hat{B}$. Since $\hat{G}$ is a central extension of G, we may choose $\hat{D}$ so that $\hat{D}Z/Z = D$.

STEP 3. We have:

(3.a) $\hat{D}Z/Z = D$. In particular, $d(\hat{B}) = d(B) + \nu|Z|$.

(3.b) $Z(\hat{D})Z/Z = Z(D)$. In particular, $\nu|Z(\hat{D})| = \nu|Z(D)| + \nu|Z|$.

(3.c) $\hat{D} \cap N = D \cap N$.

(3.d) $Z(\hat{D}) \cap N = Z(D) \cap N$.

Proof. (3.a) This is true by our choice of $\hat{D}$.

(3.b) By (3.a), $Z(\hat{D})Z/Z \le Z(D)$. In the notation of Step 2, $f^{-1}(Z(D)N) = Z \times L$. Let $U = f^{-1}(Z(D)) \cap \hat{D}$. Let Z_p be a p-Sylow subgroup of Z. It is obvious that $Z_p \le U \le Z_p \times L$. So $U = Z_p \times (U \cap L)$. Then, since $\hat{D} \le f^{-1}(DN)$ normalizes L by (2.d) and $[U, \hat{D}] \le Z$ by (3.a), we get $[U, \hat{D}] = [U \cap L, \hat{D}] \le L \cap Z = 1$. So $U \le Z(\hat{D})$ and $Z(D) \le Z(\hat{D})Z/Z$. Hence $Z(\hat{D})Z/Z = Z(D)$.

(3.c) By our choice of $\hat{D}$ (and our convention that $N_1 = N$), $\hat{D} \cap N \le D \cap N$. Since both $\hat{D} \cap N$ and $D \cap N$ are defect groups of b, we get $\hat{D} \cap N = D \cap N$.

(3.d) By (3.a), $[Z(\hat{D}) \cap N, D] = 1$. Thus $Z(\hat{D}) \cap N \le Z(D) \cap N$ by (3.c). On the other hand, $[Z(D) \cap N, \hat{D}] \le Z$ by (3.a), so $[Z(D) \cap N, \hat{D}] \le Z \cap N = 1$. Thus $Z(D) \cap N \le Z(\hat{D}) \cap N$ by (3.c) and (3.d) follows.

There is an extension $\hat{\xi}$ of ξ to $\hat{G}$ by (2.b). Then there is a unique irreducible character θ of $\hat{G}/N$ with $\hat{\chi} = \hat{\xi} \otimes \theta$. Let $\tilde{B}$ be the block of $\hat{G}/N$ to which θ belongs and let $\tilde{D}$ be a defect group of $\tilde{B}$.

STEP 4. We have:

(4.a) $$\mathrm{ht}\chi - \mathrm{ht}\xi = \mathrm{ht}\theta + d(\hat{B}) - d(\tilde{B}) - d(b).$$

(4.b) $$\mathrm{ht}\theta \le \nu|\tilde{D} : Z(\tilde{D})|.$$

Further, we may choose $\tilde{D}$ so that

(4.c) $Z(\tilde{D}) \ge Z(\hat{D})N/N$. In particular,
$$\nu|Z(\tilde{D})| \ge \nu|Z(\hat{D})| - \nu|Z(\hat{D}) \cap N|.$$

Proof. (4.a) follows from (3.a). Since $\hat{G}/N$ is a central extension of G/N, we get (4.b) by our assumption on N. Let V be an irreducible $R\hat{G}$-module affording $\hat{\xi}$. Then $\tilde{B}$ is V-dominated by $\hat{B}$. So we get (4.c) by Theorem 2.8 (ii).

STEP 5. Conclusion.

