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CONFORMAL WELDING OF ANNULI

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1. Preliminary

We take a concentric annulus with center at zero in the complex plane $\mathbf{C}$ which contains the unit circle C. We cut it along C to get two annuli

$$A = \{z; a < |z| < 1\} \text{ and } B = \{z; 1 < |z| < b\}.$$

Take a continuous function φ on $\mathbf{R}$ such that

$$\varphi(\theta_1) < \varphi(\theta_2) \text{ if } \theta_1 < \theta_2,\ \ \varphi(\theta + 2\pi) = \varphi(\theta) + 2\pi,$$

and weld A and B so that the point $\exp i\theta \in \partial A$ corresponds to the point $\exp i\varphi(\theta) \in \partial B$. The resulting doubly connected region can not always be given a conformal structure whose restrictions to A and B are the same as original ones. We call φ a conformal welding function if there exists a conformal mapping f from $A \cup B$ onto an annulus $A(\varphi, f)$ except for a Jordan curve γ such that

$$\lim_{A \ni z \to e^{i\theta}} f(z) = \lim_{B \ni z \to e^{i\varphi(\theta)}} f(z) \in \gamma,$$

where $A(\varphi, f) = \{w; 1 < |w| < e^{M(\varphi, f)}\}$. The number $M(\varphi, f)$ is called the modulus of $A(\varphi, f)$. We call f a φ-mapping. For a conformal welding function φ, the welded doubly connected region has the conformal structure induced by $A(\varphi, f)$, which is consistent with the original conformal structure A and B.

Set

$$V(\varphi) = \{M(\varphi, f); f \text{ is a } \varphi\text{-mapping}\},$$

that is, $V(\varphi)$ denotes the set of all moduli of annuli made from the welding by a fixed conformal welding function φ.

If φ is real analytic, $V(\varphi)$ is a point. For a φ-mapping f, if $\gamma = f(C)$ has a positive area, we can induce a conformal structure given by the metric $ds = |dw + t\mu d\bar{w}|$,

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where $t\mu(d\bar{w}/dw)$ is a Beltrami differential whose support is contained in γ. For the modulus $m(t)$ of the associated Riemann surface, we know the following variational formula(cf. [2]);

$$m'(0) = \frac{1}{2\pi}\iint_{\gamma}\frac{\mu(w)}{w^2}dudv, \quad (w = u + iv).$$

Since we can choose μ which satisfies $m'(0) \neq 0$, $V(\varphi)$ is not a point.

K. Oikawa asked the following question. Is there a conformal welding function φ such that $V(\varphi)$ is a point but the welded doubly connected region has different conformal structures? If the welded doubly connected region has different conformal structures, there are φ-mappings f_1, f_2 such that $\hat{F} = f_2 \circ f_1^{-1}$ is regarded as a mapping from $A(\varphi, f_1)$ to $A(\varphi, f_2)$ which is not conformal on $\gamma = f_1(C)$. There is a homeomorphism $\tilde{F}$ on the extended complex plane $\hat{\mathbf{C}}$ such that $\tilde{F} = \hat{F}$ on $A(\varphi, f)$ and $\tilde{F}$ is quasiconformal on $\hat{\mathbf{C}} - \gamma$. Further there is a quasiconformal mapping h on $\hat{\mathbf{C}}$ such that $F = h \circ \tilde{F}$ is conformal on $\hat{\mathbf{C}} - \gamma$. The mapping F is continuous on $\hat{\mathbf{C}}$ and is conformal on $\hat{\mathbf{C}} - \gamma$. This γ contains a point on which F is not conformal. We are concerned with a Jordan curve γ which allows mappings like this F.

