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ON THE TRACE NORM ESTIMATE OF THE TROTTER PRODUCT FORMULA FOR SCHRÖDINGER OPERATORS

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0. Introduction

It is well-known that for the Schrödinger operator $(-1/2)\Delta + V$ with a nonnegative continuous potential V in the space $L_2(\mathbb{R}^d)$, the Trotter product formula

$$\lim_{n\to\infty}(e^{-tV/n}e^{-t(-(1/2)\Delta)/n})^n = e^{-t(-(1/2)\Delta+V)} \tag{0.1}$$

and its variant

$$\lim_{n\to\infty}(e^{-tV/2n}e^{-t(-(1/2)\Delta)/n}e^{-tV/2n})^n = e^{-t(-(1/2)\Delta+V)} \tag{0.2}$$

hold in the strong operator topology. It has recently been discussed that if V is e.g. in C^2 and satisfies

$$\begin{aligned} V(x) &\geq c(1+|x|^2)^{\rho/2} \\ |\nabla^m V(x)| &\leq c_m(1+|x|^2)^{(\rho-m)_+/2}, \quad m=1,2 \end{aligned} \tag{0.3}$$

for some $0 \leq \rho < \infty$, $0 < c < \infty$ and $0 \leq c_1, c_2 < \infty$ (which is the condition from [2]), (0.1) and (0.2) are convergent in the L_p-operator norm ($1 \leq p \leq \infty$). More precisely, as $t \downarrow 0$

$$\|(e^{-tV/n}e^{-t(-(1/2)\Delta)/n})^n - e^{-t(-(1/2)\Delta+V)}\|_{p\to p} = \left(\frac{1}{n}\right)^{2/(2\vee\rho)} O(t^{1/2+1/(1\vee\rho)}) \tag{0.4}$$

$$\|(e^{-tV/2n}e^{-t(-(1/2)\Delta)/n}e^{-tV/2n})^n - e^{-t(-(1/2)\Delta+V)}\|_{p\to p} = \left(\frac{1}{n}\right)^{2/(2\vee\rho)} O(t^{1+2/(2\vee\rho)}), \tag{0.5}$$

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where $\|\cdot\|_{p\to p}$ stands for the L_p-operator norm. This convergence in the operator norm is proved in [4], [5], [1], [6], [2] and [10], though these works except [6] and [10] deal only with the convergence in L_2-operator norm.

In [2], however, the convergence in the trace norm is further studied. Namely, when $\rho > 0$ in (0.3), they have shown operator-theoretically that

$$(0.6)\qquad \|(e^{-tV/n}e^{-t(-(1/2)\Delta)/n})^n - e^{-t(-(1/2)\Delta+V)}\|_{\text{trace}} = O\Big(\Big(\frac{1}{n}\Big)^{2/(2\vee\rho)}\Big)$$

$$(0.7)\qquad \|(e^{-tV/2n}e^{-t(-(1/2)\Delta)/n}e^{-tV/2n})^n - e^{-t(-(1/2)\Delta+V)}\|_{\text{trace}} = O\Big(\Big(\frac{1}{n}\Big)^{2/(2\vee\rho)}\Big),$$

as $n \to \infty$, locally uniformly in $t > 0$, where $\|\cdot\|_{\text{trace}}$ stands for the trace norm. Since $\|\cdot\|_{2\to 2} \le \|\cdot\|_{\text{trace}}$, the convergence in the trace norm implies that in the L_2-operator norm, so that their result in the trace norm is better compared with the others. But they have not observed the behavior of the error bounds in (0.6) and (0.7) as $t \downarrow 0$.

The aim of this paper is to take care of this point to give another proof to the trace norm convergence of (0.1) and (0.2), that is, a probabilistic proof following the lines of [2]. It should be emphasized here that in the one-dimensional case ($d = 1$) the convergence of (0.1) and (0.2) in the trace norm may hold, locally uniformly even in $t \ge 0$.

In Section 1, the condition on V is presented, which relaxes (0.3), and Theorem is stated. Its proof is done in Section 4. For this, Sections 2 and 3 are devoted to preliminaries. Section 5 deals as a remark with the case of less regular potentials V.

1. Presentation of Theorem

First we present the condition on a scalar potential V: Let $0 < \rho < \infty$, $0 < \delta \le 1$, $0 \le C_1$, $C_2 < \infty$ and $0 \le \mu$, $\nu < \infty$. Let $V : \mathbb{R}^d \to [0,\infty)$ be a C^1-function such that

(A)$'_2$

(o) $$\liminf_{|x|\to\infty} \frac{V(x)}{|x|^\rho} > 0$$

(i) $$|\nabla V(x)| \le C_1(1 + V(x)^{1-\delta})$$

(ii) $$|\nabla V(x) - \nabla V(y)| \le C_2\Big\{V(x)^{(1-2\delta)_+}(1 + |x-y|^\mu) + 1 + |x-y|^\nu\Big\}|x-y|.$$

REMARK 1. (i) The conditions (o) and (i) in (A)$'_2$ imply $1/\delta \ge \rho$.
(ii) The condition (o) in (A)$'_2$ is equivalent to that

$$(1.1)\qquad V(x) \ge c|x|^\rho - c', \qquad x \in \mathbb{R}^d$$

for some positive constants c and c'.

REMARK 2. The condition (0.3) implies (A)$'_2$(i) and (ii) with $\delta = 1\wedge(1/\rho)$, $C_1 =$

$c_1 c^{-(1-1\wedge(1/\rho))}$, $C_2 = c_2 2^{(\rho-3)_+}(c^{-(1-2(1\wedge(1/\rho)))_+}/2 \vee 1)$, $\mu = 0$ and $\nu = (\rho-2)_+$.

In the following let V be as above. Set

$$\begin{aligned} H &:= -\frac{1}{2}\Delta + V, \\ K(t) &:= e^{-tV/2}e^{-t(-(1/2)\Delta)}e^{-tV/2}, \\ G(t) &:= e^{-tV}e^{-t(-(1/2)\Delta)}, \\ R(t) &:= e^{-t(-(1/2)\Delta)/2}e^{-tV}e^{-t(-(1/2)\Delta)/2}. \end{aligned}$$

By the condition (o), e^{-tH}, $K(t)$, $G(t)$ and $R(t)$ $(t > 0)$ are trace class operators. Let us denote by $\|\cdot\|_{\text{trace}}$ the trace norm.

Theorem. *Let $T \geq 1$ and $0 < t \leq T$. Then*

(i) *For $n \geq 2$*

$$\left\| K\left(\frac{t}{n}\right)^n - e^{-tH} \right\|_{\text{trace}} \leq \text{const}\left(\frac{1}{n}\right)^{1\wedge 2\delta} t^{1+1\wedge 2\delta - d(1/2+1/\rho)},$$

where const *depends only on C_1, C_2, δ, μ, ν, ρ, c, c', d and T (c and c' are positive constants in* (1.1)).

(ii) *For $n \geq 3$*

$$\left\| G\left(\frac{t}{n}\right)^n - e^{-tH} \right\|_{\text{trace}} \leq \text{const}\left(\frac{1}{n}\right)^{1\wedge 2\delta} t^{1/2+\delta - d(1/2+1/\rho)},$$

where const *depends only on C_1, C_2, δ, μ, ν, ρ, c, c', d and T.*

(iii) *For $n \geq 3$*

$$\left\| R\left(\frac{t}{n}\right)^n - e^{-tH} \right\|_{\text{trace}} \leq \text{const}\left(\frac{1}{n}\right)^{1\wedge 2\delta} t^{1/2+\delta - d(1/2+1/\rho)},$$

where const *depends only on C_1, C_2, δ, μ, ν, ρ, c, c', d and T.*

REMARK 3. When $d = 1$ and $\rho = 1/\delta$, $\|K(t/n)^n - e^{-tH}\|_{\text{trace}}$, $\|G(t/n)^n - e^{-tH}\|_{\text{trace}}$, $\|R(t/n)^n - e^{-tH}\|_{\text{trace}} = O((1/n)^{1\wedge 2\delta})$ as $n \to \infty$, locally uniformly in $t \geq 0$.

2. Decomposition of $K(t/n)^n - e^{-tH}$, $G(t/n)^n - e^{-tH}$ and $R(t/n)^n - e^{-tH}$

It is observed that for $n \geq 2$ and $t \geq 0$

$$K\left(\frac{t}{n}\right)^n - e^{-tH}$$

$$= \sum_{j=1}^{n} K\left(\frac{t}{n}\right)^{j-1} \left(K\left(\frac{t}{n}\right) - e^{-tH/n}\right) e^{-(n-j)tH/n}$$

$$= \sum_{1 \le j \le [n/2]} K\left(\frac{t}{n}\right)^{j-1} \left(K\left(\frac{t}{n}\right) - e^{-tH/n}\right) e^{-(n-j)tH/n}$$

$$+ \sum_{[n/2] < j \le n} K\left(\frac{t}{n}\right)^{j-1} \left(K\left(\frac{t}{n}\right) - e^{-tH/n}\right) e^{-(n-j)tH/n},$$

$$G\left(\frac{t}{n}\right)^n - e^{-tH}$$

$$= e^{-tV/2n}\left(K\left(\frac{n-1}{n}t\frac{1}{n-1}\right)^{n-1} - e^{-(n-1)tH/n}\right) e^{-tV/2n} e^{-t(-(1/2)\Delta)/n}$$

$$+ \left[e^{-tV/2n}, e^{-(n-1)tH/n}\right] e^{-tV/2n} e^{-t(-(1/2)\Delta)/n}$$

$$+ e^{-(n-1)tH/n}\left(G\left(\frac{t}{n}\right) - e^{-tH/n}\right),$$

$$R\left(\frac{t}{n}\right)^n - e^{-tH}$$

$$= e^{-t(-(1/2)\Delta)/2n} e^{-tV/2n}\left(K\left(\frac{n-1}{n}t\frac{1}{n-1}\right)^{n-1} - e^{-(n-1)tH/n}\right)$$

$$\times e^{-tV/2n} e^{-t(-(1/2)\Delta)/2n}$$

$$+ e^{-t(-(1/2)\Delta)/2n}\left[e^{-tV/2n}, e^{-(n-1)tH/n}\right] e^{-tV/2n} e^{-t(-(1/2)\Delta)/2n}$$

$$+ \left[e^{-t(-(1/2)\Delta)/2n}, e^{-(n-1)tH/n}\right] e^{-tV/n} e^{-t(-(1/2)\Delta)/2n}$$

$$+ e^{-(n-1)tH/n}\left(R\left(\frac{t}{n}\right) - e^{-tH/n}\right).$$

Here we use the following fundamental inequalities (cf. e.g. [3]):