Proof. We have

$$\begin{aligned}
\mathrm{ht}\chi - \mathrm{ht}\xi &= \mathrm{ht}\theta + d(\hat{B}) - d(\tilde{B}) - d(b) \text{ (by (4.a))}\\
&\le \nu|\tilde{D} : Z(\tilde{D})| + d(\hat{B}) - d(\tilde{B}) - d(b) \text{ (by (4.b))}\\
&= -\nu|Z(\tilde{D})| + d(\hat{B}) - d(b)\\
&\le -(\nu|Z(\hat{D})| - \nu|Z(\hat{D}) \cap N|) + d(B) + \nu|Z| - d(b)\\
&\quad \text{(by (4.c) and (3.a))}\\
&= -(\nu|Z(D)| + \nu|Z|) + \nu|Z(\hat{D}) \cap N| + d(B) + \nu|Z| - d(b)\\
&\quad \text{(by (3.b))}\\
&= d(B) - d(b) - \nu|Z(D)| + \nu|Z(D) \cap N| \text{ (by (3.d))}\\
&= \nu|D : D \cap N| - \nu|Z(D) : Z(D) \cap N|\\
&= \nu|DN/N| - \nu|Z(D)N/N|\\
&= \nu|DN : Z(D)N|.
\end{aligned}$$

Thus we get

$$\mathrm{ht}\chi - \mathrm{ht}\xi \le \nu|DN : Z(D)N|. \tag{5.a}$$

It remains to show that the equality holds in (5.a) only if χ is afforded by a $Z(D)N$-projective RG-module. Assume the equality holds in (5.a), then in the above proof of (5.a) the equality holds throughout. Hence $\tilde{D}$ is abelian by (4.b) and our assumption on N, and $Z(\tilde{D}) = Z(\hat{D})N/N$ by (4.c). Thus,

$$\tilde{D} = Z(\hat{D})N/N. \tag{5.b}$$

Let W be an $R[\hat{G}/N]$-module affording θ and V an $R\hat{G}$-module affording $\hat{\xi}$. Then $V \otimes W$ affords $\hat{\chi}$. Since W is, as an $R[\hat{G}/N]$-module, $\tilde{D}$-projective, $V \otimes W$ is $Z(\hat{D})N$-projective by (5.b). So, $V \otimes W$ is, as an RG-module, $Z(D)N$-projective by (3.b) and affords χ. This completes the proof. □

Lemma 4.4. *If* $(*)$ *is true for every block of every quasi-simple group, then it is true for every block of every finite group* G *such that* G/C *is simple for a central subgroup* C *of* G.

Proof. If G/C is of prime order, then G is abelian and $(*)$ is trivially true. Assume that G/C is non-abelian simple. Then, as is well-known, $G = G'C$, where G' is the commutator subgroup of G (which is quasi-simple). Let C_p be a p-Sylow subgroup of C and D a defect group of B. Let B' be the block of G' covered by B. Then $C_p \leq D \leq C_pG'$, so if we put $Q = D \cap G'$, then $D = C_pQ$ and Q is a defect group of B'. Let χ be an irreducible character in B. Clearly $\chi_{G'}$ is an irreducible character in B'. By assumption, we get

$$\text{(1)} \qquad \text{ht}\chi_{G'} \leq \nu|Q : Z(Q)|.$$

Since $G = G'C$ and $D \geq C_p$, $|G/G'D|$ is prime to p. This shows $\text{ht}\chi = \text{ht}\chi_{G'}$. Also, easy computation shows $\nu|D : Z(D)| = \nu|Q : Z(Q)|$. So we get

$$\text{(2)} \qquad \text{ht}\chi \leq \nu|D : Z(D)|.$$

If the equality holds in (2), then the equality holds in (1). So Q is abelian by assumption, and D is abelian. This completes the proof. □

Lemma 4.5. *Let N be a normal subgroup of a group G. Let B be a block of G with defect group D. Let χ be an irreducible character in B. Then the following are equivalent.*
(i) *χ is afforded by a $Z(D)N$-projective RG-module.*
(ii) *χ is afforded by a $Z(D)(D \cap N)$-projective RG-module.*
Further, the following are equivalent.
(iii) *χ is afforded by a $Z(D)$-projective RG-module.*
(iv) *D is abelian.*

Proof. (i) ⇒ (ii): Let U be a $Z(D)N$-projective RG-module affording χ. By Knörr [11], there is a vertex Q of U such that

$$\text{(1)} \qquad D \geq Q \geq C_D(Q) \geq Z(D).$$

We have $Q \leq {}_G Z(D)N$. So, by (1), we get $QN = Z(D)N$ and $Q = Z(D)(Q \cap N) \leq Z(D)(D \cap N)$. Thus (ii) holds.

