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ON THE COMPLETE SYSTEM OF FINITE ORDER FOR CR MAPPINGS AND ITS APPLICATION

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1. Introduction

The present study is related to the following problem.

Problem. Classify CR mappings between CR manifolds in terms of CR geometry (for example, Levi form, type of point, minimality and so on).

There are at least following three directions in classifying CR mappings. Let $F\ M \to \tilde{M}$ be an arbitrary CR mapping between CR manifolds.
(1) If the Levi form of M satisfies certain conditions, then F is the restriction of a holomorphic mapping. [18], [6], [7], [20], [3], [4], [8], [5].
(2) If the type of points in M and $\tilde{M}$ satisfies certain conditions, then F is constant. [5], [15], [16], [17].
(3) If the system of vectors derived from the mapping F and the tangential Cauchy-Riemann vector fields on M satisfy certain conditions, then F satisfies a complete system of finite order. [11], [12], [13], [14].
In this paper, we consider (3). First we give a definition of a complete system.

Complete System ([13]). A function F is said to satisfy a complete system of order K if, for each multi-index α with $|\alpha| = K$, there exists a real analytic function H_α such that

$$D^\alpha F = H_\alpha(z, D^\beta F\ ;\ |\beta| \leq K - 1).$$

Thus if a function of class C^K satisfies a complete system of order K, then it is a real analytic function. We say that a mapping $(F_1, \ldots, F_n)$ satisfies a complete system of order K if its component F_j does.

Previously, C.K. Han mentioned the following conjecture and, in this paper, we shall give a partial answer to it.

Conjecture ([11]). Suppose that M_1 and M_2 are germs of C^ω pseudoconvex hypersurfaces in $\mathbb{C}^{n+1}$ and $F\ M_1 \to M_2$ is a CR equivalence. If there is no complex subvariety contained in M_1, then there exists an integer k such that $F \in C^k$ implies $F \in C^\omega$.

Recently he gave a partial answer to this conjecture.

Theorem (C. K. Han) [14]. *Let M^{2m+1} be a C^ω CR manifold of nondegenerate Levi form. Let $\{L_1, \dots, L_m\}$ be C^ω independent sections of the CR structure bundle of ν. Let N be a C^ω real hypersurface in $\mathbb{C}^{n+1}$, $n \geq m$, defined by $r(z, \bar{z}) = 0$, where $r(z, \bar{z})$ is normalized as Chern-Moser style (see* (2.2) *in* §2). *Let $f : M \to N$ be a CR mapping. Suppose for some positive integer K the vectors $\{L^\alpha f : |\alpha| \leq K\}$ together with $(0, \dots, 0, 1)$ span $\mathbb{C}^{n+1}$. Then f satisfies a complete system of order $2K + 1$. Thus, f is determined by $2K$-jet at a point and f is C^ω provided that $f \in C^{2K+1}$.*

By observing the proof of the Han Theorem carefully, we can generalize the Han Theorem to the case of the CR mappings from the hypersurface with degenerate Levi form to the one with nondegenerate Levi form.

Let M and $\tilde{M}$ be real hypersurfaces containing the origin in $\mathbb{C}^{n+1}$. Denote the variables in $\mathbb{C}^{n+1}$ as $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $w = s + it \in \mathbb{C}$. We say that a CR mapping $F : M \to \tilde{M}$ between hypersurfaces satisfies the Hopf lemma property at $p \in M$ [1] if the component of F normal to $\tilde{M}$ has a nonzero derivative at p in the normal direction to M. For functions $f_1, \dots, f_n$ of class C^m, the symbol $\mathrm{sp}\langle f_1, \dots, f_n\rangle_{\mathbb{C}} \not\ni 0 \pmod{\mathcal{I}^{m+1}}$ means that there does not exist $(a_1, \dots, a_n) \in \mathbb{C}^n \backslash (0, \dots, 0)$ such that $a_1 f_1 + \dots + a_n f_n \equiv 0 \pmod{\mathcal{I}^{m+1}}$, where $\mathcal{I}$ is an ideal generated by z, $\bar{z}$, s. In the following theorem, we assume that M and $\tilde{M}$ are real analytic hypersurfaces containing the origin and that $(f, f_{n+1}) = (f_1, \dots, f_n, f_{n+1}) : M \to \tilde{M}$ is a CR mapping of class C^m preserving the origin. In case (I), we assume that $m \geq l + 1$ and in case (II), m is large enough (but not necessarily infinite). Type of point is in the sense of T. Bloom–I. Graham [6] ($l < +\infty$).