(i) For trace class operators A and B on $L_2(\mathbb{R}^d)$

$$\|A + B\|_{\text{trace}} \le \|A\|_{\text{trace}} + \|B\|_{\text{trace}}. \tag{2.1}$$

(ii) For a trace class operator A and a bounded operator B on $L_2(\mathbb{R}^d)$

$$\begin{aligned} \|AB\|_{\text{trace}} &\le \|A\|_{\text{trace}} \, \|B\|_{2\to 2}, \\ \|BA\|_{\text{trace}} &\le \|B\|_{2\to 2} \, \|A\|_{\text{trace}} \end{aligned} \tag{2.2}$$

where $\|\cdot\|_{2\to 2}$ denotes the L_2-operator norm. By (2.1), (2.2) and the contraction property of e^{-tV} and $e^{-t(-(1/2)\Delta)}$, we have

$$\left\|K\left(\frac{t}{n}\right)^n - e^{-tH}\right\|_{\text{trace}} \tag{2.3}$$

$$\leq \left\|K\left(\frac{t}{n}\right) - e^{-tH/n}\right\|_{2\to 2}$$
$$\times \left\{ \sum_{1\leq j\leq [n/2]} \|e^{-(n-j)tH/n}\|_{\mathrm{trace}} + \sum_{[n/2]<j\leq n} \left\|K\left(\frac{j-1}{n}\frac{t}{j-1}\right)^{j-1}\right\|_{\mathrm{trace}} \right\},$$

(2.4)
$$\left\|G\left(\frac{t}{n}\right)^n - e^{-tH}\right\|_{\mathrm{trace}}$$
$$\leq \left\|K\left(\frac{n-1}{n}t\frac{1}{n-1}\right)^{n-1} - e^{-(n-1)tH/n}\right\|_{\mathrm{trace}}$$
$$+ \left\|\left[e^{-tV/2n}, e^{-(n-1)tH/n}\right]\right\|_{\mathrm{trace}}$$
$$+ \left\|e^{-(n-1)tH/n}\right\|_{\mathrm{trace}} \left\|G\left(\frac{t}{n}\right) - e^{-tH/n}\right\|_{2\to 2},$$

(2.5)
$$\left\|R\left(\frac{t}{n}\right)^n - e^{-tH}\right\|_{\mathrm{trace}}$$
$$\leq \left\|K\left(\frac{n-1}{n}t\frac{1}{n-1}\right)^{n-1} - e^{-(n-1)tH/n}\right\|_{\mathrm{trace}}$$
$$+ \left\|\left[e^{-tV/2n}, e^{-(n-1)tH/n}\right]\right\|_{\mathrm{trace}}$$
$$+ \left\|\left[e^{-t(-(1/2)\Delta)/2n}, e^{-(n-1)tH/n}\right]\right\|_{\mathrm{trace}}$$
$$+ \left\|e^{-(n-1)tH/n}\right\|_{\mathrm{trace}} \left\|R\left(\frac{t}{n}\right) - e^{-tH/n}\right\|_{2\to 2}.$$

3. Kernels of e^{-tH}, $K(t/n)^n$, $G(t/n)^n$ and $R(t/n)^n$

Let (W, P_0) be a d-dimensional Wiener space: W is the totality of all continuous functions $w : [0,1] \to \mathbb{R}^d$ such that $w(0) = 0$ with the topology of uniform convergence and P_0 is the Wiener measure on W. Set

$$X(t,w) := w(t),$$
$$X_0(t,w) := X(t,w) - tX(1,w) = w(t) - tw(1),$$

and

$$p(t,x) := P_0(w(t) \in dx)/dx = \left(\frac{1}{2\pi t}\right)^{d/2} \exp\left\{-\frac{|x|^2}{2t}\right\}.$$

Note that $(X_0(t))_{0\leq t\leq 1}$ is the Brownian bridge, i.e., the probability law of $X_0(\cdot)$ coincides with $P_0(\cdot\,|X(1) = 0)$. By using this, the integral kernels of e^{-tH}, $K(t/n)^n$, $G(t/n)^n$ and $R(t/n)^n$ are expressed as follows (cf. [8], [9]):

Proposition 1.

$$e^{-tH}(x,y) = p(t,x-y) \times E_0\Big[\exp\Big\{-t\int_0^1 V(x+s(y-x)+\sqrt{t}X_0(s))ds\Big\}\Big], \tag{3.1}$$

$$K\Big(\frac{t}{n}\Big)^n(x,y) = p(t,x-y) \times E_0\Big[\exp\Big\{-\frac{t}{2}\Big(\int_0^1 V(x+s_n^-(y-x)+\sqrt{t}X_0(s_n^-))ds + \int_0^1 V(x+s_n^+(y-x)+\sqrt{t}X_0(s_n^+))ds\Big)\Big\}\Big], \tag{3.2}$$

$$G\Big(\frac{t}{n}\Big)^n(x,y) = p(t,x-y) \times E_0\Big[\exp\Big\{-t\int_0^1 V(x+s_n^-(y-x)+\sqrt{t}X_0(s_n^-))ds\Big\}\Big], \tag{3.3}$$

$$R\Big(\frac{t}{n}\Big)^n(x,y) = p(t,x-y) \times E_0\Big[\exp\Big\{-t\int_0^1 V(x+\frac{1}{2}(s_n^-+s_n^+)(y-x)+\sqrt{t}X_0(\frac{1}{2}(s_n^-+s_n^+)))ds\Big\}\Big]. \tag{3.4}$$

Here $s_n^+ := ([ns]+1)/n$ *and* $s_n^- := [ns]/n$.

There is another description of the Brownian bridge. For ξ, $\eta \in \mathbb{R}^d$ and $0 < t_0 \leq 1$, let $(X_\xi^{t_0,\eta}(t))_{0\leq t\leq t_0}$ be the solution of the following SDE (cf. [7], p. 243–244):

$$\begin{cases} dx_t = dw_t + \dfrac{\eta - x_t}{t_0 - t}\,dt, & 0 \leq t \leq t_0 \\ x_0 = \xi. \end{cases} \tag{3.5}$$

Then $(X_\xi^{t_0,\eta}(t))_{0\leq t\leq t_0} \overset{\mathcal{L}}{\sim} (\xi + (t/t_0)(\eta-\xi) + w(t) - (t/t_0)w(t_0))_{0\leq t\leq t_0}$. In particular, $(X_0^{1,0}(t))_{0\leq t\leq 1} \overset{\mathcal{L}}{\sim} (X_0(t))_{0\leq t\leq 1}$. By this and the scaling property, the expressions in Proposition 1 are rewritten as follows:

Proposition 2.

$$e^{-tH}(x,y) = p(t,x-y)E_0\Big[\exp\Big\{-\int_0^t V(X_x^{t,y}(s))ds\Big\}\Big], \tag{3.6}$$

$$K\Big(\frac{t}{n}\Big)^n(x,y) = p(t,x-y) \tag{3.7}$$

$$\times E_0\Big[\exp\Big\{-\frac{1}{2}\Big(\int_0^t V\Big(X_x^{t,y}\Big(\Big(\frac{s}{t}\Big)_n^- t\Big)\Big)ds + \int_0^t V\Big(X_x^{t,y}\Big(\Big(\frac{s}{t}\Big)_n^+ t\Big)\Big)ds\Big)\Big\}\Big],$$

$$G\Big(\frac{t}{n}\Big)^n(x,y) = p(t,x-y)E_0\Big[\exp\Big\{-\int_0^t V\Big(X_x^{t,y}\Big(\Big(\frac{s}{t}\Big)_n^- t\Big)\Big)ds\Big\}\Big], \tag{3.8}$$

$$R\Big(\frac{t}{n}\Big)^n(x,y) = p(t,x-y)E_0\Big[\exp\Big\{-\int_0^t V\Big(X_x^{t,y}\Big(\frac{1}{2}\Big(\Big(\frac{s}{t}\Big)_n^- + \Big(\frac{s}{t}\Big)_n^+\Big)t\Big)\Big)ds\Big\}\Big]. \tag{3.9}$$

4. Proof of Theorem

CLAIM 1. Let $t \geq 0$. Then

(i) $\|K(t) - e^{-tH}\|_{2\to 2} \leq$ const $\{C_1^2(t^3 + t^{1+2\delta}) + C_2(t^2 + t^{1+1\wedge 2\delta} + t^{1+\mu/2+1\wedge 2\delta} + t^{2+\nu/2})\}$, where const depends only on δ, μ, ν and d.

(ii) $\|G(t) - e^{-tH}\|_{2\to 2} \leq$ const $\{C_1(t^{3/2} + t^{1/2+\delta}) + C_1^2(t^3 + t^{1+2\delta}) + C_2(t^2 + t^{1+1\wedge 2\delta})\}$, where const depends only on δ and d.

(iii) $\|R(t) - e^{-tH}\|_{2\to 2} \leq$ const $\{C_1(t^{3/2} + t^{1/2+\delta}) + C_1^2(t^3 + t^{1+2\delta}) + C_2(t^2 + t^{1+1\wedge 2\delta})\}$, where const depends only on δ and d.