(ii) ⇒ (i): This is trivial.

(iii) ⇒ (iv): Let U be a $Z(D)$-projective RG-module affording χ. There is a vertex Q of U such that (1) above holds. Then, since $Q \leq {}_G Z(D)$, we get, by (1), $Q = Z(D) = D$. So D is abelian.

(iv) ⇒ (iii): This is trivial. □

Theorem 4.6. *If $(*)$ is true for every block of every quasi-simple group, then it is true for every block of every finite group.*

Proof. Let B a block of G with a defect group D. The proof is done by induction on $|G/Z(G)|$. If $G = Z(G)$, then $(*)$ is trivially true. Assume $G > Z(G)$ and let $N/Z(G)$ be a maximal normal subgroup of $G/Z(G)$. We claim that N is a normal subgroup of G satisfying the condition in Theorem 4.3. Let H be a subgroup such that $N \leq H \leq G$ and let L be a central extension of H/N. If $H < G$, then $|L/Z(L)| \leq |H/N| < |G/N| \leq |G/Z(G)|$, so $(*)$ is true for every block of L by induction. On the other hand, if $H = G$, then $(*)$ is true for every block of L by Lemma 4.4 and assumption. So the claim is proved. Thus we may apply Theorem 4.3 to conclude that for every irreducible character χ in B and an irreducible constituent ξ of χ_N, we have

$$\mathrm{ht}\chi - \mathrm{ht}\xi \leq \nu|DN : Z(D)N|. \tag{1}$$

Let b be the block of N to which ξ belongs. Since $|N/Z(N)| < |G/Z(G)|$, we get by induction,

$$\mathrm{ht}\xi \leq \nu|\delta : Z(\delta)|, \tag{2}$$

where δ is a defect group of b. Replacing ξ by a G-conjugate of it if necessary, we may assume $\delta = D \cap N$ by Knörr [10]. Thus, by (1) and (2),

$$\begin{aligned} \mathrm{ht}\chi &\leq \nu|DN : Z(D)N| + \nu|\delta : Z(\delta)| \\ &= \nu|D : Z(D)| + \nu|Z(D) \cap N| - \nu|Z(\delta)| \\ &\leq \nu|D : Z(D)| \text{ (since } Z(D) \cap N \leq Z(\delta)). \end{aligned} \tag{3}$$

Hence

$$\mathrm{ht}\chi \leq \nu|D : Z(D)|. \tag{4}$$

If the equality holds in (4), then equality holds throughout. So, by (1) and Theorem 4.3, we see that χ is afforded by a $Z(D)N$-projective RG-module. Further, we get $\delta \leq Z(D)$ by (2), (3) and induction. Now, $Z(D)(D \cap N) = Z(D)\delta = Z(D)$. So, by Lemma 4.5, we see that D is abelian. Thus the proof is complete. □

The following is a "relative version" of the results of Fong [7, (3C)] and Watanabe [18, Proposition].

Corollary 4.7. *Let N be a normal subgroup of a group G such that G/N is p-solvable. Let B be a p-block of G with defect group D. Let χ be an irreducible character in B and let ξ be an irreducible constituent of χ_N. Then*

$$\mathrm{ht}\chi - \mathrm{ht}\xi \leq \nu|DN : Z(D)N|$$

and the equality holds if and only if χ is afforded by a $Z(D)N$-projective RG-module.

Proof. Since $(*)$ is true for every block of a p-solvable quasi-simple group, because a p-solvable quasi-simple group is a p'-group, $(*)$ is true for every block of a p-solvable group, cf. the proof of Theorem 4.6. Then the assertion follows from Theorem 4.3. □

REMARK. (1) If $N = 1$, the corollary above boils down to the results of Fong [7] and Watanabe [18], cf. Lemma 4.5.

(2) The modular version of Corollary 4.7 is also true.

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2-27 Meiji-machi
Izumi Toki-shi
Gifu-ken 509-5146
Japan