It is said that a compact set E is of class N_{SB} if there exists no bounded univalent analytic function on $\hat{\mathbf{C}} - E$ and is of class N_D if there exists no non-constant analytic function with a finite Dirichlet integral on $\hat{\mathbf{C}} - E$. Each univalent meromorphic function on $\hat{\mathbf{C}} - E$ is continuously extendable to $\hat{\mathbf{C}}$ if $E \in N_{SB}$ and it is a Möbius Transformation if $E \in N_D$(cf. [7]). Let f_h and f_v be the extremal horizontal and vertical slit mappings for $\hat{\mathbf{C}} - E$ respectively, which are normalized as follows

$$f_h = z + \sum_{n=1}^{\infty} a_n z^{-n},$$

$$f_v = z + \sum_{n=1}^{\infty} b_n z^{-n}.$$

For $E \in N_{SB} - N_D$, $f_v + f_h$ is a mapping which is continuous on $\hat{\mathbf{C}}$ and conformal on $\hat{\mathbf{C}} - E$, and $(f_v + f_h)(E)$ has positive area. Hence if γ contains a set in the class $N_{SB} - N_D$, there is a mapping such that it is continuous on $\hat{\mathbf{C}}$, is conformal on $\hat{\mathbf{C}} - \gamma$, but is not conformal on a point of γ.

Under such a background, in the first half we investigate, for the concerned mapping F, the behavior of a mapping $F(z) + tz$ with a complex parameter t, and in the second half we check the set of parameter t for which $f_v + tf_h$ is univalent. This article is a stepping stone to get answer for Oikawa's question. If the set of parameters t for which $F(z) + tz$ is univalent has interior points, γ is transformed to a Jordan curve of positive area. Hence, if it is shown that the refered set has always interior points, $V(\varphi)$ can't be a point in the case the doubly connected region welded by φ has different conformal structures. As an example we remark that the set of parameters t for which $f_v + tf_h$ is univalent becomes a half plane.

2. Behavior of F on the Jordan curve γ

Consider the situation in section 1, namely, the mapping F is continuous on $\hat{\mathbf{C}}$ and is conformal on $\hat{\mathbf{C}} - \gamma$ and γ contains a point on which F is not conformal. Let φ_1 be a conformal mapping from the exterior of the unit disk to the exterior of Jordan curve γ, whose Laurent development is

$$\varphi_1(z) = \sum_{n=-1}^{\infty} a_n z^{-n}$$

and φ_2 be a conformal mapping from the interior of the unit disk to the interior of γ, whose Taylor development is

$$\varphi_2(z) = \sum_{n=0}^{\infty} b_n z^n.$$

For $R > 1$, let G_R denote the bounded region enclosed by $\varphi_1(|z| = R)$. Then the area of G_R is

$$\begin{aligned}
|G_R| &= \frac{i}{2}\iint_{G_R} d\zeta d\bar{\zeta} = \frac{i}{2}\int_{\partial G_R} \zeta d\bar{\zeta} = \frac{i}{2}\int_{|z|=R} \varphi_1(z)\overline{\varphi_1'(z)}d\bar{z} \\
&= \frac{1}{2}\int_0^{2\pi} \left(\sum_{n=-1}^{\infty}\right)\left(\sum_{m=-1}^{\infty}\right) - m a_n \overline{a_m} R^{-n-m} e^{(-n+m)\theta i} d\theta \\
&= \pi \sum_{n=-1}^{\infty} -n|a_n|^2 R^{-2n}.
\end{aligned}$$

That is

$$|G_R| = \pi\left\{|a_{-1}|^2 R^2 - \sum_{n=1}^{\infty} n\frac{|a_n|^2}{R^{2n}}\right\}.$$

The area $|G_R|$ is decreasing as R decreases and

$$\lim_{R\to 1+} |G_R| = \pi\left\{|a_{-1}|^2 - \sum_{n=1}^{\infty} n|a_n|^2\right\}.$$

We denote this by $|\overline{G_1}|$. Similarly, for $r < 1$, let G_r denote the bounded region enclosed by $\varphi_2(|z| = r)$. Then the area of G_r is

$$|G_r| = \frac{i}{2}\int_{|z|=r} \varphi_2(z)\overline{\varphi_2'(z)}d\bar{z} = \frac{1}{2}\int_0^{2\pi}\left(\sum_{n=0}^{\infty}\right)\left(\sum_{m=0}^{\infty}\right) m b_n \overline{b_m} r^{n+m} e^{(n-m)\theta i} d\theta$$

$$= \pi \sum_{n=1}^{\infty} n|b_n|^2 r^{2n}.$$

We write

$$|\underline{G}_1| = \lim_{r \to 1-} |G_r| = \pi \sum_{n=1}^{\infty} n|b_n|^2.$$