Proof. We here show the estimate in the kernel level: As for (i)

$$\Big|E_0\Big[\exp\Big\{-\int_0^t V(X_x^{t,y}(s))ds\Big\}\Big] - \exp\Big\{-\frac{t}{2}\Big(V(x)+V(y)\Big)\Big\}\Big|$$

$$\begin{aligned}\leq \text{const}\,\Big\{&C_1^2(t^3 + t^{1+2\delta}) + C_2(t|x-y|^2 + t^2 + t^{1\wedge 2\delta}|x-y|^2 + t^{1+1\wedge 2\delta}\\ &+ t^{1\wedge 2\delta}|x-y|^{2+\mu} + t^{1+\mu/2+1\wedge 2\delta} + t|x-y|^{2+\nu} + t^{2+\nu/2})\Big\},\end{aligned} \tag{4.1}$$

where const depends only on δ, μ, ν and d. As for (ii)

$$\Big|E_0\Big[\exp\Big\{-\int_0^t V(X_x^{t,y}(s))ds\Big\}\Big] - e^{-tV(x)}\Big|$$

$$\begin{aligned}\leq \text{const}\,\Big\{&C_1(t|x-y| + t^{3/2} + t^{\delta}|x-y| + t^{1/2+\delta}) + C_1^2(t^3 + t^{1+2\delta})\\ &+ C_2(t^2 + t^{1+1\wedge 2\delta})\Big\},\end{aligned} \tag{4.2}$$

where const depends only on δ and d. As for (iii)

$$\Big|E_0\Big[\exp\Big\{-\int_0^t V(X_x^{t,y}(s))ds\Big\}\Big] - E_0\Big[\exp\Big\{-tV\Big(X_x^{t,y}\Big(\frac{t}{2}\Big)\Big)\Big\}\Big]\Big|$$

$$\begin{aligned}\leq \text{const}\,\Big\{&C_1(t|x-y| + t^{3/2} + t^{\delta}|x-y| + t^{1/2+\delta}) + C_1^2(t^3 + t^{1+2\delta})\\ &+ C_2(t^2 + t^{1+1\wedge 2\delta})\Big\},\end{aligned} \tag{4.3}$$

where const depends only on δ and d. From these estimates Claim 1 follows immediately.

First suppose that $V:\mathbb{R}^d\to[0,\infty)$ is a C^2-function and satisfies (A)$'_2$(i) and (ii). Let $0<T_1\le T$. By noting (3.5), Itô's formula gives us that

$$\begin{aligned}
&\exp\Big\{-\int_0^{T_1} V(X_x^{T,y}(s))ds\Big\}\exp\Big\{-\frac{T-T_1}{2}\Big(V(X_x^{T,y}(T_1))+V(y)\Big)\Big\}\\
&\quad-\exp\Big\{-\frac{T}{2}\Big(V(x)+V(y)\Big)\Big\}\\
&=\sum_{i=1}^d\int_0^{T_1}\exp\Big\{-\int_0^t V(X_x^{T,y}(s))ds\Big\}\exp\Big\{-\frac{T-t}{2}\Big(V(X_x^{T,y}(t))+V(y)\Big)\Big\}\\
&\qquad\qquad\times\Big(-\frac{T-t}{2}\Big)\partial_iV(X_x^{T,y}(t))dw_t^i\\
&\quad+\int_0^{T_1}\exp\Big\{-\int_0^t V(X_x^{T,y}(s))ds\Big\}\exp\Big\{-\frac{T-t}{2}\Big(V(X_x^{T,y}(t))+V(y)\Big)\Big\}\\
&\qquad\qquad\times\Big\{\frac{1}{2}(V(y)-V(X_x^{T,y}(t))-\langle\nabla V(X_x^{T,y}(t)),y-X_x^{T,y}(t)\rangle)\\
&\qquad\qquad-\frac{T-t}{4}\Delta V(X_x^{T,y}(t))+\frac{(T-t)^2}{8}|\nabla V(X_x^{T,y}(t))|^2\Big\}dt,
\end{aligned}$$

$$\begin{aligned}
&\exp\Big\{-\int_0^{T_1} V(X_x^{T,y}(s))ds\Big\}\exp\Big\{-(T-T_1)V(X_x^{T,y}(T_1))\Big\}-e^{-TV(x)}\\
&=\sum_{i=1}^d\int_0^{T_1}\exp\Big\{-\int_0^t V(X_x^{T,y}(s))ds\Big\}\exp\Big\{-(T-t)V(X_x^{T,y}(t))\Big\}\\
&\qquad\qquad\times(-(T-t))\partial_iV(X_x^{T,y}(t))dw_t^i\\
&\quad+\int_0^{T_1}\exp\Big\{-\int_0^t V(X_x^{T,y}(s))ds\Big\}\exp\Big\{-(T-t)V(X_x^{T,y}(t))\Big\}\\
&\qquad\qquad\times\Big\{-\langle\nabla V(X_x^{T,y}(t)),y-X_x^{T,y}(t)\rangle\\
&\qquad\qquad-\frac{T-t}{2}\Delta V(X_x^{T,y}(t))+\frac{(T-t)^2}{2}|\nabla V(X_x^{T,y}(t))|^2\Big\}dt.
\end{aligned}$$

Hence, by taking expectation

$$\begin{aligned}
(4.4)\qquad &E_0\Big[\exp\Big\{-\int_0^T V(X_x^{T,y}(s))ds\Big\}\Big]-\exp\Big\{-\frac{T}{2}\Big(V(x)+V(y)\Big)\Big\}\\
&=\int_0^T E_0\Big[\exp\Big\{-\int_0^t V(X_x^{T,y}(s))ds\Big\}\exp\Big\{-\frac{T-t}{2}V(y)\Big\}
\end{aligned}$$

$$\times \Big(\frac{1}{2}\int_0^1 \langle \nabla V(\xi+\theta\eta)-\nabla V(\xi),\eta\rangle d\theta \ \exp\Big\{-\frac{T-t}{2}V(\xi)\Big\}$$

$$-\frac{T-t}{4}\Delta V(\xi)\exp\Big\{-\frac{T-t}{2}V(\xi)\Big\}$$

$$+\frac{(T-t)^2}{8}|\nabla V(\xi)|^2\exp\Big\{-\frac{T-t}{2}V(\xi)\Big\}\Big)\Big|_{\xi=X_x^{T,y}(t),\eta=y-X_x^{T,y}(t)}\Big]dt,$$

$$(4.5)\quad E_0\Big[\exp\Big\{-\int_0^T V(X_x^{T,y}(s))ds\Big\}\Big]-e^{-TV(x)}$$

$$=\int_0^T E_0\Big[\exp\Big\{-\int_0^t V(X_x^{T,y}(s))ds\Big\}$$

$$\times\Big(-\langle\nabla V(\xi),\eta\rangle e^{-(T-t)V(\xi)}-\frac{T-t}{2}\Delta V(\xi)e^{-(T-t)V(\xi)}$$

$$+\frac{(T-t)^2}{2}|\nabla V(\xi)|^2e^{-(T-t)V(\xi)}\Big)\Big|_{\xi=X_x^{T,y}(t),\eta=y-X_x^{T,y}(t)}\Big]dt.$$

By (A)$'_2$(i) and (ii), and the inequality: $t^be^{-t}\le(b/e)^b, t\ge 0, b\ge 0$ (where $(0/e)^0:=1$), it is observed that for ξ, $\eta\in\mathbb{R}^d$ and $\tau>0$

$$\Big|\int_0^1\langle\nabla V(\xi+\theta\eta)-\nabla V(\xi),\eta\rangle d\theta\ e^{-\tau V(\xi)}\Big|$$

$$\le C_2\Big\{\Big(\frac{(1-2\delta)_+}{e}\Big)^{(1-2\delta)_+}\tau^{-(1-2\delta)_+}(|\eta|^2+|\eta|^{2+\mu})+|\eta|^2+|\eta|^{2+\nu}\Big\},$$

$$|\Delta V(\xi)e^{-\tau V(\xi)}|\le 2C_2d\Big\{1+\Big(\frac{(1-2\delta)_+}{e}\Big)^{(1-2\delta)_+}\tau^{-(1-2\delta)_+}\Big\},$$

$$|\langle\nabla V(\xi),\eta\rangle e^{-\tau V(\xi)}|\le C_1\Big\{1+\Big(\frac{(1-\delta)}{e}\Big)^{1-\delta}\tau^{-1+\delta}\Big\}|\eta|,$$

$$|\nabla V(\xi)|^2e^{-\tau V(\xi)}\le 2C_1^2\Big\{1+\Big(\frac{2(1-\delta)}{e}\Big)^{2(1-\delta)}\tau^{-2+2\delta}\Big\}.$$

Using these estimates in (4.4) and (4.5), we have

$$\Big|E_0\Big[\exp\Big\{-\int_0^T V(X_x^{T,y}(s))ds\Big\}\Big]-\exp\Big\{-\frac{T}{2}\Big(V(x)+V(y)\Big)\Big\}\Big|$$

$$\le\int_0^T\Big[\frac{C_2}{2}\Big(\Big(\frac{2(1-2\delta)_+}{e}\Big)^{(1-2\delta)_+}(T-t)^{-(1-2\delta)_+}$$

$$\times(E_0[|y-X_x^{T,y}(t)|^2]+E_0[|y-X_x^{T,y}(t)|^{2+\mu}])$$

$$+E_0[|y-X_x^{T,y}(t)|^2]+E_0[|y-X_x^{T,y}(t)|^{2+\nu}]\Big)$$

$$+\frac{C_2 d}{2}\Big(\Big(\frac{2(1-2\delta)_+}{e}\Big)^{(1-2\delta)_+}(T-t)^{1\wedge 2\delta}+T-t\Big)$$
$$+\frac{C_1^2}{4}\Big(\Big(\frac{4(1-\delta)}{e}\Big)^{2(1-\delta)}(T-t)^{2\delta}+(T-t)^2\Big)\Big]dt,$$

$$\Big|E_0\Big[\exp\Big\{-\int_0^T V(X_x^{T,y}(s))ds\Big\}\Big]-e^{-TV(x)}\Big|$$
$$\leq \int_0^T\Big[C_1\Big(1+\Big(\frac{(1-\delta)}{e}\Big)^{1-\delta}(T-t)^{-1+\delta}\Big)E_0[|y-X_x^{T,y}(t)|]$$
$$+C_2 d\Big(\Big(\frac{(1-2\delta)_+}{e}\Big)^{(1-2\delta)_+}(T-t)^{1\wedge 2\delta}+T-t\Big)$$
$$+C_1^2\Big(\Big(\frac{2(1-\delta)}{e}\Big)^{2(1-\delta)}(T-t)^{2\delta}+(T-t)^2\Big)\Big]dt.$$