MainTheorem. *Let M and $\tilde{M}$ be real hypersurfaces in $\mathbb{C}^{n+1}$ and $(f, f_{n+1}) : M \to \tilde{M}$ a CR mapping. Suppose that $\tilde{M}$ has a nondegenerate Levi form at the origin and that the origin in M is a point of type l ($< +\infty$). Consider the following two cases;*

(I) *M has a nondegenerate Levi form at the origin ($l = 2$), or M has a degenerate Levi form at the origin and $n = 1$.*

(II) *M has a degenerate Levi form at the origin and $n \geq 2$.*

In case (I), *if (f, f_{n+1}) satisfies the Hopf lemma property at the origin, then it satisfies a complete system of order $l + 1$.*

In case (II), *if (f, f_{n+1}) satisfies $sp\langle f_1, \dots, f_n\rangle_{\mathbb{C}} \not\ni 0 \pmod{\mathcal{I}^{m+1}}$, then it satis-*

fies a complete system of finite order.

As a corollary to this theorem, we can prove a holomorphic extendability theorem. Many of the holomorphic extendability theorems for CR mappings proved before have an assumption that the CR mappings are of class C^∞. For example, M.S. Baouendi and L.P. Rothschild proved the following theorem.

Theorem (M.S. Baouendi–L.P. Rothschild) [2]. *Let $F : M \to M'$ be a smooth CR mapping, where M and M' are real analytic hypersurfaces in $\mathbb{C}^{n+1}$. Let $p_0 \in M$ and $p_0' = F(p_0)$. If either one of the following conditions is satisfied, then F is the restriction of a holomorphic mapping from a neighborhood of p_0 in $\mathbb{C}^{n+1}$ into $\mathbb{C}^{n+1}$.*

(1) *The mapping F is of finite multiplicity at p_0, and M' is essentially finite at p_0'.*

(2) *M is essentially finite at p_0 and F satisfies $F'(\mathbb{C}T_{p_0}M) \not\subset H^{\mathbb{C}}_{p_0'}(M')$.*

On the other hand, by the help of the study of the extension problem for proper holomorphic mappings, the holomorphic extendability theorem for CR mappings of class C^m $(m < +\infty)$ were proved by using the argument of papers [19] and [10]. We give another extension theorem for such mappings as a corollary to main theorem.

Corollary. *Let the notation be the same as in the main theorem. Then a CR mapping (f, f_{n+1}) is the restriction of a holomorphic mapping on a neighborhood of the origin if one of the following conditions holds.*

(1) *M has a nondegenerate Levi form at the origin. The mapping (f, f_{n+1}) is of class C^3 and satisfies the Hopf lemma property at the origin.*

(2) *M has a degenerate Levi form at the origin and $n = 1$. The mapping (f_1, f_2) is a mapping of class C^{l+1} and satisfies the Hopf lemma property at the origin.*

(3) *M has a degenerate Levi form at the origin and $n \geq 2$. The mapping (f, f_{n+1}) is of class C^m (m is large enough, but not necessarily infinite) and satisfies $sp\langle f_1, \dots, f_n\rangle_{\mathbb{C}} \not\ni 0 \pmod{\mathcal{I}^{m+1}}$.*

This paper is organized as follows. In §2, we give some notation, basic results on the tangential Cauchy-Riemann vector fields and on expansions of CR functions. In §3, we prove the main theorem.