If the area of γ vanishes, then $|\overline{G}_1| = |\underline{G}_1|$. Therefore we have

$$|a_{-1}|^2 = \sum_{n=1}^{\infty} n(|a_n|^2 + |b_n|^2).$$

We assume that there exists a homeomorphism F of the extended complex plane which is analytic off γ, has a non-differentiable point on γ, and fix the infinity. Further assume that the area of $F(\gamma)$ vanishes. Then $\psi_1 = F \circ \varphi_1$ is a conformal mapping from the exterior of the unit disk to the exterior of $F(\gamma)$, whose Laurent development is

$$\psi_1(z) = \sum_{n=-1}^{\infty} A_n z^{-n}$$

and $\psi_2 = F \circ \varphi_2$ is a conformal mapping from the interior of the unit disk to the interior of $F(\gamma)$, whose Taylor development is

$$\psi_2(z) = \sum_{n=0}^{\infty} B_n z^n.$$

For $R > 1$, let Ω_R denote the bounded region enclosed by $\psi_1(|z| = R)$ and for $r < 1$, let Ω_r denote the bounded region enclosed by $\psi_2(|z| = r)$. Then we have

$$|\Omega_R| = \pi \left\{ |A_{-1}|^2 R^2 - \sum_{n=1}^{\infty} n \frac{|A_n|^2}{R^{2n}} \right\},$$

$$|\overline{\Omega}_1| = \lim_{R \to 1+} |\Omega_R| = \pi \left\{ |A_{-1}|^2 - \sum_{n=1}^{\infty} n|A_n|^2 \right\},$$

$$|\Omega_r| = \pi \sum_{n=-1}^{\infty} n|B_n|^2 r^{2n},$$

$$|\underline{\Omega}_1| = \lim_{r \to 1-} |\Omega_r| = \pi \sum_{n=1}^{\infty} n|B_n|^2.$$

Since the area of $F(\gamma)$ vanishes, we have also

$$|A_{-1}|^2 = \sum_{n=1}^{\infty} n(|A_n|^2 + |B_n|^2).$$

Let a parameter $t \in \mathbf{C}$ be fixed. Consider the following function

$$g(\zeta) = F(\zeta) + t\zeta,$$

and for $R > 1$, consider its Dirichlet integral over $G_R - \gamma$ which is now represented as the Dirichlet inner product

$$(dg, dg)_{G_R-\gamma} = (dF, dF)_{G_R-\gamma} + 2\Re\,\bar{t}(dF, d\zeta)_{G_R-\gamma} + |t|^2(d\zeta, d\zeta)_{G_R-\gamma}.$$

As for the last term, we have

$$\begin{aligned}
(d\zeta, d\zeta)_{G_R-\gamma} &= \lim_{R'\to 1+}(d\zeta, d\zeta)_{G_R-G_{R'}} + \lim_{r\to 1-}(d\zeta, d\zeta)_{G_r} \\
&= 2\left\{\lim_{R'\to 1+}(|G_R| - |G_{R'}|) + \lim_{r\to 1-}|G_r|\right\} = 2\{|G_R| - |\overline{G}_1| + |\underline{G}_1|\} \\
&= 2\pi\left\{|a_{-1}|^2(R^2-1) - \sum_{n=1}^{\infty} n|a_n|^2(R^{-2n}-1) + \sum_{n=1}^{\infty} n|b_n|^2\right\} \\
&= 2\pi\left\{|a_{-1}|^2R^2 - \sum_{n=1}^{\infty} n|a_n|^2R^{-2n}\right\}.
\end{aligned}$$