Therefore, from the moment estimate:

$$\text{(4.6)}\qquad E_0[|y-X_x^{T,y}(t)|^a]$$
$$\leq 3^{(a-1)_+}\Big\{\Big(\frac{T-t}{T}\Big)^a|x-y|^a+2C(a,d)|T-t|^{a/2}\Big\},\quad a\geq 0$$

where $C(a,d):=E_0[|X(1)|^a]=\int_{\mathbb{R}^d}|y|^a p(1,y)dy$, it follows that

$$\Big|E_0\Big[\exp\Big\{-\int_0^T V(X_x^{T,y}(s))ds\Big\}\Big]-\exp\Big\{-\frac{T}{2}\Big(V(x)+V(y)\Big)\Big\}\Big|$$
$$\leq \text{const}\ \Big\{C_1^2(T^3+T^{1+2\delta})+C_2(T^2+T|x-y|^2+T^{1+1\wedge 2\delta}+T^{1\wedge 2\delta}|x-y|^2$$
$$+T^{1+\mu/2+1\wedge 2\delta}+T^{1\wedge 2\delta}|x-y|^{2+\mu}+T^{2+\nu/2}+T|x-y|^{2+\nu})\Big\},$$

where const depends only on δ, μ, ν and d, and

$$\Big|E_0\Big[\exp\Big\{-\int_0^T V(X_x^{T,y}(s))ds\Big\}\Big]-e^{-TV(x)}\Big|$$
$$\leq \text{const}\ \Big\{C_1(T^{3/2}+T|x-y|+T^{1/2+\delta}+T^{\delta}|x-y|)+C_1^2(T^3+T^{1+2\delta})$$
$$+C_2(T^2+T^{1+1\wedge 2\delta})\Big\},$$

where const depends only on δ and d. These are just (4.1) and (4.2).

Next we consider the general case that $V:\mathbb{R}^d\to[0,\infty)$ is a C^1-function satisfying $(A)_2'$(i) and (ii). To this end take a $\psi\in C_0^\infty(\mathbb{R}^d\to[0,\infty))$ such that $\int_{\mathbb{R}^d}\psi(x)dx=1$ and set $V_\varepsilon(x):=\int_{\mathbb{R}^d}\psi_\varepsilon(x-y)V(y)dy$ where $\psi_\varepsilon(z):=(1/\varepsilon)^d\psi(z/\varepsilon)$ $(\varepsilon>0)$. Then V_ε is smooth and satisfies $(A)_2'$(i) and (ii) with the same constants as V has. From what was seen above, (4.1) and (4.2) hold for V_ε. Since $V_\varepsilon \to V$ compact uniformly as $\varepsilon \downarrow 0$, these estimates are valid for V.

It remains to show (4.3). We note the following: For ξ, $\eta \in \mathbb{R}^d$ and $0 \le t_1 < t_0 \le 1$,

$$P_0\Big(X_\xi^{t_0,\eta}(\cdot + t_1) \in * \,\Big|\, \mathcal{F}_{t_1}\Big) = P_0\Big(X_{\xi_1}^{t_0-t_1,\eta}(\cdot) \in *\Big)\Big|_{\xi_1 = X_\xi^{t_0,\eta}(t_1)} \tag{4.7}$$

where $\mathcal{F}_\tau$ is the sub σ-field generated by $w(t)$, $0 \le t \le \tau$, and

$$\Big(X_\xi^{t_0,\eta}(t)\Big)_{0\le t\le t_0} \overset{\mathcal{L}}{\sim} \Big(X_\eta^{t_0,\xi}(t_0 - t)\Big)_{0\le t\le t_0}. \tag{4.8}$$

By (4.7) we have

$$\begin{aligned}
&E_0\Big[\exp\Big\{-\int_0^t V(X_x^{t,y}(s))ds\Big\}\Big] \\
&= E_0\Big[\exp\Big\{-\int_0^{t/2} V(X_x^{t,y}(s))ds\Big\}\exp\Big\{-\int_0^{t/2} V\Big(X_x^{t,y}\Big(s+\frac{t}{2}\Big)\Big)ds\Big\}\Big] \\
&= E_0\Big[\exp\Big\{-\int_0^{t/2} V(X_x^{t,y}(s))ds\Big\}E_0\Big[\exp\Big\{-\int_0^{t/2} V(X_\xi^{t/2,y}(s))ds\Big\}\Big]\Big|_{\xi=X_x^{t,y}(t/2)}\Big].
\end{aligned}$$

By this together with (4.8) and (4.7) we see

$$\begin{aligned}
&E_0\Big[\exp\Big\{-\int_0^t V(X_x^{t,y}(s))ds\Big\}\Big] - E_0\Big[\exp\Big\{-tV\Big(X_x^{t,y}\Big(\frac{t}{2}\Big)\Big)\Big\}\Big] \\
&= E_0\Big[\Big(\exp\Big\{-\int_0^{t/2} V(X_x^{t,y}(s))ds\Big\}E_0\Big[\exp\Big\{-\int_0^{t/2} V(X_\xi^{t/2,y}(s))ds\Big\}\Big] \\
&\qquad - e^{-tV(\xi)/2}e^{-tV(\xi)/2}\Big)\Big|_{\xi=X_x^{t,y}(t/2)}\Big] \\
&= E_0\Big[\exp\Big\{-\int_0^{t/2} V(X_x^{t,y}(s))ds\Big\} \\
&\qquad \times E_0\Big[\exp\Big\{-\int_0^{t/2} V(X_\xi^{t/2,y}(s))ds\Big\} - e^{-tV(\xi)/2}\Big]\Big|_{\xi=X_x^{t,y}(t/2)}\Big] \\
&\quad + E_0\Big[\Big(\exp\Big\{-\int_0^{t/2} V(X_y^{t,x}(t-s))ds\Big\} - \exp\Big\{-\frac{t}{2}V\Big(X_y^{t,x}\Big(\frac{t}{2}\Big)\Big)\Big\}\Big) \\
&\qquad \times \exp\Big\{-\frac{t}{2}V\Big(X_y^{t,x}\Big(\frac{t}{2}\Big)\Big)\Big\}\Big] \\
&= E_0\Big[\exp\Big\{-\int_0^{t/2} V(X_x^{t,y}(s))ds\Big\}
\end{aligned}$$

$$\times E_0\Big[\exp\Big\{-\int_0^{t/2} V(X_\xi^{t/2,y}(s))ds\Big\} - e^{-tV(\xi)/2}\Big]\Big|_{\xi=X_x^{t,y}(t/2)}\Big]$$

$$+ E_0\Big[\Big(\exp\Big\{-\int_0^{t/2} V\Big(X_y^{t,x}\Big(s+\frac{t}{2}\Big)\Big)ds\Big\} - \exp\Big\{-\frac{t}{2}V\Big(X_y^{t,x}\Big(\frac{t}{2}\Big)\Big)\Big\}\Big)$$

$$\times \exp\Big\{-\frac{t}{2}V\Big(X_y^{t,x}\Big(\frac{t}{2}\Big)\Big)\Big\}\Big]$$

$$= E_0\Big[\exp\Big\{-\int_0^{t/2} V(X_x^{t,y}(s))ds\Big\}$$

$$\times E_0\Big[\exp\Big\{-\int_0^{t/2} V(X_\xi^{t/2,y}(s))ds\Big\} - e^{-tV(\xi)/2}\Big]\Big|_{\xi=X_x^{t,y}(t/2)}\Big]$$

$$+ E_0\Big[E_0\Big[\exp\Big\{-\int_0^{t/2} V(X_\eta^{t/2,x}(s))ds\Big\} - e^{-tV(\eta)/2}\Big]\Big|_{\eta=X_y^{t,x}(t/2)}$$

$$\times \exp\Big\{-\frac{t}{2}V\Big(X_y^{t,x}\Big(\frac{t}{2}\Big)\Big)\Big\}\Big].$$

Hence

$$\Big|E_0\Big[\exp\Big\{-\int_0^{t} V(X_x^{t,y}(s))ds\Big\}\Big] - E_0\Big[\exp\Big\{-tV\Big(X_x^{t,y}\Big(\frac{t}{2}\Big)\Big)\Big\}\Big]\Big|$$

$$\leq E_0\Big[\Big|E_0\Big[\exp\Big\{-\int_0^{t/2} V(X_\xi^{t/2,y}(s))ds\Big\} - e^{-tV(\xi)/2}\Big]\Big|\ \Big|_{\xi=X_x^{t,y}(t/2)}\Big]$$

$$+ E_0\Big[\Big|E_0\Big[\exp\Big\{-\int_0^{t/2} V(X_\eta^{t/2,x}(s))ds\Big\} - e^{-tV(\eta)/2}\Big]\Big|\ \Big|_{\eta=X_y^{t,x}(t/2)}\Big]$$

$$\leq \text{const } E_0\Big[C_1\Big(\frac{t}{2}\Big|X_x^{t,y}\Big(\frac{t}{2}\Big)-y\Big| + \Big(\frac{t}{2}\Big)^3 + \Big(\frac{t}{2}\Big)^\delta\Big|X_x^{t,y}\Big(\frac{t}{2}\Big)-y\Big| + \Big(\frac{t}{2}\Big)^{\delta+1/2}\Big)$$

$$+ C_1^2\Big(\Big(\frac{t}{2}\Big)^3 + \Big(\frac{t}{2}\Big)^{1+2\delta}\Big) + C_2\Big(\Big(\frac{t}{2}\Big)^{1+1\wedge 2\delta} + \Big(\frac{t}{2}\Big)^2\Big)\Big]$$

$$+ \text{const } E_0\Big[C_1\Big(\frac{t}{2}\Big|X_y^{t,x}\Big(\frac{t}{2}\Big)-x\Big| + \Big(\frac{t}{2}\Big)^3 + \Big(\frac{t}{2}\Big)^\delta\Big|X_y^{t,x}\Big(\frac{t}{2}\Big)-x\Big| + \Big(\frac{t}{2}\Big)^{\delta+1/2}\Big)$$

$$+ C_1^2\Big(\Big(\frac{t}{2}\Big)^3 + \Big(\frac{t}{2}\Big)^{1+2\delta}\Big) + C_2\Big(\Big(\frac{t}{2}\Big)^{1+1\wedge 2\delta} + \Big(\frac{t}{2}\Big)^2\Big)\Big],$$

where in the last inequality we have used the estimate (4.2). Combining this with (4.6) we have (4.3) at once, and the proof is complete. □

REMARK 4. The estimates (4.1) and (4.2) are a little better than the ones in [10] (cf. [6]). To prove them we have used Itô's formula. This treatment seems to be more stochastic analytic than the one in [10]. The present proof is slightly simpler and probably more elegant.

