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LENGTH SPECTRUM OF CIRCLES IN A COMPLEX PROJECTIVE SPACE

TOSHIAKI ADACHI and SADAHIRO MAEDA

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1. Introduction

A smooth curve $\gamma : \mathbb{R} \longrightarrow M$ in a complete Riemannian manifold M is called a *circle* of geodesic curvature $\kappa(\geq 0)$ if it is parametrized by its arc-length t and satisfies the following equation:

$$\nabla_t \nabla_t \dot{\gamma}(t) = -\kappa^2 \dot{\gamma}(t),$$

where ∇_t denotes the covariant differentiation along γ with respect to the Riemannian connection ∇ of M. Since γ is paremetrized by its arc-length we have $\|\nabla_t \dot{\gamma}\| = \kappa$, hence this equation is equivalent to the equation of geodesics when $\kappa = 0$. Thus the notion of circles is a natural extension of the notion of geodesics. Although this notion was settled by Nomizu-Yano[11] in 1974, there are very few results about their geometric properties. In this paper we propose to study their length spectrum in connection with some geometry of the base manifold.

We call a circle γ closed if there exists a nonzero constant T with

$$\gamma(T) = \gamma(0), \dot{\gamma}(T) = \dot{\gamma}(0), \nabla_t \dot{\gamma}(T) = \nabla_t \dot{\gamma}(0).$$

This condition is equivalent to the condition that $\gamma(t + T) = \gamma(t)$ for every t. The minimum positive constant with this property is called the prime period of γ and denoted by $length(\gamma)$. For an open circle, a circle which is not closed, we put its prime period as $length(\gamma) = \infty$. In order to get rid of the influence of the action of the full isometry group, we shall consider the moduli space of circles under the action of isometries. We say that two circles γ_1 and γ_2 are congruent each other if there exist an isometry φ and a constant t_0 with $\gamma_2(t) = \varphi \circ \gamma_1(t + t_0)$ for every t. The moduli space $Cir(M)$ of circles is the quotient space of the set of all circles in M under this congruence relation. The *length spectrum* of circles in M is the map $\mathcal{L} : Cir(M) \longrightarrow \mathbb{R} \cup \{\infty\}$ defined by $\mathcal{L}([\gamma]) = length(\gamma)$. Sometimes we also call the image $LSpec(M) = \mathcal{L}(Cir(M)) \cap \mathbb{R}$ in the real line the length spectrum of circles

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on M. For $\lambda \in LSpec(M)$ the cardinality $\sharp\left(\mathcal{L}^{-1}(\lambda)\right)$ of the set $\mathcal{L}^{-1}(\lambda)$ is called the multiplicity of the length spectrum $\mathcal{L}$ at λ. When $\sharp\left(\mathcal{L}^{-1}(\lambda)\right) = 1$ we say λ is simple for $\mathcal{L}$. If the multiplicity of $\mathcal{L}$ is greater than 1 at some point, this means that we can find circles which are not congruent each other but have the same length.

The moduli space of circles have a natural stratification by their geodesic curvature. We denote by $Cir_\kappa(M)$ the moduli space of circles of curvature κ in M and by $\mathcal{L}_\kappa$ the restriction of $\mathcal{L}$ on this space. The length spectrum $\mathcal{L}_0$ is hence the length spectrum of geodesics. For length spectrum of geodesics many results are known. For example, every geodesic in a compact rank one symmetric sapce is closed with the same length. Needless to say these geodesics are congruent each other, so that the length spectrum $\mathcal{L}_0$ takes one value and is simple at there. In a complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature c, the length spectrum of geodesics is $2\pi/\sqrt{c}$. Being concerned with circles, we know that two closed circles in a real space form are congruent if and only if they have the same length. The study of length spectrum of circles may give us more information on geometric properties of a base manifold.