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2. The tangential Cauchy-Riemann vector fields and power series of CR functions

Let M be a real analytic hypersurface containing the origin as a point of type l ($< \infty$) in the sense of T. Bloom and I. Graham [6], [8]. Then, after a suitable coordinate change, we may assume that M has a local defining function $r = t - h(z, \bar{z}, s)$, where h is a real analytic function and is expanded in a neighborhood of the origin as

$$h(z, \bar{z}, s) = \sum_{\substack{|\nu| + |\mu| \geq l \\ |\nu|, |\mu| \geq 1, \ \tau \geq 0}} h_{\nu,\mu,\tau} z^{\nu} \bar{z}^{\mu} s^{\tau}. \tag{2.1}$$

In particular, if M has a nondegenerate Levi form at the origin, then we may assume that h is expanded as

$$h(z, \bar{z}, s) = \sum_{j=1}^{n} \lambda_j |z_j|^2 + \sum_{\substack{|\nu|, |\mu| \geq 2 \\ \tau \geq 0}} h_{\nu,\mu,\tau} z^{\nu} \bar{z}^{\mu} s^{\tau}, \tag{2.2}$$

where $\lambda_j = +1$ or -1. In case $n = 1$, we assume $\lambda_1 = +1$. This expansion is due to Chern-Moser [9]. Therefore, after a suitable coordinate change, we may assume that, for a sufficiently small neighborhood U of the origin, we have $(\{0\}^n \times \mathbb{R}) \cap U \subset M$. Notation for $\tilde{M}$ will be denoted by 'tilde' style.

In this notation, we write down the tangential Cauchy-Riemann vector field L_k as

$$L_k = \frac{\partial}{\partial z_k} + i \frac{h_{z_k}(z, \bar{z}, s)}{1 - i h_s(z, \bar{z}, s)} \frac{\partial}{\partial s}, \quad \text{for } k = 1, \ldots, n, \tag{2.3}$$

where $h_{z_k}(z, \bar{z}, s)$ (resp. $h_s(z, \bar{z}, s)$) stands for the derivative of h in z_k (resp. s).

Lemma 2.1. *Let $\alpha_1, \ldots, \alpha_n$ be nonnegative integers. Then*

$$L_1^{\alpha_1} \ldots L_n^{\alpha_n} \big|_{(z,\bar{z})=(0,0)} = \frac{\partial^{\alpha_1 + \ldots + \alpha_n}}{\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n}}.$$

Proof. This follows from the form of expansion h. □

Any CR mapping between hypersurfaces defined by (2.1) and its 'tilde' style is expanded as a power series. To show it, we need the following lemma.

Lemma 2.2 ([2]). *Any function f on M of class C^m satisfying $\bar{L}_k f \equiv 0$ (mod $\mathcal{I}^{m+1}$) and $f|_{\{y=0\}} \equiv 0$ (mod $\mathcal{I}^{m+1}$) is an identically zero function in* mod $\mathcal{I}^{m+1}$.

Proof. We write

$$2\bar{L}_k = \frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k} + a_k(x,y,s)\frac{\partial}{\partial s}.$$

Expand f on M as

$$f(x,y,s) \equiv \sum_{|\alpha|\geq 0} f_\alpha(x,s)y^\alpha, \ (\mathrm{mod}\ \mathcal{I}^{m+1})$$

so that

$$f(x,0,s) \equiv f_0(x,s) \equiv 0 \ (\mathrm{mod}\ \mathcal{I}^{m+1}).$$

Then since the coefficient of y^α in $\bar{L}_k f(x,y,s) \equiv 0$ (mod $\mathcal{I}^{m+1}$) satisfies

$$\frac{\partial f_{\alpha_1,\ldots,\alpha_n}}{\partial x_k} + i(\alpha_k+1)f_{\alpha_1,\ldots,\alpha_k+1,\ldots,\alpha_n} + a_k(x,y,s)\frac{\partial f_{\alpha_1,\ldots,\alpha_n}}{\partial s} \equiv 0 \ (\mathrm{mod}\ \mathcal{I}^{m+1})$$

for $\alpha_1 + \ldots + \alpha_n \leq m-1$, we have $f \equiv 0$ (mod $\mathcal{I}^{m+1}$). □

Using this lemma, we now prove the following proposition.