CLAIM 2.
Let $t > 0$ and $n \in \mathbb{N}$. Then

$$\left\|e^{-tH}\right\|_{\text{trace}} \leq \left(\sqrt{\frac{2}{\pi}}c^{-1/\rho}\right)^{d} t^{-d(1/2+1/\rho)} e^{tc'} \int_{\mathbb{R}^d} e^{-|x|^\rho} dx + \frac{2^{d/2}}{\Gamma(d/2+1)} E_0\Big[\max_{0\leq s\leq 1} |X_0(s)|^d\Big],$$

$$\left\|K\Big(\frac{t}{n}\Big)^n\right\|_{\text{trace}} \leq \left(\sqrt{\frac{2}{\pi}}c^{-1/\rho}\right)^{d} t^{-d(1/2+1/\rho)} e^{tc'} \int_{\mathbb{R}^d} e^{-|x|^\rho} dx + \frac{2^{d/2}}{\Gamma(d/2+1)} E_0\Big[\max_{0\leq s\leq 1} |X_0(s)|^d\Big].$$

Proof. By the expressions (3.1) and (3.2),

$$\left\|e^{-tH}\right\|_{\text{trace}} = \int_{\mathbb{R}^d} e^{-tH}(x,x)dx = \Big(\frac{1}{2\pi t}\Big)^{d/2} \int_{\mathbb{R}^d} E_0\Big[\exp\Big\{-t\int_0^1 V(x+\sqrt{t}X_0(s))ds\Big\}\Big]dx, \tag{4.9}$$

$$\left\|K\Big(\frac{t}{n}\Big)^n\right\|_{\text{trace}} = \int_{\mathbb{R}^d} K\Big(\frac{t}{n}\Big)^n(x,x)dx = \Big(\frac{1}{2\pi t}\Big)^{d/2} \int_{\mathbb{R}^d} E_0\Big[\exp\Big\{-\frac{t}{2}\Big(\int_0^1 V(x+\sqrt{t}X_0(s_n^-))ds + \int_0^1 V(x+\sqrt{t}X_0(s_n^+))ds\Big)\Big\}\Big]dx. \tag{4.10}$$

By (1.1), it is clear that on $\{\max_{0\leq s\leq 1}|\sqrt{t}X_0(s)| < |x|/2\}$

$$\int_0^1 V(x+\sqrt{t}X_0(s_n^\pm))ds \geq c\int_0^1 \Big||x| - |\sqrt{t}X_0(s_n^\pm)|\Big|^\rho ds - c' \geq c\int_0^1 \Big(\frac{|x|}{2}\Big)^\rho ds - c' = c\Big(\frac{|x|}{2}\Big)^\rho - c',$$

$$\int_0^1 V(x+\sqrt{t}X_0(s))ds \geq c\int_0^1 \Big||x| - |\sqrt{t}X_0(s)|\Big|^\rho ds - c' \geq c\int_0^1 \Big(\frac{|x|}{2}\Big)^\rho ds - c' = c\Big(\frac{|x|}{2}\Big)^\rho - c'.$$

Hence, substituting these inequalities into (4.9) and (4.10), we have

$$\text{(4.11)} \qquad \left\| e^{-tH} \right\|_{\text{trace}} \leq \left(\frac{1}{2\pi t}\right)^{d/2} \int_{\mathbb{R}^d} \exp\left\{-t\left(c\left(\frac{|x|}{2}\right)^{\rho} - c'\right)\right\} dx + \left(\frac{1}{2\pi t}\right)^{d/2} \int_{\mathbb{R}^d} P_0\left(\max_{0\leq s\leq 1} |\sqrt{t}X_0(s)| \geq \frac{|x|}{2}\right) dx,$$

$$\text{(4.12)} \qquad \left\| K\left(\frac{t}{n}\right)^n \right\|_{\text{trace}} \leq \left(\frac{1}{2\pi t}\right)^{d/2} \int_{\mathbb{R}^d} \exp\left\{-t\left(c\left(\frac{|x|}{2}\right)^{\rho} - c'\right)\right\} dx + \left(\frac{1}{2\pi t}\right)^{d/2} \int_{\mathbb{R}^d} P_0\left(\max_{0\leq s\leq 1} |\sqrt{t}X_0(s)| \geq \frac{|x|}{2}\right) dx.$$

Each term on the RHS of (4.11) and (4.12) is computed as follows:

$$\begin{aligned}
\text{The first term} &= \left(\frac{1}{2\pi t}\right)^{d/2} e^{tc'} \int_{\mathbb{R}^d} e^{-tc|y|^{\rho}} 2^d dy \\
&= \left(\frac{2}{\pi}\right)^{d/2} t^{-d/2} e^{tc'} \int_{\mathbb{R}^d} e^{-|z|^{\rho}} (tc)^{-d/\rho} dz \\
&= \left(\frac{2}{\pi}\right)^{d/2} \int_{\mathbb{R}^d} e^{-|z|^{\rho}} dz \; c^{-d/\rho} t^{-d(1/2+1/\rho)} e^{tc'},
\end{aligned}$$

$$\begin{aligned}
\text{The second term} &= \left(\frac{1}{2\pi t}\right)^{d/2} \int_{\mathbb{R}^d} P_0\left(2\max_{0\leq s\leq 1} |X_0(s)| \geq \left|\frac{x}{\sqrt{t}}\right|\right) dx \\
&= \left(\frac{1}{2\pi}\right)^{d/2} \int_{\mathbb{R}^d} P_0\left(2\max_{0\leq s\leq 1} |X_0(s)| \geq |y|\right) dy \\
&= \left(\frac{1}{2\pi}\right)^{d/2} \int_{S^{d-1}} d\omega \int_0^{\infty} r^{d-1} P_0\left(2\max_{0\leq s\leq 1} |X_0(s)| \geq r\right) dr \\
&= \left(\frac{1}{2\pi}\right)^{d/2} \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^{\infty} r^{d-1} P_0\left(2\max_{0\leq s\leq 1} |X_0(s)| \geq r\right) dr \\
&= \left(\frac{1}{2}\right)^{d/2-1} \frac{1}{\Gamma(d/2)} E_0\left[\int_0^{\infty} r^{d-1} 1_{2\max_{0\leq s\leq 1}|X_0(s)|\geq r} dr\right] \\
&= \left(\frac{1}{2}\right)^{d/2-1} \frac{1}{\Gamma(d/2)} E_0\left[\int_0^{2\max_{0\leq s\leq 1}|X_0(s)|} \left(\frac{r^d}{d}\right)' dr\right] \\
&= 2^{-d/2+1} \frac{1}{\Gamma(d/2)} E_0\left[\frac{1}{d}(2\max_{0\leq s\leq 1} |X_0(s)|)^d\right] \\
&= 2^{d/2} \frac{1}{(d/2)\Gamma(d/2)} E_0\left[(\max_{0\leq s\leq 1} |X_0(s)|)^d\right] \\
&= \frac{2^{d/2}}{\Gamma(d/2+1)} E_0\left[\max_{0\leq s\leq 1} |X_0(s)|^d\right].
\end{aligned}$$

The proof is complete. □

Proof of Theorem (i). Let $T \geq 1$, $0 < t \leq T$ and $n \geq 2$. By Claim 1,

$$\left\| K\Big(\frac{t}{n}\Big) - e^{-tH/n} \right\|_{2\to 2} \leq \text{const } T^{2\vee(\mu/2)\vee(1+\nu/2)}(C_1^2 + C_2)\Big(\frac{1}{n}\Big)^{1+1\wedge 2\delta} t^{1+1\wedge 2\delta}, \tag{4.13}$$

where const depends only on δ, μ, ν and d. Note that

$$\frac{n-j}{n} \geq \frac{1}{2} \text{ for } 1 \leq j \leq \Big[\frac{n}{2}\Big],$$
$$\frac{j-1}{n} \geq \frac{1}{3} \text{ for } \Big[\frac{n}{2}\Big] < j \leq n.$$