The equation of geodesics descirbes a uniform rectilineal motion of a particle. From this physical point of view, we can interprete some circles as a motion of a charged particle under an action of a magnetic field. In view of classical magnetic fields on $\mathbb{R}^3$, we call a closed 2-form $\mathbf{B}$ on M a magnetic field. A smooth curve γ which satisfies $\nabla_t\dot{\gamma} = \Omega(\dot{\gamma})$ is called a trajectory for $\mathbf{B}$, where Ω denotes the skew-symmetric operator on the tangent bundle TM with $\mathbf{B}(u, v) = \langle u, \Omega(v)\rangle$ for every u, $v \in TM$. When Ω satisfies $\Omega^2 = -I$ and has constant strength $\|\Omega\| \equiv \kappa$, trajectories for $\mathbf{B}$ are circles of curvature κ. One of the important examples of magnetic fields is a Kähler magnetic field, a constant multiple of the Kähler form, on a Kähler manifold. Trajectories for a Kähler magnetic field satisfy the equation $\nabla_t\dot{\gamma} = \pm\kappa J\dot{\gamma}$ with complex structure J, hence they are circles. It is well known that the length spectrum of geodesics is deeply related to the properties of the geodesic flow. For given a magnetic field $\mathbf{B}$ on M we can also consider the magnetic flow associated to $\mathbf{B}$ on the unit tangent bundle of M. By theory of Anosov flows we get an asymptotic behaviour of the distribution of length spectrum under a condition of hyperbolicity for the magnetic flow (see [7], [12]).

Along this context, one of the most important objects in the study of circles should be the set of circles in a Kähler manifold. In a Kähler manifold we have another index for circles which is called the *complex torsion*. The complex torsion for a circle γ is given by $\tau = \langle\dot{\gamma}, J\nabla_t\dot{\gamma}\rangle/\|\nabla_t\dot{\gamma}\|$, which does not depend on t and satisfies $|\tau| \leq 1$. This suggests us that properties of circles in a Kähler manifold are related to the complex structure. Circles of complex torsion ± 1, which are trajectories of some Kähler magnetic field, are called *holomorphic circles*, and circles of null complex torsion are called *totally real circles*. By using this index we get other stratification of the moduli space of circles. We denote by $Cir^\tau(M)$ the moduli space of circles with complex torsion τ, by $\mathcal{L}^\tau$ the restricition of $\mathcal{L}$ onto this space, and by $Cir_\kappa^\tau(M)$ the moduli space of those with curvature κ and complex torsion τ. In this paper we investigate the length spectrum of circles in a complex projective space $\mathbb{C}P^n$ of complex dimension $n \geq 2$, which is a model space in the study of circles. As a direct consequence of Maeda-Ohnita[10], for a complex projective space the moduli space $Cir_\kappa^\tau(\mathbb{C}P^n)$ consisits of a single point, hence we study the structure of $\mathcal{L}_\kappa$, $\mathcal{L}^\tau$, and $\mathcal{L}$ of a complex projective space. Our main result is the following:

Theorem. *For a complex projective space $\mathbb{C}P^n(c)$ $(n \geq 2)$ of constant holomorphic sectional curvature c, the length spectrum has the following properties.*

1) *Both the sets*

$$LSpec_\kappa(\mathbb{C}P^n(c)) = \mathcal{L}(Cir_\kappa(\mathbb{C}P^n(c))) \cap \mathbb{R}$$

and

$$LSpec^\tau(\mathbb{C}P^n(c)) = \mathcal{L}(Cir^\tau(\mathbb{C}P^n(c))) \cap \mathbb{R}$$

are unbounded discrete subsets of $\mathbb{R}$ for $\kappa > 0$ and $0 < \tau < 1$.

2) *The length spectrum $LSpec(\mathbb{C}P^n(c))$ of circles coinsides with the real positive line $(0, \infty)$.*
3) *For $\kappa > 0$ the bottom of $LSpec_\kappa(\mathbb{C}P^n(c))$ is $2\pi/\sqrt{\kappa^2+c}$, which is the length of the holomorphic circle of curvature κ. The second lowest spectrum of $LSpec_\kappa(\mathbb{C}P^n(c))$ is $4\pi/\sqrt{4\kappa^2+c}$, which is the length of the totally real circle of curvature κ. They are simple for $\mathcal{L}_\kappa$.*
4) *The multiplicity of $\mathcal{L}$ is finite at each point $\lambda \in \mathbb{R}$.*
5) *$\lambda(\in \mathbb{R})$ is simple for $\mathcal{L}$ if and only if $\lambda \in ((2/\sqrt{c})\pi, (4/3)\sqrt{5/c}\pi]$.*
6) *The multiplicity of $\mathcal{L}_\kappa$ $(\kappa > 0)$ is not uniformly bounded :*

$$\limsup_{\lambda\to\infty} \sharp(\mathcal{L}_\kappa^{-1}(\lambda)) = \infty.$$

By our result, we can conclude that even if we restrict ourselves on the moduli space $Cir_\kappa(\mathbb{C}P^n(c))$ of circles of curvature κ we can not distinguish congruence classes by their length spectrum.

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2. The length spectrum of circles of curvature κ

In this section we study the length spectrum of circles of curvature κ in a complex projective space $\mathbb{C}P^n(c)$ $(n \geq 2)$ of constant holomorphic sectional curvature c. We first review our preceding results on prime periods of circles in a complex projective space.