Next

$$\begin{aligned}
(dF, dF)_{G_R-G_{R'}} &= (dF\circ\varphi_1, dF\circ\varphi_1)_{\{z:R'<|z|<R\}} = (d\psi_1, d\psi_1)_{\{z:R'<|z|<R\}} \\
&= 2\pi\left\{|A_{-1}|^2(R^2-R'^2) - \sum_{n=1}^{\infty} n|A_n|^2(R^{-2n}-R'^{-2n})\right\},
\end{aligned}$$

and

$$\begin{aligned}
(dF, dF)_{G_r} &= (dF\circ\varphi_2, dF\circ\varphi_2)_{\{z:|z|<r\}} = (d\psi_2, d\psi_2)_{\{z:|z|<r\}} \\
&= 2\pi\sum_{n=1}^{\infty} n|B_n|^2r^{2n}.
\end{aligned}$$

Therefore

$$\begin{aligned}
(dF, dF)_{G_R-\gamma} &= 2\pi\left\{|A_{-1}|^2(R^2-1) - \sum_{n=1}^{\infty} n|A_n|^2(R^{-2n}-1) + \sum_{n=1}^{\infty} n|B_n|^2\right\} \\
&= 2\pi\left\{|A_{-1}|^2R^2 - \sum_{n=1}^{\infty} n|A_n|^2R^{-2n}\right\}.
\end{aligned}$$

Similarly

$$(dF, d\zeta)_{G_R-G_{R'}} = (dF\circ\varphi_1, d\varphi_1)_{\{z:R'<|z|<R\}} = (d\psi_1, d\varphi_1)_{\{z:R'<|z|<R\}}$$

$$= i\left\{\int_{|z|=R}\psi_1\overline{d\varphi_1} - \int_{|z|=R'}\psi_1\overline{d\varphi_1}\right\}$$

$$= \int_0^{2\pi}\left(\sum_{n=-1}^{\infty}\right)\left(\sum_{m=-1}^{\infty}\right) - mA_n\overline{a_m}(R^{-n-m} - R'^{-n-m})e^{(-n+m)\theta i}d\theta$$

$$= 2\pi\left\{A_{-1}\overline{a_{-1}}(R^2 - R'^2) - \sum_{n=1}^{\infty} nA_n\overline{a_n}(R^{-2n} - R'^{-2n})\right\},$$

and

$$(dF, d\zeta)_{G_r} = (dF\circ\varphi_2, d\varphi_2)_{\{z:|z|<r\}} = (d\psi_2, d\varphi_2)_{\{z:|z|<r\}} = i\int_{|z|=r}\psi_2\overline{d\varphi_2}$$

$$= \int_0^{2\pi}\left(\sum_{n=0}^{\infty}\right)\left(\sum_{m=0}^{\infty}\right)mB_n\overline{b_m}r^{n+m}e^{(n-m)\theta i}d\theta$$

$$= 2\pi\sum_{n=1}^{\infty} nB_n\overline{b_n}r^{2n},$$

so that

$$(dF, d\zeta)_{G_R-\gamma} = 2\pi\left\{A_{-1}\overline{a_{-1}}(R^2 - 1) - \sum_{n=1}^{\infty} nA_n\overline{a_n}(R^{-2n} - 1) + \sum_{n=1}^{\infty} nB_n\overline{b_n}\right\}.$$

By combining the results we obtain

$$\begin{aligned}(dg, dg)_{G_R-\gamma} = 2\pi\Bigg\{&|A_{-1}|^2R^2 - \sum_{n=1}^{\infty} n|A_n|^2R^{-2n}\\ &+ |t|^2\left(|a_{-1}|^2R^2 - \sum_{n=1}^{\infty} n|a_n|^2R^{-2n}\right)\\ &+2\Re\,\bar{t}\left[A_{-1}\overline{a_{-1}}(R^2 - 1) - \sum_{n=1}^{\infty} nA_n\overline{a_n}(R^{-2n} - 1) + \sum_{n=1}^{\infty} nB_n\overline{b_n}\right]\Bigg\}\\ = 2\pi\Bigg\{&-\sum_{n=-1}^{\infty} n|A_n + ta_n|^2R^{-2n} + 2\Re\,\bar{t}\left(\sum_{n=-1}^{\infty} nA_n\overline{a_n} + \sum_{n=1}^{\infty} nB_n\overline{b_n}\right)\Bigg\}.\end{aligned}$$

Set

$$Q = \sum_{n=-1}^{\infty} nA_n\overline{a_n} + \sum_{n=1}^{\infty} nB_n\overline{b_n}.$$

Then we can write

$$(dg, dg)_{G_R-\gamma} = 2\pi\left\{2\Re\,\bar{t}Q - \sum_{n=-1}^{\infty} n|A_n + ta_n|^2R^{-2n}\right\}.$$

We remark the following.