By this and Claim 2,

$$\begin{aligned}\left\| e^{-(n-j)tH/n} \right\|_{\text{trace}} \leq{}& \Big(\sqrt{\frac{2}{\pi}}c^{-1/\rho}\Big)^d \Big(\frac{t}{2}\Big)^{-d(1/2+1/\rho)} e^{Tc'} \int_{\mathbb{R}^d} e^{-|x|^\rho} dx \\ &+ \frac{2^{d/2}}{\Gamma(d/2+1)} E_0\Big[\max_{0\leq s\leq 1} |X_0(s)|^d\Big] \quad \text{for } 1 \leq j \leq \Big[\frac{n}{2}\Big],\end{aligned}$$

$$\begin{aligned}\left\| K\Big(\frac{j-1}{n}\frac{t}{j-1}\Big)^{j-1} \right\|_{\text{trace}} \leq{}& \Big(\sqrt{\frac{2}{\pi}}c^{-1/\rho}\Big)^d \Big(\frac{t}{3}\Big)^{-d(1/2+1/\rho)} e^{Tc'} \int_{\mathbb{R}^d} e^{-|x|^\rho} dx \\ &+ \frac{2^{d/2}}{\Gamma(d/2+1)} E_0\Big[\max_{0\leq s\leq 1} |X_0(s)|^d\Big] \quad \text{for } \Big[\frac{n}{2}\Big] < j \leq n.\end{aligned}$$

Combining these with (4.13) we have by (2.3)

$$\begin{aligned}&\left\| K\Big(\frac{t}{n}\Big)^n - e^{-tH} \right\|_{\text{trace}} \\ &\leq \text{const } T^{2\vee(\mu/2)\vee(1+\nu/2)}(C_1^2 + C_2)\Big(\frac{1}{n}\Big)^{1+1\wedge 2\delta} t^{1+1\wedge 2\delta} \\ &\quad \times n \times \Big\{\Big(\sqrt{\frac{2}{\pi}}c^{-1/\rho}\Big)^d 3^{d(1/2+1/\rho)} e^{Tc'} \int_{\mathbb{R}^d} e^{-|x|^\rho} dx \; t^{-d(1/2+1/\rho)} \\ &\qquad + \frac{2^{d/2}}{\Gamma(d/2+1)} E_0\Big[\max_{0\leq s\leq 1} |X_0(s)|^d\Big]\Big\} \\ &\leq \text{const } \Big(\frac{1}{n}\Big)^{1\wedge 2\delta} t^{1+1\wedge 2\delta - d(1/2+1/\rho)},\end{aligned}$$

where const in the last line depends only on C_1, C_2, δ, μ, ν, ρ, c, c', d and T. The proof of Theorem (i) is complete. □

CLAIM 3. (i) For $t > 0$

$\|[e^{-tH}, V]\|_{2\to 2}$

$$\leq C_2 d\Big(t + \Big(\frac{(1-2\delta)_+}{e}\Big)^{(1-2\delta)_+} t^{1\wedge 2\delta}\Big) + C_1^2\Big(t^2 + \Big(\frac{2(1-\delta)}{e}\Big)^{2(1-\delta)} t^{2\delta}\Big)$$
$$+ C_1 C(1,d)\Big(t^{1/2} + \Big(\frac{1-\delta}{e}\Big)^{1-\delta} t^{-1/2+\delta}\Big),$$

where $C(a,d) := E_0[|X(1)|^a] = \int_{\mathbb{R}^d} |y|^a p(1,y)dy$, $a \geq 0$.

(ii) For $t > 0$ and $s \geq 0$,

$$\|[e^{-sV}, e^{-tH}]\|_{\text{trace}} \leq 2s\|[e^{-tH/2}, V]\|_{2\to 2}\ \|e^{-tH/2}\|_{\text{trace}},$$
$$\|[e^{-s(-(1/2)\Delta)}, e^{-tH}]\|_{\text{trace}} \leq 2s\|[e^{-tH/2}, V]\|_{2\to 2}\ \|e^{-tH/2}\|_{\text{trace}}.$$

Proof. As for (ii), note that

$$[e^{-sV}, e^{-tH}] = \int_0^s e^{-uV}[e^{-tH}, V]e^{-(s-u)V}du,$$
$$[e^{-s(-(1/2)\Delta)}, e^{-tH}] = -\int_0^s e^{-u(-(1/2)\Delta)}\Big[e^{-tH}, \frac{1}{2}\Delta\Big]e^{-(s-u)(-(1/2)\Delta)}du$$
$$= -\int_0^s e^{-u(-(1/2)\Delta)}[e^{-tH}, V]e^{-(s-u)(-(1/2)\Delta)}du,$$
$$[e^{-tH}, V] = [e^{-tH/2}, V]e^{-tH/2} + e^{-tH/2}[e^{-tH/2}, V].$$

By (2.1) and (2.2), these expressions give us the estimate described in (ii).

As for (i), note that the integral kernel of $[e^{-tH}, V]$ is expressed as

$$\text{(4.14)}\quad [e^{-tH}, V](x,y) = p(t, x-y)$$
$$\times (V(y) - V(x))E_0\Big[\exp\Big\{-t\int_0^1 V(x + s(y-x) + \sqrt{t}X_0(s))ds\Big\}\Big].$$

If we show the following estimate:

$$\text{(4.15)}\quad \Big|(V(y) - V(x))E_0\Big[\exp\Big\{-t\int_0^1 V(x + s(y-x) + \sqrt{t}X_0(s))ds\Big\}\Big]\Big|$$
$$\leq C_2 d\Big(t + \Big(\frac{(1-2\delta)_+}{e}\Big)^{(1-2\delta)_+} t^{1\wedge 2\delta}\Big) + C_1^2\Big(t^2 + \Big(\frac{2(1-\delta)}{e}\Big)^{2(1-\delta)} t^{2\delta}\Big)$$
$$+ C_1|x-y|\Big(1 + \Big(\frac{1-\delta}{e}\Big)^{1-\delta} t^{-1+\delta}\Big),$$

the estimate in (i) follows immediately from this and (4.14).

In the following we show (4.15). To this end we may suppose without loss of generality that $V : \mathbb{R}^d \to [0,\infty)$ is a C^2-function and satisfies $(A)_2'$(i) and (ii) (cf. the proof of Claim 1).

First of all we note that for $f \in \mathcal{S}(\mathbb{R}^d)$ and $t \geq 0$

$$e^{-tH}\Big(-\frac{1}{2}\Delta + V\Big)f = \Big(-\frac{1}{2}\Delta + V\Big)e^{-tH}f. \tag{4.16}$$

Let $T > 0$, $0 < t < T$ and $y \in \mathbb{R}^d$. By letting $f = p(T-t, \cdot, y)$ in (4.16), the Feynman-Kac formula gives us that

$$\begin{aligned}
&E_0\Big[\exp\Big\{-\int_0^t V(x+X_s)ds\Big\}V(x+X_t)p(T-t, x+X_t-y)\Big] \\
&\quad - V(x)E_0\Big[\exp\Big\{-\int_0^t V(x+X_s)ds\Big\}p(T-t, x+X_t-y)\Big] \\
&= -\frac{1}{2}\Delta_x E_0\Big[\exp\Big\{-\int_0^t V(x+X_s)ds\Big\}p(T-t, x+X_t-y)\Big] \\
&\quad + E_0\Big[\exp\Big\{-\int_0^t V(x+X_s)ds\Big\}\frac{1}{2}\Delta p(T-t, x+X_t-y)\Big] \\
&= E_0\Big[\exp\Big\{-\int_0^t V(x+X_s)ds\Big\} \\
&\qquad \times \Big\{\Big(-\frac{1}{2}\Big|\int_0^t \nabla V(x+X_s)ds\Big|^2 + \int_0^t \frac{1}{2}\Delta V(x+X_s)ds\Big)p(T-t, x+X_t-y) \\
&\qquad + \Big\langle\int_0^t \nabla V(x+X_s)ds, \nabla p(T-t, x+X_t-y)\Big\rangle\Big\}\Big].
\end{aligned}$$

By using the Brownian bridge $(X_0(s))_{0\leq s\leq 1}$, this is rewritten as

$$\begin{aligned}
&\int_{\mathbb{R}^d} E_0\Big[\exp\Big\{-t\int_0^1 V(x+s(z-x)+\sqrt{t}X_0(s))ds\Big\}\Big] \\
&\qquad \times V(z)p(T-t, z-y)p(t, x-z)dz \\
&\qquad - V(x)\int_{\mathbb{R}^d} E_0\Big[\exp\Big\{-t\int_0^1 V(x+s(z-x)+\sqrt{t}X_0(s))ds\Big\}\Big] \\
&\qquad\qquad \times p(T-t, z-y)p(t, x-z)dz \\
&\quad = \int_{\mathbb{R}^d} E_0\Big[\exp\Big\{-t\int_0^1 V(x+s(z-x)+\sqrt{t}X_0(s))ds\Big\} \\
&\qquad\qquad \times \Big\{-\frac{t^2}{2}\Big|\int_0^1 \nabla V(x+s(z-x)+\sqrt{t}X_0(s))ds\Big|^2 \\
&\qquad\qquad + \frac{t}{2}\int_0^1 \Delta V(x+s(z-x)+\sqrt{t}X_0(s))ds\Big\}\Big] \\
&\qquad \times p(T-t, z-y)p(t, x-z)dz
\end{aligned}$$

$$+\sum_{i=1}^{d}\int_{\mathbb{R}^d} E_0\Big[\exp\Big\{-t\int_0^1 V(x+s(z-x)+\sqrt{t}X_0(s))ds\Big\}$$
$$\times t\int_0^1 \partial_i V(x+s(z-x)+\sqrt{t}X_0(s))ds\Big]$$
$$\times \partial_i p(T-t,z-y)p(t,x-z)dz.$$

By integration by parts and the formula: $\partial/\partial z_i\ p(t,x-z)=p(t,x-z)(x_i-z_i)/t$, the second term on the RHS is further computed, so that

$$\text{The RHS} = \int_{\mathbb{R}^d} E_0\Big[\exp\Big\{-t\int_0^1 V(x+s(z-x)+\sqrt{t}X_0(s))ds\Big\}$$
$$\times\Big\{-\frac{t^2}{2}\Big\langle\int_0^1 \nabla V(x+s(z-x)+\sqrt{t}X_0(s))ds,$$
$$\int_0^1 (1-2s)\nabla V(x+s(z-x)+\sqrt{t}X_0(s))ds\Big\rangle$$
$$+\frac{t}{2}\int_0^1 (1-2s)\Delta V(x+s(z-x)+\sqrt{t}X_0(s))ds$$
$$-\Big\langle\int_0^1 \nabla V(x+s(z-x)+\sqrt{t}X_0(s))ds, x-z\Big\rangle ds\Big\}\Big]$$
$$\times p(T-t,z-y)p(t,x-z)dz.$$