Fact 1 ([3]). 1) Every holomorphic circle and totally real circle in $\mathbb{C}P^n(c)$ are closed. Their prime periods are $2\pi/\sqrt{\kappa^2+c}$ and $4\pi/\sqrt{4\kappa^2+c}$ respectively.

2) Let $a_{\kappa,\tau}$, $b_{\kappa,\tau}$, $d_{\kappa,\tau}(a_{\kappa,\tau} < b_{\kappa,\tau} < d_{\kappa,\tau})$ denote the solutions for the cubic equation

$$c\lambda^3 - (4\kappa^2 + c)\lambda + 2\sqrt{c}\kappa\tau = 0.$$

A circle γ of curvature κ and complex torsion τ ($\neq \pm 1$) in $\mathbb{C}P^n(c)$ is closed if and only if one of the ratios $a_{\kappa,\tau}/b_{\kappa,\tau}$, $b_{\kappa,\tau}/d_{\kappa,\tau}$, $d_{\kappa,\tau}/a_{\kappa,\tau}$ is rational. The condition that one of these three ratios is rational is equivalent to the condition that each of these ratios is rational. In this case the prime period of this closed circle is

$$length(\gamma) = \frac{4\pi}{\sqrt{c}} \cdot \text{L.C.M.}\left(\frac{1}{b_{\kappa,\tau} - a_{\kappa,\tau}}, \frac{1}{d_{\kappa,\tau} - a_{\kappa,\tau}}\right).$$

Here for two real numbers α, β the least common multiple L.C.M.(α, β) is the minimum value of the set $\{\alpha m \mid m = 1, 2, 3, \cdots\} \cap \{\beta m \mid m = 1, 2, 3, \cdots\}$.

For the special case that $\kappa = \sqrt{2c}/4$ we can also calculate prime periods of circles in the following manner.

Fact 2 ([3]). In a complex projective space $\mathbb{C}P^n(c)$ a circle γ of curvature $\sqrt{2c}/4$ and complex torsion $\tau = 3\alpha - 4\alpha^3(0 < |\alpha| < 1/2)$ is closed if and only if $\sqrt{(1-\alpha^2)/(3\alpha^2)}$ is rational. In this case if we denote $\sqrt{(1-\alpha^2)/(3\alpha^2)} = p/q$ by mutually prime positive integers p, q, then its prime period is

$$length(\gamma) = \begin{cases} \dfrac{4}{3\sqrt{c}}\pi\sqrt{2(3p^2+q^2)}, & \text{if } pq \text{ is even,} \\ \dfrac{2}{3\sqrt{c}}\pi\sqrt{2(3p^2+q^2)}, & \text{if } pq \text{ is odd.} \end{cases}$$

In a complex projective space two circles are congruent if and only if they have the same curvatures and the same absolute values of complex torsions (see [3]). Therefore the moduli space $Cir(\mathbb{C}P^n(c))$ of circles on $\mathbb{C}P^n(c)$ is bijective to the set $[0, \infty) \times [0, 1]/\sim$, where (κ, τ) and (κ', τ') are equivalent if and only if $(\kappa, \tau) = (\kappa', \tau')$ or $\kappa = \kappa' = 0$. We denote by $[\gamma_{\kappa,\tau}]$ the congruency class of circles of curvature κ and complex torsion $\tau(\geq 0)$ in $\mathbb{C}P^n(c)$. For a positive constant κ we define a canonical transformation

$$\Phi_\kappa : Cir_\kappa(\mathbb{C}P^n(c)) \setminus \{[\gamma_{\kappa,1}]\} \longrightarrow Cir_{\sqrt{2c}/4}(\mathbb{C}P^n(c)) \setminus \{[\gamma_{\sqrt{2c}/4,1}]\}$$

by

$$\Phi_\kappa([\gamma_{\kappa,\tau}]) = [\gamma_{\sqrt{2c}/4, 3\sqrt{3}c\kappa\tau(4\kappa^2+c)^{-3/2}}].$$

Lemma. *The canonical transformation satisfies*

$$\mathcal{L}([\gamma_{\kappa,\tau}]) = C_\kappa \cdot \mathcal{L} \circ \Phi_\kappa([\gamma_{\kappa,\tau}])$$

for every $\tau (0 \le \tau < 1)$*, where* $C_\kappa = \sqrt{3c/(2(4\kappa^2 + c))}$.