Lemma 1.

$$Q = \sum_{n=-1}^{\infty} nA_n\overline{a_n} + \sum_{n=1}^{\infty} nB_n\overline{b_n} \neq 0.$$

Proof. Since ψ_1 and φ_1 are conformal at infinity, A_{-1} and a_{-1} don't vanish. Let $t_0 = -(A_{-1}/a_{-1})$ and set

$$g_0(\zeta) = F(\zeta) + t_0\zeta.$$

Then

$$\begin{aligned} 0 &< (dg_0, dg_0)_{G_R - \gamma} \\ &= 2\pi\left\{2\Re\,\overline{t_0}Q - \sum_{n=-1}^{\infty} n|A_n + t_0a_n|^2R^{-2n}\right\}. \end{aligned}$$

Hence

$$4\pi\Re\,\overline{t_0}Q = \lim_{R\to\infty}(dg_0, dg_0)_{G_R-\gamma} > 0,$$

and $Q \neq 0$. □

For $t \neq t_0$ and sufficiently large R, $g\circ\varphi_1(\{z : |z| = R\})$ becomes a Jordan curve and the interior region is denoted by $\Omega_{g,R}$. We have

$$\begin{aligned} |\Omega_{g,R}| &= \frac{i}{2}\iint_{\Omega_{g,R}} d\omega d\bar{\omega} = \frac{i}{2}\int_{\partial\Omega_{g,R}} g\circ\varphi_1(z)d\overline{g\circ\varphi_1(z)} \\ &= \frac{1}{2}\int_0^{2\pi}\left(\sum_{n=-1}^{\infty}\right)\left(\sum_{m=-1}^{\infty}\right) - m(A_n + ta_n)\overline{(A_m + ta_m)}R^{-n-m}e^{-(n-m)\theta i}d\theta \\ &= \pi\sum_{n=-1}^{\infty} -n|A_n + ta_n|^2R^{-2n}. \end{aligned}$$

Therefore we have the following.

Lemma 2.

$$|\Omega_{g,R}| - \frac{1}{2}(dg, dg)_{G_R-\gamma} = -2\pi\Re\,\bar{t}Q.$$

Let

$$S(F) = \left\{ \frac{F(\zeta_1) - F(\zeta_2)}{\zeta_1 - \zeta_2} : (\zeta_1, \zeta_2) \in \mathbf{C} \times \mathbf{C} - \{(\zeta, \zeta)\}_{\zeta \in \mathbf{C}} \right\}.$$

Then we have the following assertion.

Theorem 1. *The set S(F) contains a half plane;*

$$\{t : \Re \bar{t}Q < 0\} \subset S(F).$$

Proof. Note that $(1/2)(dg, dg)_{G_R - \gamma}$ is the image area of $G_R - \gamma$ by g, counting multiplicity. Hence if $2\pi \Re \bar{t}Q > 0$, by Lemma 2, there is an image point mapped from at least two points ζ_1, ζ_2. When $t = t_0$, by Wermer's lemma $g(\gamma) = g(\hat{\mathbf{C}})$ (cf. [1]), we can also find two points ζ_1, ζ_2 with the same property. Then

$$F(\zeta_1) + t\zeta_1 = F(\zeta_2) + t\zeta_2,$$

or

$$-t = \frac{F(\zeta_1) - F(\zeta_2)}{\zeta_1 - \zeta_2} \in S(F).$$

If $-t$ does not belong to $S(F)$, $F(\zeta_1) + t\zeta_1 \neq F(\zeta_2) + t\zeta_2$ for every pair (ζ_1, ζ_2), $\zeta_1 \neq \zeta_2$. Then $g(\zeta) = F(\zeta) + t\zeta$ is univalent. When $-t \in \mathbf{C} - S(F)$ and $\Re \bar{t}Q < 0$, g is univalent and the image area of γ by g is positive. Then $g(\gamma)$ contains a compact set whose complement belongs to the class $N_{SB} - N_D$ (cf. [7]). If the closure of $S(F)$ doesn't contain $-t$, then $\Re \bar{t}Q < 0$. Hence g is univalent and the image area $g(\gamma)$ is positive. □