Proposition 2.3 ([2]). *Let*

$$(f_1,\ldots,f_n,g) = (f,g)\ M \to \tilde{M}$$

be a CR mapping of class C^m with $(f,g)(0,0) = (0,0)$. Suppose that there exists a sufficiently small neighborhood U of the origin such that the mapping satisfies the property;

$$\begin{cases} f_j((\{0\}^n \times \mathbb{R}) \cap U) \equiv 0 \quad (\mathrm{mod}\ \mathcal{I}^{m+1}), \\ \mathrm{Im} g((\{0\}^n \times \mathbb{R}) \cap U) \equiv 0 \quad (\mathrm{mod}\ \mathcal{I}^{m+1}). \end{cases} \tag{2.4}$$

Then f_j and g can be expanded as

$$f_j(z,\bar{z},s) \equiv \sum_{|\alpha|\geq 1, p\geq 0} a^j_{\alpha,p} z^\alpha \big(s + ih(z,\bar{z},s)\big)^p \qquad (\mathrm{mod}\ \mathcal{I}^{m+1}), \tag{2.5}$$

$$g(z,\bar{z},s) \equiv \sum_{q=1}^{\infty} b_{0,q}\big(s + ih(z,\bar{z},s)\big)^q \qquad (\mathrm{mod}\ \mathcal{I}^{m+1}). \tag{2.6}$$

Proof. Expand f_j on $M \cap \{y=0\}$ as

$$f_j(x,x,s) \equiv \sum_{|\alpha|+p\geq 1} \tilde{a}^j_{\alpha,p} x^\alpha s^p \qquad (\mathrm{mod}\ \mathcal{I}^{m+1}).$$

Then we can find $a^j_{\alpha,p}$ inductivity so that

$$\sum_{|\alpha|+p\geq 1} \tilde{a}^j_{\alpha,p} x^\alpha s^p \equiv \sum_{|\alpha|+p\geq 1} a^j_{\alpha,p} x^\alpha \big(s+ih(x,x,s)\big)^p \qquad (\mathrm{mod}\ \mathcal{I}^{m+1}).$$

Let $F_j(z,\bar{z},s)$ be a power series as

$$F_j(z,\bar{z},s) \equiv \sum_{|\alpha|+p\geq 1} a^j_{\alpha,p} z^\alpha \big(s+ih(z,\bar{z},s)\big)^p \qquad (\mathrm{mod}\ \mathcal{I}^{m+1}).$$

Since we have
(1) $(f_j - F_j)|_{\{y=0\}} \equiv 0 \qquad (\mathrm{mod}\ \mathcal{I}^{m+1})$
(2) $\bar{L}_k(f_j - F_j) \equiv 0 \qquad (\mathrm{mod}\ \mathcal{I}^{m+1})$
and by Lemma 2.2, we obtain an expansion of f_j. By the same argument, we get an expansion of g. Namely, without the property (2.4), f_j and g are expanded as

$$f_j(z,\bar{z},s) \equiv \sum_{|\alpha|+p\geq 1} a^j_{\alpha,p} z^\alpha \big(s+ih(z,\bar{z},s)\big)^p \qquad (\mathrm{mod}\ \mathcal{I}^{m+1}) \tag{2.7}$$

and

$$g(z,\bar{z},s) \equiv \sum_{|\beta|+q\geq 1} b_{\beta,q} z^\beta \big(s+ih(z,\bar{z},s)\big)^q \qquad (\mathrm{mod}\ \mathcal{I}^{m+1}). \tag{2.8}$$

The property (2.4) implies $a^j_{0,p} = 0$ for $1 \leq p \leq m$, which gives a desired result for f_j, and $b_{0,q} \in \mathbb{R}$ for $1 \leq q \leq m$. Substitute $a^j_{0,p} = 0$ and $b_{0,q} \in \mathbb{R}$ into (2.7) and (2.8), then resulting power series satisfy