Proof. Consider the cubic equation $c\lambda^3 - (4\kappa^2 + c)\lambda + 2\sqrt{c}\kappa\tau = 0$. By putting $\Lambda = C_\kappa \lambda$ we find it is equivalent to $c\Lambda^3 - (3c/2)\Lambda + 2\sqrt{c}\kappa\tau C_\kappa^3 = 0$. Since $0 \le 2\sqrt{2/c}\kappa\tau C_\kappa^3 < 1$, this means that

$$a_{\sqrt{2c}/4,\mu} = C_\kappa a_{\kappa,\tau},\ b_{\sqrt{2c}/4,\mu} = C_\kappa b_{\kappa,\tau},\ d_{\sqrt{2c}/4,\mu} = C_\kappa d_{\kappa,\tau},$$

with $\mu = 2\sqrt{2/c}\kappa\tau C_\kappa^3 = 3\sqrt{3}c\kappa\tau(4\kappa^2 + c)^{-3/2}$. We hence get the conclusion with Fact 1. $\square$

This lemma gurantees that the structure of the length spectrum $\mathcal{L}_\kappa$ of circles of curvature κ essentially does not depend on κ. We shall study about $\mathcal{L}_{\sqrt{2c}/4}$. Here we rewrite the assertion of Fact 2. When α runs on the open interval $(0, 1/2)$, we find that $\tau = 3\alpha - 4\alpha^3$ is monotone increasing and runs on the open interval $(0, 1)$, and that $\sqrt{(1-\alpha^2)/(3\alpha^2)}$ is monotone decreasing and runs on the open interval $(1, \infty)$. Therefore a circle γ of curvature $\sqrt{2c}/4$ and complex torsion τ in $\mathbb{C}P^n(c)$ is closed if and only if

$$\tau = \tau(p, q) = q(9p^2 - q^2)(3p^2 + q^2)^{-3/2},$$

for some mutually prime positive integers p, q with $p > q$. Here we should note that $\tau(p, q) \neq \tau(p', q')$ if $(p, q) \neq (p', q')$ because of the uniquness of the representaion of a positive rational number $\sqrt{(1-\alpha^2)/(3\alpha^2)}$ by mutually prime numbers. Thus we have

$$\begin{aligned} LSpec_{\sqrt{2c}/4}(\mathbb{C}P^n(c)) = &\left\{ \frac{4}{3}\sqrt{\frac{2}{c}}\pi, \frac{4}{3}\sqrt{\frac{6}{c}}\pi \right\} \\ &\bigcup \left\{ \frac{4}{3\sqrt{c}}\pi\sqrt{2(3p^2 + q^2)} \mid p > q, pq \text{ is even}, \right. \\ &\qquad\qquad \left. p \text{ and } q \text{ are mutually prime} \right\} \\ &\bigcup \left\{ \frac{2}{3\sqrt{c}}\pi\sqrt{2(3p^2 + q^2)} \mid p > q, pq \text{ is odd}, \right. \\ &\qquad\qquad \left. p \text{ and } q \text{ are mutually prime} \right\} \end{aligned}$$

As a direct consequence we get the following:

(1) The set $LSpec_{\sqrt{2c}/4}(\mathbb{C}P^n(c))$ is an unbounded discrete subset of $\mathbb{R}$.
(2) The bottom of $LSpec_{\sqrt{2c}/4}(\mathbb{C}P^n(c))$ is $(4/3)\sqrt{2/c}\,\pi$.
(3) The bottom of the set $LSpec_{\sqrt{2c}/4}(\mathbb{C}P^n(c))\setminus\{(4/3)\sqrt{2/c}\,\pi\}$ is $(4/3)\sqrt{6/c}\,\pi$.
(4) These two spectrum $(4/3)\sqrt{2/c}\,\pi$ and $(4/3)\sqrt{6/c}\,\pi$ for $\mathcal{L}_{\sqrt{2c}/4}$ are simple.

Since the multiplicity of $\mathcal{L}_{\sqrt{2c}/4}$ at $\lambda(> (4/3)\sqrt{6/c}\,\pi)$ is equal to the cardinality of the set

$$\Big\{(p,q)\in\mathbb{N}\times\mathbb{N} \mid p>q, pq \text{ is even},$$
$$p \text{ and } q \text{ are mutually prime}, \frac{4}{3\sqrt{c}}\pi\sqrt{2(3p^2+q^2)}=\lambda\Big\}$$
$$\bigcup\Big\{(p,q)\in\mathbb{N}\times\mathbb{N} \mid p>q, pq \text{ is odd},$$
$$p \text{ and } q \text{ are mutually prime}, \frac{2}{3\sqrt{c}}\pi\sqrt{2(3p^2+q^2)}=\lambda\Big\},$$

we also find by classical number theory the following (see for example [8]):
(5) The multiplicity of $\mathcal{L}_{\sqrt{2c}/4}$ is finite at each $\lambda\in\mathbb{R}$ but not uniformly bounded;

$$\limsup_{\lambda\to\infty}\sharp(\mathcal{L}^{-1}_{\sqrt{2c}/4}(\lambda))=\infty.$$