Theorem 2. *If* $(\mathbf{C} - S(F)) \bigcap \{t; \Re \bar{t}Q > 0\} \neq \emptyset$, *there exists a homeomorphism* g *on* $\mathbf{C}$ *such that* g *is conformal on* $\mathbf{C} - \gamma$ *and area of* $g(\gamma)$ *is positive.*

3. Linear combination of the extremal horizontal and vertical slit mappings

Let G be a region in the extended complex plane which allows non-constant analytic function with a finite Dirichlet integral. Assume that infinity is contained in G.

Theorem 3. *Let* f_h, f_v *be the extremal horizontal slit mapping and the extremal vertical slit mapping on G. Assume that* f_h, f_v *are normalized such that*

$$f_h = \zeta + \sum_{n=1}^{\infty} a_n \zeta^{-n}$$

$$f_v = \zeta + \sum_{n=1}^{\infty} b_n \zeta^{-n}$$

at a neighborhood of infinity. Then $f_v - tf_h$ is univalent on G if $\Re t \leq 0$.

Proof. Assume that G is a multiply connected regular region. Then $w = f(z) = f_v \circ f_h^{-1}(z)$ is a conformal mapping from a multiply connected horizontal slit region H to a vertical slit region V. The function $f(z)$ has the following Laurent developement,

$$f = z + \sum_{n=1}^{\infty} A_n z^{-n}$$

on a neighborhood of ∞. The function $f(z)$ has an analytic extension to every component of the boundary except for 4 points and $\Re f'(z) = 0$ on $\partial H - \{$a finite number of points$\}$. The function $f'(z)$ can be regarded as a meromorphic function on the doubled surface. The doubled surface is compact and the total order of f' is twice of the number of slits. Hence the inverse image of the imaginary axis by f' consists of the boundary slits of H. Therefore the real part of f' doesn't vanish on any interior points of H. Further by the normalization of $f; f'(\infty) = 1$, it follows that the real part of f' is positive on H. Let A be the left end point of a boundary horizontal slit of H, C its right end point, B the point which is mapped on the top end point of a boundary vertical slit of V, and D the point which is mapped on the bottom end point of a boundary vertical slit of V. Then note that

$$f'(B) = f'(D) = 0, \ f'(A) = f'(C) = \infty.$$

It follows that B lies on the upper side of the horizontal slit and D lies on the under side of the horizontal slit. Further, we have

$$\Im f'(z) > 0, \quad \text{on} \quad (A, B) \cup (D, C),$$
$$\Im f'(z) < 0, \quad \text{on} \quad (B, C) \cup (A, D).$$

By the monotonous change of $\Im f'(z)$,

$$\Im f''(z) \leq 0, \quad \text{on} \quad (A, B) \cup (B, C),$$
$$\Im f''(z) \geq 0, \quad \text{on} \quad (D, C) \cup (A, D),$$

where (A, B), (B, C) lie on the upper side of the horizontal slit and (D, C), (A, D) lie on the under side of the horizontal slit. Let $g(z) = f(z) + z = u + iv$. The curvature of the boundary curve $g(\partial H)$ is

$$\frac{\dot{u}\ddot{v} - \dot{v}\ddot{u}}{(\dot{u}^2 + \dot{v}^2)^{\frac{3}{2}}} = \Im \frac{g''\bar{g'}}{|g'|^3} = \Im \frac{f''\bar{f'} + f''}{|f' + 1|^3}$$
$$= \Im \frac{f''}{|f' + 1|^3}$$

$$\begin{cases} \leq 0 & \text{on } (A,B) \cup (B,C) \\ \geq 0 & \text{on } (A,D) \cup (D,C). \end{cases}$$