$$\begin{aligned}
&\frac{1}{2i}\left[\sum_{|\beta|+q\geq 1} b_{\beta,q} z^\beta (s+ih)^q - \sum_{|\beta|+q\geq 1} \bar{b}_{\beta,q} \bar{z}^\beta (s-ih)^q\right] \\
&\equiv \sum_{\substack{|\nu|+|\mu|\geq \tilde{l} \\ |\nu|,|\mu|\geq 1,\ \tau\geq 0}} \tilde{h}_{\nu,\mu,\tau}\left[\sum_{|\alpha|\geq 1, p\geq 0} a_{\alpha,p} z^\alpha (s+ih)^p\right]^\nu \left[\sum_{|\alpha|\geq 1, p\geq 0} \bar{a}_{\alpha,p} \bar{z}^\alpha (s-ih)^p\right]^\mu \\
&\quad \times\left[\frac{1}{2}\left\{\sum_{|\beta|+q\geq 1} b_{\beta,q} z^\beta (s+ih)^q + \sum_{|\beta|+q\geq 1} \bar{b}_{\beta,q} \bar{z}^\beta (s-ih)^q\right\}\right]^\tau \qquad (\mathrm{mod}\ \mathcal{I}^{m+1}).
\end{aligned}$$

Since, in the above equality, the sum of the terms that are not multiplies of $z_i \bar{z}_k$ satisfies

$$\sum_{|\beta|\geq 1, q\geq 0} b_{\beta,q} z^\beta s^q - \sum_{|\beta|\geq 1, q\geq 0} \bar{b}_{\beta,q} \bar{z}^\beta s^q \equiv 0 \qquad (\mathrm{mod}\ \mathcal{I}^{m+1}),$$

we get

$$b_{\beta,q}=0 \quad \text{for} \quad |\beta|+q\le m, \ |\beta|\ge 1.$$

Therefore the terms with coefficients $b_{\beta,q}$ with indices $|\beta|+q\ge m+1$, $|\beta|\ge 1$ and $|\beta|=0$, $q\ge 1$ remain. Since we consider (2.8) in mod $\mathcal{I}^{m+1}$, only the terms with coefficients $b_{0,q}$ for $q\ge 1$ remain. This gives a desired expansion of g. □

Lemma 2.4. *Under the same notation as in Proposition* 2.3, *we have* $(\partial g/\partial s)(0) = b_{0,1}\in\mathbb{R}$.

This lemma, which plays an important role in the proof of the main theorem, holds without the property (2.4).

3. Proof of Main Theorem

In this section, we give a proof of the main theorem, which was stated in §1. Assume that defining functions for real analytic hypersurfaces are normalized as (2.1) or (2.2). Recall that a CR mapping (f, f_{n+1}) satisfies the Hopf lemma property at the origin if $(\partial f_{n+1}/\partial s)(0)\neq 0$. Divide case (I) into two parts;
(I-1) M has a nondegenerate Levi form at the origin ($l=2$).
(I-2) M has a degenerate Levi form at the origin and $n=1$.

We shall prove (I-1) $K=1$, (I-2) $K=l/2$ and (II) $K<+\infty$, where K is an integer in the Han Theorem.

Proof of Case (I). For a sufficiently small neighborhood U of the origin, take a real analytic curve γ such that it approximates the curve $(f, f_{n+1})((\{0\}^n\times\mathbb{R})\cap U)$ up to order m at the origin. Since the CR mapping (f, f_{n+1}) satisfies the Hopf Lemma property at the origin, γ is transversal to $H_0(\tilde{M})$, the holomorphic tangent space at the origin. Therefore by [9] §3, after a suitable coordinate change, we may assume that $(f, f_{n+1})((\{0\}^n\times\mathbb{R})\cap U)$ is tangent to $\{0\}^n\times\mathbb{R}$ at the origin up to order m. Namely, the CR mapping satisfies the property (2.4) in Proposition 2.3 and therefore f_j and f_{n+1} are expanded as (2.5) and (2.6) respectively. Note that the form of the defining function for M is invariant under this coordinate change.

CASE (I-1). To show $K=1$, we prove that

$$\det\begin{pmatrix} L_1 f_1(0) & \dots & L_1 f_n(0)\\ \vdots & \ddots & \vdots\\ L_n f_1(0) & \dots & L_n f_n(0)\end{pmatrix} = \det\begin{pmatrix} a^1_{\alpha_1,0} & \cdots & a^n_{\alpha_1,0}\\ \vdots & \ddots & \vdots\\ a^1_{\alpha_n,0} & \cdots & a^n_{\alpha_n,0}\end{pmatrix}\neq 0$$

for $\alpha_k=(0,\dots,1,\dots,0)$ (only k-th component is 1). The first equality follows from the simple calculation. For simplicity, we write the left hand side as det(Jac$(f)(0)$).