Hence, as $t\uparrow T$, we have

$$(V(y)-V(x))E_0\Big[\exp\Big\{-T\int_0^1 V(x+s(y-x)+\sqrt{T}X_0(s))ds\Big\}\Big]$$
$$= E_0\Big[\exp\Big\{-T\int_0^1 V(x+s(y-x)+\sqrt{T}X_0(s))ds\Big\}$$
$$\times\Big\{-\frac{T^2}{2}\Big\langle\int_0^1 \nabla V(x+s(y-x)+\sqrt{T}X_0(s))ds,$$
$$\int_0^1 (1-2s)\nabla V(x+s(y-x)+\sqrt{T}X_0(s))ds\Big\rangle$$
$$+\frac{T}{2}\int_0^1 (1-2s)\Delta V(x+s(y-x)+\sqrt{t}X_0(s))ds$$
$$-\int_0^1 \langle\nabla V(x+s(y-x)+\sqrt{T}X_0(s)), x-y\rangle ds\Big\}\Big].$$

Finally, using the estimates:

$$|\nabla V(x)| \le C_1(1+V(x)^{1-\delta}),$$

$$|\Delta V(x)| \leq 2C_2 d(V(x)^{(1-2\delta)_+} + 1)$$

and then applying Jensen's inequality, we obtain (4.15) and the proof is complete. □

Proof of Theorem (ii) and (iii). Let $T \geq 1$, $0 < t \leq T$ and $n \geq 3$. Note that $1/(n-1) \leq (3/2)(1/n)$, so that $(n-1)/n \geq 2/3$. First, by Theorem (i)

$$\begin{aligned} (4.17) \qquad & \left\| K\left(\frac{n-1}{n}t \frac{1}{n-1}\right)^{n-1} - e^{-(n-1)tH/n} \right\|_{\text{trace}} \\ & \leq \text{const} \left(\frac{1}{n-1}\right)^{1\wedge 2\delta} \left(\frac{n-1}{n}t\right)^{1+1\wedge 2\delta - d(1/2+1/\rho)} \\ & \leq \text{const} \left(\frac{3}{2}\right)^{1\wedge 2\delta + d(1/2+1/\rho)} \left(\frac{1}{n}\right)^{1\wedge 2\delta} t^{1+1\wedge 2\delta - d(1/2+1/\rho)}, \end{aligned}$$

where const depends only on C_1, C_2, δ, μ, ν, ρ, c, c', d and T. Second, by Claims 1 and 2

$$\begin{aligned} (4.18) \qquad & \left\| e^{-(n-1)tH/n} \right\|_{\text{trace}} \times \left\| G\left(\frac{t}{n}\right) - e^{-tH/n} \right\|_{2\to 2}, \\ & \left\| e^{-(n-1)tH/n} \right\|_{\text{trace}} \times \left\| R\left(\frac{t}{n}\right) - e^{-tH/n} \right\|_{2\to 2} \\ & \leq \left\{ \left(\sqrt{\frac{2}{\pi}} c^{-1/\rho}\right)^d \left(\frac{3}{2}\right)^{d(1/2+1/\rho)} t^{-d(1/2+1/\rho)} e^{Tc'} \int_{\mathbb{R}^d} e^{-|x|^\rho} dx \right. \\ & \qquad \left. + \frac{2^{d/2}}{\Gamma(d/2+1)} E_0 \left[\max_{0\leq s\leq 1} |X_0(s)|^d \right] \right\} \\ & \qquad \times \text{const } T^{5/2}(C_1 + C_1^2 + C_2) \left(\frac{1}{n}\right)^{1/2+\delta} t^{1/2+\delta} \\ & \leq \text{const} \left(\frac{1}{n}\right)^{1/2+\delta} t^{1/2+\delta - d(1/2+1/\rho)}, \end{aligned}$$

where const in the last line depends only on C_1, C_2, δ, ρ, c, c', d and T. Third, by Claim 3 and Claim 2

$$\begin{aligned} (4.19) \qquad & \left\| \left[e^{-tV/2n}, e^{-(n-1)tH/n} \right] \right\|_{\text{trace}}, \quad \left\| \left[e^{-t(-(1/2)\Delta)/2n}, e^{-(n-1)tH/n} \right] \right\|_{\text{trace}} \\ & \leq \frac{t}{n} \left\| \left[e^{-(n-1)tH/2n}, V \right] \right\|_{2\to 2} \left\| e^{-(n-1)tH/2n} \right\|_{\text{trace}} \\ & \leq \frac{t}{n} \left\{ C_2 d \left(\frac{(1-2\delta)_+}{e}\right)^{(1-2\delta)_+} \left(\frac{n-1}{2n}t\right)^{1\wedge 2\delta} + C_2 d \, \frac{n-1}{2n}t \right. \\ & \qquad + C_1^2 \left(\frac{2(1-\delta)}{e}\right)^{2(1-\delta)} \left(\frac{n-1}{2n}t\right)^{2\delta} + C_1^2 \left(\frac{n-1}{2n}t\right)^2 \end{aligned}$$

$$
\begin{aligned}
&+C_1C(1,d)\Big(\frac{1-\delta}{e}\Big)^{1-\delta}\Big(\frac{n-1}{2n}t\Big)^{-1/2+\delta}\\
&+C_1C(1,d)\Big(\frac{n-1}{2n}t\Big)^{1/2}\Big\}\\
&\times\Big\{\Big(\sqrt{\frac{2}{\pi}}c^{-1/\rho}\Big)^d\Big(\frac{n-1}{2n}t\Big)^{-d(1/2+1/\rho)}e^{(n-1)tc'/2n}\int_{\mathbb{R}^d}e^{-|x|^\rho}dx\\
&+\frac{2^{d/2}}{\Gamma(d/2+1)}E_0\Big[\max_{0\le s\le 1}|X_0(s)|^d\Big]\Big\}\\
\le\ &\frac{1}{n}\Big\{C_2d\ \Big(\frac{(1-2\delta)_+}{e}\Big)^{(1-2\delta)_+}\Big(\frac{1}{2}\Big)^{1\wedge 2\delta}t^{1+1\wedge 2\delta}+C_2d\ \frac{1}{2}t^2\\
&+C_1^2\Big(\frac{2(1-\delta)}{e}\Big)^{2(1-\delta)}\Big(\frac{1}{2}\Big)^{2\delta}t^{1+2\delta}+C_1^2\Big(\frac{1}{2}\Big)^2t^3\\
&+C_1C(1,d)\Big(\frac{1-\delta}{e}\Big)^{1-\delta}\Big(\Big(\frac{1}{2}\Big)^{-1/2+\delta}\vee\Big(\frac{1}{3}\Big)^{-1/2+\delta}\Big)t^{1/2+\delta}\\
&+C_1C(1,d)\Big(\frac{1}{2}\Big)^{1/2}t^{3/2}\Big\}\\
&\times\Big\{\Big(\sqrt{\frac{2}{\pi}}c^{-1/\rho}\Big)^d3^{d(1/2+1/\rho)}e^{Tc'/2}\int_{\mathbb{R}^d}e^{-|x|^\rho}dx\ t^{-d(1/2+1/\rho)}\\
&+\frac{2^{d/2}}{\Gamma(d/2+1)}E_0\Big[\max_{0\le s\le 1}|X_0(s)|^d\Big]\Big\}\\
\le\ &\frac{1}{n}\Big\{C_2d\ \Big(\frac{(1-2\delta)_+}{e}\Big)^{(1-2\delta)_+}\Big(\frac{1}{2}\Big)^{1\wedge 2\delta}+C_2d\ \frac{1}{2}\\
&+C_1^2\Big(\frac{2(1-\delta)}{e}\Big)^{2(1-\delta)}\Big(\frac{1}{2}\Big)^{2\delta}+C_1^2\Big(\frac{1}{2}\Big)^2\\
&+C_1C(1,d)\Big(\frac{1-\delta}{e}\Big)^{1-\delta}\Big(\Big(\frac{1}{2}\Big)^{-1/2+\delta}\vee\Big(\frac{1}{3}\Big)^{-1/2+\delta}\Big)\\
&+C_1C(1,d)\Big(\frac{1}{2}\Big)^{1/2}\Big\}\\
&\times\Big\{\Big(\sqrt{\frac{2}{\pi}}c^{-1/\rho}\Big)^d3^{d(1/2+1/\rho)}e^{Tc'/2}\int_{\mathbb{R}^d}e^{-|x|^\rho}dx\\
&+\frac{2^{d/2}}{\Gamma(d/2+1)}E_0\Big[\max_{0\le s\le 1}|X_0(s)|^d\Big]T^{d(1/2+1/\rho)}\Big\}\\
&\times T^{5/2}t^{1/2+\delta-d(1/2+1/\rho)}\\
&(\text{since } 1/2+\delta<1+1\wedge 2\delta\le 1+2\delta,\ 2)\\
\le\ &\text{const}\ \frac{1}{n}\ t^{1/2+\delta-d(1/2+1/\rho)},
\end{aligned}
$$

where const depends only on C_1, C_2, δ, ρ, c, c', d and T. Therefore, combining

(4.17), (4.18) and (4.19), we have by (2.4) and (2.5)

$$\begin{aligned}
&\left\|G\left(\frac{t}{n}\right)^n - e^{-tH}\right\|_{\text{trace}}, \ \left\|R\left(\frac{t}{n}\right)^n - e^{-tH}\right\|_{\text{trace}} \\
&\le \text{const} \left(\frac{1}{n}\right)^{1\wedge 2\delta} t^{1+1\wedge 2\delta - d(1/2+1/\rho)} + \text{const} \ \frac{1}{n} \ t^{1/2+\delta-d(1/2+1/\rho)} \\
&\quad + \text{const}\left(\frac{1}{n}\right)^{1/2+\delta} t^{1/2+\delta-d(1/2+1/\rho)} \\
&\le \text{const} \left(\frac{1}{n}\right)^{1\wedge 2\delta} t^{1/2+\delta-d(1/2+1/\rho)} \\
&\quad (\text{since } 1\wedge 2\delta \le 1/2+\delta, \ 1)
\end{aligned}$$

and the proof is complete. □

5. Remark

The condition $(A)_2'$ is just $(A)_2$ in [10] plus the condition (o). So for $(A)_0$ and $(A)_1$ in [10] we can consider $(A)_0'$ and $(A)_1'$ respectively:

$(A)_0'$ $\quad V : \mathbb{R}^d \to [0,\infty)$ is a function such that

(o) $\liminf_{|x|\to\infty} \dfrac{V(x)}{|x|^\rho} > 0$

(i) $|V(x) - V(y)| \le C_1|x-y|^\gamma$,

$(A)_1'$ $\quad V : \mathbb{R}^d \to [0,\infty)$ is a C^1-function such that

(o) $\liminf_{|x|\to\infty} \dfrac{V(x)}{|x|^\rho} > 0$

(i) $|\nabla V(x)| \le C_1(1+V(x)^{1-\delta})$

(ii) $|\nabla V(x) - \nabla V(y)| \le C_2|x-y|^\kappa$.

Here $0<\rho<\infty$, $0<\gamma\le 1$, $0\le C_1$, $C_2<\infty$, $0<\delta\le 1$ and $0\le\kappa\le 1$.

Under these conditions Claims 1 and 3(i) are restated as follows:

CLAIM 4. Let $t \ge 0$.

(i) Under $(A)_0'$

$$\|K(t)-e^{-tH}\|_{2\to 2}, \ \|G(t)-e^{-tH}\|_{2\to 2}, \ \|R(t)-e^{-tH}\|_{2\to 2} \le \text{const}(\gamma,d)C_1 t^{1+\gamma/2}.$$

(ii) Under $(A)_1'$

$$\begin{aligned}
&\|K(t)-e^{-tH}\|_{2\to 2} \\
&\le \text{const}(\delta,\kappa,d)\Big\{C_1^2(t^3+t^{1+2\delta}) + C_2 t^{1+(1+\kappa)/2} + C_2^2 t^{2+1+\kappa}\Big\},
\end{aligned}$$

$$\|G(t) - e^{-tH}\|_{2\to 2},\ \|R(t) - e^{-tH}\|_{2\to 2}$$
$$\le \mathrm{const}(\delta,\kappa,d)\Big\{C_1(t^{3/2} + t^{1/2+\delta}) + C_1^2(t^3 + t^{1+2\delta}) + C_2 t^{1+(1+\kappa)/2} + C_2^2 t^{2+1+\kappa}\Big\}.$$

Claim 4 can be shown in the same way as in [10] or [6] without using Itô's formula (cf. Remark 4).

CLAIM 5. For $t > 0$

$$\|[e^{-tH}, V]\|_{2\to 2} \le \begin{cases} \mathrm{const}(\gamma, d)C_1 t^{\gamma/2} & \text{under (A)}_0' \\ \mathrm{const}(\delta,\kappa,d)\Big\{C_1(t^{1/2} + t^{-1/2+\delta}) + C_2 t^{(1+\kappa)/2}\Big\} & \text{under (A)}_1'. \end{cases}$$

As in the proof of Claim 3(i), Claim 5 follows from the following estimate:

$$|V(y) - V(x)|\exp\Big\{-t\int_0^1 V(x + s(y-x) + \sqrt{t}X_0(s))ds\Big\}$$
$$\le \begin{cases} C_1|x-y|^{\gamma} & \text{under (A)}_0' \\ C_1\Big(1 + \Big(\dfrac{1-\delta}{e}\Big)^{1-\delta}\Big)|x-y| + C_2\displaystyle\int_0^1 |X_0(s)|^{\kappa}ds\ t^{\kappa/2}|x-y| & \text{under (A)}_1'. \end{cases}$$

The former estimate is clear by the condition $(\mathrm{A})_0'$(i). The latter is easily seen from the following:

$$|V(y) - V(x)|$$
$$= \Big|\int_0^1 \Big\langle \nabla V(x + s(y-x) + \sqrt{t}X_0(s)), y - x\Big\rangle ds$$
$$- \int_0^1 \Big\langle \nabla V(x + s(y-x) + \sqrt{t}X_0(s)) - \nabla V(x + s(y-x)), y - x\Big\rangle ds\Big|$$
$$\le \int_0^1 |\nabla V(x + s(y-x) + \sqrt{t}X_0(s))|ds|x-y|$$
$$+ \int_0^1 |\nabla V(x + s(y-x) + \sqrt{t}X_0(s)) - \nabla V(x + s(y-x))|ds|x-y|$$
$$\le \int_0^1 C_1(1 + V(x + s(y-x) + \sqrt{t}X_0(s))^{1-\delta})ds|x-y|$$
$$+ \int_0^1 C_2 t^{\kappa/2}|X_0(s)|^{\kappa}ds|x-y|$$
$$\le C_1\Big(1 + \Big(\int_0^1 V(x + s(y-x) + \sqrt{t}X_0(s)ds\Big)^{1-\delta}\Big)|x-y|$$

$$+ C_2 \int_0^1 |X_0(s)|^\kappa ds\ t^{\kappa/2}|x-y|$$

where the last inequality is due to Jensen's inequality.

Now let us look over the proof of Theorem. This time we use Claims 4 and 5 instead of Claims 1 and 3(i), so that we have the following theorem:

Theorem$'$. *Let $T \geq 1$ and $0 < t \leq T$.*

(i) *For $n \geq 2$*

$$\left\|K\left(\frac{t}{n}\right)^n - e^{-tH}\right\|_{\text{trace}} \leq \begin{cases} \text{const}(C_1, \gamma, \rho, c, c', d, T)\left(\frac{1}{n}\right)^{\gamma/2} t^{1+\gamma/2-d(1/2+1/\rho)} & \textit{under } (\text{A})'_0 \\ \text{const}(C_1, C_2, \delta, \kappa, \rho, c, c', d, T)\left(\frac{1}{n}\right)^{2\delta\wedge((1+\kappa)/2)} t^{1+2\delta\wedge((1+\kappa)/2)-d(1/2+1/\rho)} & \textit{under } (\text{A})'_1. \end{cases}$$

(ii) *For $n \geq 3$*

$$\left\|G\left(\frac{t}{n}\right)^n - e^{-tH}\right\|_{\text{trace}}, \left\|R\left(\frac{t}{n}\right)^n - e^{-tH}\right\|_{\text{trace}} \leq \begin{cases} \text{const}(C_1, \gamma, \rho, c, c', d, T)\left(\frac{1}{n}\right)^{\gamma/2} t^{1+\gamma/2-d(1/2+1/\rho)} & \textit{under } (\text{A})'_0 \\ \text{const}(C_1, C_2, \delta, \kappa, \rho, c, c', d, T)\left(\frac{1}{n}\right)^{2\delta\wedge((1+\kappa)/2)} t^{1/2+\delta-d(1/2+1/\rho)} & \textit{under } (\text{A})'_1. \end{cases}$$

As an example of our conditions $(\text{A})'_0$, $(\text{A})'_1$ and $(\text{A})'_2$ we give the following: Let $V_\rho(x) = |x|^\rho$ $(0 < \rho < \infty)$. Then

(i) if $0 < \rho \leq 1$, $(\text{A})'_0$ holds with $C_1 = 1$, $\gamma = \rho$,

(ii) if $1 < \rho < 2$, $(\text{A})'_1$ holds with $C_1 = \rho$, $\delta = 1/\rho$, $C_2 = \rho 2^{\rho-2}$ and $\kappa = \rho - 1$,

(iii) if $\rho \geq 2$, $(\text{A})'_2$ holds with $C_1 = \rho$, $\delta = 1/\rho$, $C_2 = \rho(\rho-1)2^{(\rho-3)_+}$, $\mu = 0$ and $\nu = \rho - 2$.

Thus it turns out that the conditions $(\text{A})'_0$ and $(\text{A})'_1$ treat the case of less regular potentials V than the condition $(\text{A})'_2$.

References

[1] B.O. Dia and M. Schatzman: *An estimate on the Kac transfer operator*, J. Funct. Anal. **145** (1997), 108–135.

[2] A. Doumeki, T. Ichinose and H. Tamura: *Error bound on exponential product formulas for Schrödinger operators*, J. Math. Soc. Japan **50** (1998), 359–377.

[3] N. Dunford and J.T. Schwartz: Linear operators part II, Interscience Publ. 1963.

[4] B. Helffer: *Correlation decay and gap of the transfer operator* (in English), Algebra i Analiz (St. Petersburg Math. J.) **8** (1996), 192–210.

[5] B. Helffer: *Around the transfer operator and the Trotter-Kato formula*, Operator Theory: Advances and Appl. **78**, 161–174, Birkhäuser, Basel, 1995.

[6] T. Ichinose and S. Takanobu: *Estimate of the difference between the Kac operator and the Schrödinger semigroup*, Commun. Math. Phys. **186** (1997), 167–197.

[7] N. Ikeda and S. Watanabe: Stochastic differential equations and diffusion processes, 2nd ed. North-Holland/Kodansha, Amsterdam/Tokyo, 1988.

[8] B. Simon: Functional integration and quantum physics, Academic Press, New York, 1979.

[9] B. Simon: *Schrödinger semigroups*, Bull. Amer. Math. Soc. **7** (1982), 447–526.

[10] S. Takanobu: *On the error estimate of the integral kernel for the Trotter product formula for Schrödinger operators*, Ann. Probab. **25** (1997), 1895–1952.

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