Hence $g(\partial H)$ surrounds a convex region, if it is univalent. Similarly for $h(z) = f(z) - z$ the curvature of the boundary curve $h(\partial H)$

$$\Im \frac{h'' \bar{h}'}{|h'|^3} = \Im \frac{-f''}{|f' - 1|^3}.$$

For z_1, $(\neq) z_2 \in H$, let $\rho = |z_2 - z_1|$, $e^{i\theta} = (z_2 - z_1)/(|z_2 - z_1|)$. Assume that $0 < \theta < \pi$. Then

$$f(z_2) - f(z_1) = \int_0^\rho f'(z_1 + re^{i\theta}) e^{i\theta} dr + \sum \{f(z_i^+) - f(z_i^-)\},$$

where z_i^+ denotes the point on the upper side of a boundary horizontal slit of H which meets the line segment $[z_1, z_2]$ and z_i^- denotes the point on its under side. We have

$$\Re \frac{f(z_2) - f(z_1)}{z_2 - z_1} = \frac{1}{\rho} \int_0^\rho \Re f'(z_1 + re^{i\theta}) dr + \frac{1}{\rho} \Re \sum \{f(z_i^+) - f(z_i^-)\} e^{-i\theta}.$$

Since $f(z_i^+)$ and $f(z_i^-)$ lie on a vertical slit,

$$f(z_i^+) - f(z_i^-) = i\{\Im f(z_i^+) - \Im f(z_i^-)\}.$$

Suppose that $\{\Im f(z_i^+) - \Im f(z_i^-)\} \leq 0$. The function $\{\Im f(z_i^+) - \Im f(z_i^-)\}$ is continuous on the horizontal slit and it is positive near the end point. There is a point z_0 such that

$$\{\Im f(z_0^+) - \Im f(z_0^-)\} = 0.$$

Then $f(z_0^+) = f(z_0^-)$ and

$$g(z_0^+) = f(z_0^+) + z_0 = f(z_0^-) + z_0 = g(z_0^-).$$

If we use the fact that $g(z)$ is univalent and the complement of the image domain consists of convex sets, this gives a contradiction. It follows that

$$\{\Im f(z_i^+) - \Im f(z_i^-)\} > 0.$$

Now we have

$$\Re \sum \{f(z_i^+) - f(z_i^-)\} e^{-i\theta} = \Re \; e^{(\pi/2 - \theta)i} \sum \Im \{f(z_i^+) - f(z_i^-)\} > 0.$$

Therefore

$$\Re \frac{f(z_2) - f(z_1)}{z_2 - z_1} > 0.$$

It is clear in the case $\theta = 0$ or the second term vanishes. For $t \in \mathbf{C}$ ($\Re t \leq 0$),

$$\frac{f(z_2) - f(z_1)}{z_2 - z_1} \neq t \quad i.e. \quad f(z_2) - tz_2 \neq f(z_1) - tz_1.$$

We have $F(z) = f(z) - tz$ is an univalent function on H if $\Re t \leq 0$. It follows the result holds on a multiply connected regular region.

Let G be an arbitrary region in the complex plane and $\{G_n\}$ be a regular exhaustion. Let f_h and f_v be the extremal horizontal and vertical slit mappings for G respectively, which are normalized as follows

$$f_h = z + \sum_{n=1}^{\infty} b_n z^{-n},$$
$$f_v = z + \sum_{n=1}^{\infty} a_n z^{-n}.$$

Let f_{nh} and f_{nv} be those normalized extremal slit mappings for G_n. Then $\{f_{nh}\}$ converges to f_h and $\{f_{nv}\}$ converges to f_v. Hence $f_{nv} - tf_{nh}$ converges to $f_h - tf_v$. Therefore, by Hurwitz's theorem $f_v - tf_h$ is univalent if $\Re t \leq 0$. The theorem is valid for an arbitrary plane region. □

REMARK. C. FitzGerald commented me that this was not published. But he knew the fact and remarked that it may be valid for the rectilinear slit mapping with arbitrary directions.