First we consider the equality $\mathrm{Im} f_{n+1} \equiv \tilde{h}(f, \bar{f}, \mathrm{Re} f_{n+1}) \pmod{\mathcal{I}^{m+1}}$, namely,

$$(3.1)\ \frac{1}{2i}\left[\sum_{q\geq 1} b_{0,q}\left(s+ih(z,\bar{z},s)\right)^q - \sum_{q\geq 1} \bar{b}_{0,q}\left(s-ih(z,\bar{z},s)\right)^q\right]$$
$$\equiv \sum_{j=1}^{n}\tilde{\lambda}_j\left[\sum_{|\alpha|\geq 1, p\geq 0} a^j_{\alpha,p} z^\alpha\left(s+ih(z,\bar{z},s)\right)^p\right]\left[\sum_{|\alpha|\geq 1, p\geq 0} \bar{a}^j_{\alpha,p} \bar{z}^\alpha\left(s-ih(z,\bar{z},s)\right)^p\right]$$
$$+\text{higher terms} \pmod{\mathcal{I}^{m+1}}.$$

Since $h(z,\bar{z},s)$ is expanded as (2.2), comparing the coefficients of $z_i\bar{z}_k$ $(i\neq k)$ on the both sides of (3.1), we get

$$\sum_{j=1}^{n}\tilde{\lambda}_j\left(a^j_{\alpha_i,0}\right)\left(\bar{a}^j_{\alpha_k,0}\right) = 0.$$

This is vacuous if $n = 1$.

Comparing the coefficients of $|z_k|^2$ on both sides of (3.1), we get

$$\frac{1}{2}\lambda_k(b_{0,1}+\bar{b}_{0,1}) = b_{0,1}\lambda_k \quad \text{(by Lemma 2.4)}$$
$$= \sum_{j=1}^{n}\tilde{\lambda}_j|a^j_{\alpha_k,0}|^2.$$

Therefore we obtain the relation;

$$\begin{pmatrix} \bar{a}^1_{\alpha_1,0} & \cdots & \bar{a}^n_{\alpha_1,0} \\ \vdots & \ddots & \vdots \\ \bar{a}^1_{\alpha_n,0} & \cdots & \bar{a}^n_{\alpha_n,0} \end{pmatrix}\begin{pmatrix} \tilde{\lambda}_1 a^1_{\alpha_1,0} & \cdots & \tilde{\lambda}_1 a^1_{\alpha_n,0} \\ \vdots & \ddots & \vdots \\ \tilde{\lambda}_n a^n_{\alpha_1,0} & \cdots & \tilde{\lambda}_n a^n_{\alpha_n,0} \end{pmatrix} = b_{0,1}\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

Thus $\det(\mathrm{Jac}(f)(0))\neq 0$ if and only if

$$b_{0,1} = \frac{\partial f_{n+1}}{\partial s}(0) \neq 0,$$

which implies the assertion.

CASE (I-2). It is sufficient to show $L_1^{l/2} f_1(0,0) = a^1_{l/2,0} \neq 0$. Note that $l/2 \in \mathbb{N}$ by the previous paper [16]. As in the case (I-1), the components of the CR mapping (f_1, f_2) are expanded as in Proposition 2.3. We start with the following equality,

$$(3.2)\quad \frac{1}{2i}\left[\sum_{q=1}^{\infty} b_{0,q}\left(s+ih(z,\bar{z},s)\right)^q - \sum_{q=1}^{\infty} \bar{b}_{0,q}\left(s-ih(z,\bar{z},s)\right)^q\right]$$

$$\equiv \left[\sum_{\alpha\geq 1, p\geq 0} a^1_{\alpha,p} z^\alpha \big(s + ih(z,\bar z, s)\big)^p \right] \left[\sum_{\alpha\geq 1, p\geq 0} \bar a^1_{\alpha,p} \bar z^\alpha \big(s - ih(z,\bar z, s)\big)^p \right]$$
$$+\text{higher terms} \qquad (\text{mod } \mathcal{I}^{m+1}).$$

First we consider $l \geq 4$. Comparing the terms of degree smaller than $l-1$ in z and $\bar z$ on both sides of the above equality, we get

$$a^1_{\alpha,p} = 0 \qquad \text{for } 1 \leq \alpha \leq \frac{l}{2} - 1,\ p \geq 0,\ \alpha + p \leq m.$$

Let denote $h^{(l)}$ by the homogeneous polynomial of degree l in z and $\bar z$ in h. Substitute $a^1_{\alpha,p} = 0$, $1 \leq \alpha \leq (l/2) - 1$, $p \geq 0$, $\alpha + p \leq m$ into (3.2). Then picking up the terms of degree l in z and $\bar z$ from the resulting equality, we obtain the equality;

$$\frac{1}{2}\sum_{q\geq 1}(b_{0,q} + \bar b_{0,q}) q s^{q-1} h^{(l)}$$
$$\equiv \sum_{p\geq 0} a^1_{l,p} z^l s^p \sum_{p\geq 1} \bar a^1_{0,p} s^p + \sum_{\alpha=1}^{(l/2)-1} \left(\sum_{p\geq 0} a^1_{l-\alpha,p} z^{l-\alpha} s^p\right)\left(\sum_{p\geq m-\alpha+1} \bar a^1_{\alpha,p} \bar z^\alpha s^p\right)$$
$$+ \sum_{p\geq 0} a^1_{l/2,p} z^{l/2} s^p \sum_{p\geq 0} \bar a^1_{l/2,p} \bar z^{l/2} s^p$$
$$+ \sum_{\alpha=1}^{(l/2)-1} \left(\sum_{p\geq m-\alpha+1} a^1_{\alpha,p} z^\alpha s^p\right)\left(\sum_{p\geq 0} \bar a^1_{l-\alpha,p} \bar z^{l-\alpha} s^p\right) + \sum_{p\geq 0} \bar a^1_{l,p} \bar z^l s^p \sum_{p\geq 1} a^1_{0,p} s^p$$
$$(\text{mod } \mathcal{I}^{m+1}).$$

Comparing the terms that are not multiplies of s in the above equality, we have

$$\frac{1}{2}(b_{01} + \bar b_{0,1}) h^{(l)} = |a^1_{l/2,0}|^2 |z|^l.$$

Since the homogeneous polynomial $h^{(l)}$ has an expansion;

$$h^{(l)}(z,\bar z) = \sum_{\substack{\nu+\mu=l \\ \nu,\mu\geq 1}} h_{\nu,\mu,0} z^\nu \bar z^\mu,$$

we get $h_{\nu,\mu,0} = 0$, for $(\nu,\mu) \neq (l/2, l/2)$. If we have $h_{l/2,l/2,0} = 0$, then we get $h^{(l)} = 0$, and this implies that the origin in M is not of finite type l. Therefore we conclude that $h^{(l)}(z,\bar z) = h_{l/2,l/2,0}|z|^l$ and $h_{l/2,l/2,0} \neq 0$. By Lemma 2.4, we obtain

$$|a^1_{l/2,0}|^2 = h_{l/2,l/2,0} b_{0,1} = h_{l/2,l/2,0} \frac{\partial f_2}{\partial s}(0).$$

Therefore, by the assumption that the mapping (f_1, f_2) satisfies the Hopf lemma property, we have $a^1_{l/2,0} \neq 0$, which implies the assertion of this theorem.

Next we consider $l = 2$. Comparing the coefficients of the terms of degree 2 in z and $\bar{z}$ on both sides of (3.2) and noting that $h_{1,1,0} = 1$ by (2.1) and (2.2), we have $2|a_{1,0}|^2 = b_{0,1} + \bar{b}_{0,1}$, which completes the theorem by using the same argument as above. □