Corollary 1. *If G is a multiply connected regular region, then* $\{t; f_v - tf_h \textit{ is not univalent}\}$ *is dense in the right half complex plane* $\{t; \Re t > 0\}$.

Proof. Let $f = f_v \circ f_h^{-1}$. The closure of $S(f)$ contains $f'(z)$ and the right half complex plane. □

REMARK. (1) N. Suita kindly teaches me the following. For a multiply connected (not simply connected) regular region G,

$\{t; f_v - tf_h \textit{ is univalent}\}$ *is precisely the left half complex plane* $\{t; \Re t \leq 0\}$.

It is shown in the proof of Theorem 3 that $\Re f'$ covers the right half complex plane $\{t; \Re t > 0\}$. Therefore $g_t(z) = f(z) - tz$ has a vanishing derivative if $\Re t > 0$ and is not univalent. Further the curvature of the boundary curve $g_t(\partial H)$ is $-\Re \bar{t}\, \Im f''/|f'-t|^3$. If $\Re t < 0$, it is in the same situation as the proof of Theorem 3

$$\begin{cases} \leq 0 & \text{on } (A,B) \cup (B,C) \\ \geq 0 & \text{on } (A,D) \cup (D,C). \end{cases}$$

Hence the complement of the image of g_t consists of a finite number of convex regions. If $\Re t = 0$, the curvature vanishes. Hence the complement of the image of g_t consists of a finite number of line segments. If $\Re t > 0$, it is

$$\begin{cases} \geq 0 & \text{on } (A,B) \cup (B,C) \\ \leq 0 & \text{on } (A,D) \cup (D,C). \end{cases}$$

Hence the every component of boundary curve $g_t(\partial H)$ surrounds a part of the image $g_t(H)$ convexly, which contains a branch point.

(2) M. Shiba also remarks me the following. By the extremal property of the coefficients a_1 and b_1, the linear combination $f_v - tf_h$ is not univalent for any $t \neq 1$ in the right half plane, if $a_1 \neq b_1$.

(3) M. Sakai[6] showed that $f_v - f_h$ is univalent iff G is conformally equivalent to $\{z : |z| > 1\} \bigcup \{\infty\} - E$, where E is a set satisfying $E \bigcap K \in N_D$ for every compact subset K of $\{z : |z| > 1\} \bigcup \{\infty\}$.

By the proof of Theorem 1, we have the following.

Theorem 4. *If $\hat{\mathbf{C}} - G$ is of class $N_{SB} - N_D$, then $\{t; f_v - tf_h \text{ is univalent}\} = \{t; \Re t \leq 0\}$.*

EXAMPLE. The set $E = \hat{\mathbf{C}} - f_h(G)$ has a vanishing area. There is a Jordan curve γ which contains E and has a vanishing area. Under the assumption of Theorem 4, the function $f_v \circ f_h^{-1}$ is a homeomorphism on $\hat{\mathbf{C}}$ and is conformal on $\hat{\mathbf{C}} - \gamma$. The set $S(f_v \circ f_h^{-1})$ is the right half plane.

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References

[1] A. Browder: Introduction to function algebras, Benjamin, New York, 1969.

[2] F. Gardiner: *Schiffer's interior variation and quasiconformal mappings*, Duke Math. J. **42** (1975), 373–380.
[3] K. Nishikawa and F. Maitani: *Moduli of ring domains obtained by a conformal sewing*, Kodai Math. J. **20** (1997), 161–171.
[4] K. Oikawa: *Welding of polygons and the type of Riemann surfaces*, Kodai Math. Sem. Rep. **13** (1961), 35–52.
[5] K. Oikawa: *Module of ring domains obtained by conformal sewing*, lecture manuscript, (1990), 197–207.
[6] M. Sakai: *On basic domains of extremal functions*, Kodai Math. Sem. Rep. **24** (1972), 251–258.
[7] L. Sario and K. Oikawa: Capacity functions, Springer-Verlag, 1969.
[8] M. Touki: *Moduli of tori obtained by a conformal sewing*, preprint.

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