Proof of Case (II). To show that $K < +\infty$, it is sufficient to show that there exist n-linearly independent vectors in

$$\text{(3.3)} \qquad \begin{cases} L^{\theta_1} f(0,s) &= \left(L^{\theta_1} f_1(0,s), \dots, L^{\theta_1} f_n(0,s)\right) \\ L^{\theta_2} f(0,s) &= \left(L^{\theta_2} f_1(0,s), \dots, L^{\theta_2} f_n(0,s)\right) \\ &\vdots \\ L^{\theta_n} f(0,s) &= \left(L^{\theta_n} f_1(0,s), \dots, L^{\theta_n} f_n(0,s)\right) \\ &\vdots \\ L^{\theta} f(0,s) &= \left(L^{\theta} f_1(0,s), \dots, L^{\theta} f_n(0,s)\right) \end{cases}$$

for real variable s. Here $\theta_1, \dots, \theta_n, \dots, \theta$ are multi-indices with $|\theta_1| \leq \dots \leq |\theta_n| \leq \dots \leq |\theta|$ and $|\theta|$ $(< \infty)$ is large enough.

Assume that there exist only k-independent vectors $(k < n)$. Then, after renumbering if necessary, there exist $c_1^{k+1}, \dots, c_k^{k+1}, \dots, c_1^n, \dots, c_k^n \in \mathbb{C}$ such that they satisfy $(n-k)$ equations;

$$\text{(3.4.k+1)} \qquad L^\alpha f_{k+1}(0,s) = \sum_{j=1}^{k} c_j^{k+1} L^\alpha f_j(0,s)$$

$$\vdots$$

$$\text{(3.4.n)} \qquad L^\alpha f_n(0,s) = \sum_{j=1}^{k} c_j^{n} L^\alpha f_j(0,s)$$

for any multi-index α with $1 \leq |\alpha| \leq |\theta|$. Since we have

$$L^\alpha f_j(0,s) \equiv \sum_{p \geq 0} \alpha! a^j_{\alpha,p} s^p \qquad (\text{mod } \mathcal{I}^{m+1}),$$

equations (3.4.k+1), ..., (3.4.n) become

$$\sum_{p \geq 0} a^{k+1}_{\alpha,p} s^p \equiv \sum_{j=1}^{k} \sum_{p \geq 0} c_j^{k+1} a^j_{\alpha,p} s^p \qquad (\text{mod } \mathcal{I}^{m+1})$$

$$\vdots$$

$$\sum_{p \geq 0} a^{n}_{\alpha,p} s^p \equiv \sum_{j=1}^{k} \sum_{p \geq 0} c_j^{n} a^j_{\alpha,p} s^p \qquad (\text{mod } \mathcal{I}^{m+1}).$$

Regarding these as power series in s, we get the equalities

$$a^{k+1}_{\alpha,p} = \sum_{j=1}^{k} c^{k+1}_j a^j_{\alpha,p}\ ,\ldots,\ a^n_{\alpha,p} = \sum_{j=1}^{k} c^n_j a^j_{\alpha,p}$$

for $0 \le p \le m - |\alpha|$. These imply the following relations,

$$\begin{aligned} f_{k+1}(z,\bar z,s) &\equiv \sum_{|\alpha|+p\ge 1} a^{k+1}_{\alpha,p} z^\alpha (s+ih)^p \qquad (\bmod\ \mathcal{I}^{m+1}) \\ &\equiv \sum_{|\alpha|+p\ge 1} \left\{ \sum_{j=1}^{k} c^{k+1}_j a^j_{\alpha,p} \right\} z^\alpha (s+ih)^p \qquad (\bmod\ \mathcal{I}^{m+1}) \\ &\equiv \sum_{j=1}^{k} c^{k+1}_j f_j(z,\bar z,s) \qquad (\bmod\ \mathcal{I}^{m+1}). \end{aligned}$$

Similarly, we get the relations,

$$f_{k+2}(z,\bar z,s) \equiv \sum_{j=1}^{k} c^{k+2}_j f_j(z,\bar z,s),\ldots, f_n(z,\bar z,s) \equiv \sum_{j=1}^{k} c^n_j f_j(z,\bar z,s) \quad (\bmod\ \mathcal{I}^{m+1}).$$

These relations contradict to the assumption. Therefore there are n-independent vectors in (3.3). Then the proof of case (II) is complete by putting $s = 0$ in (3.3). □

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