EXAMPLES. We find that the length spectrum $\mathcal{L}_{\sqrt{2c}/4}$ is not simple at the following points:

(i) Let γ_1 be a circle of curvature $\sqrt{2c}/4$ and complex torsion $\tau=\tau(27,7)=5698/(559\sqrt{559})$ and γ_2 be a circle of curvature $\sqrt{2c}/4$ and complex torsion $\tau=\tau(25,19)=12502/(559\sqrt{559})$. These two closed circles have the same curvature and the same length $(4\sqrt{1118})/(3\sqrt{c})\pi$. But they are not congruent each other.
(ii) Let γ_i be a circle of the same curvature $\sqrt{2c}/4$ and complex torsion $\tau_i=\tau(p_i,q_i)$, $i=1,2,3$. We set $(p_1,q_1)=(129,71)$, $(p_2,q_2)=(131,59)$ and $(p_3,q_3)=(135,17)$. Note that $3p_i^2+q_i^2=54964$ for $i=1,2,3$. Then these three circles have the same curvature and the same length. But these three circles are not congruent each other.

For general κ, since the canonical transformation Φ_κ is injective and its image is

$$\left\{[\gamma_{\sqrt{2c}/4,\mu}] \mid 0\le\mu<\frac{3\sqrt{3}c\kappa}{(4\kappa^2+c)^{3/2}}\right\}$$

our Lemma yields that

$$LSpec_\kappa(\mathbb{C}P^n(c)) = \left\{ \frac{2\pi}{\sqrt{\kappa^2+c}}, \frac{4\pi}{\sqrt{4\kappa^2+c}} \right\}$$
$$\bigcup \left\{ \sqrt{\frac{3c}{2(4\kappa^2+c)}} \times \frac{4}{3\sqrt{c}}\pi\sqrt{2(3p^2+q^2)} \mid p > \delta q, pq \text{ iseven} \right.$$
$$\left. p \text{ and } q \text{ are mutually prime} \right\}$$
$$\bigcup \left\{ \sqrt{\frac{3c}{2(4\kappa^2+c)}} \times \frac{2}{3\sqrt{c}}\pi\sqrt{2(3p^2+q^2)} \mid p > \delta q, pq \text{ is odd} \right.$$
$$\left. p \text{ and } q \text{ are mutually prime} \right\},$$

where δ (> 1) denotes the number with

$$\frac{3\sqrt{3}c\kappa}{(4\kappa^2+c)^{3/2}} = \frac{9\delta^2-1}{(3\delta^2+1)^{3/2}}.$$

We can therefore conclude the following on the length spectrum of circles of curvature κ in a complex projective space $\mathbb{C}P^n(c)$ $(n \geq 2)$:

Proposition 1. 1) *The set $LSpec_\kappa(\mathbb{C}P^n(c))$ is an unbounded discrete subset of $\mathbb{R}$.*

2) *The bottom of $LSpec_\kappa(\mathbb{C}P^n(c))$ is $2\pi/\sqrt{\kappa^2+c}$, and the second lowest spectrum in $LSpec_\kappa(\mathbb{C}P^n(c))$ is $4\pi/\sqrt{4\kappa^2+c}$ when $\kappa > 0$.*

3) *These two spectrum $2\pi/\sqrt{\kappa^2+c}$ and $4\pi/\sqrt{4\kappa^2+c}$ are simple for $\mathcal{L}_\kappa$.*

4) *The multiplicity of $\mathcal{L}_\kappa$ is finite at each point $\lambda \in \mathbb{R}$, but not uniformly bounded if $\kappa > 0$; $\limsup_{\lambda\to\infty} \sharp(\mathcal{L}_\kappa^{-1}(\lambda)) = \infty$.*

3. The length spectrum of circles of complex torion τ

In this section we study the length spectrum $LSpec^\tau(\mathbb{C}P^n(c))$ of circles of complex torsion τ in a complex projective space $\mathbb{C}P^n(c)$ $(n \geq 2)$. It is clear that $LSpec^1(\mathbb{C}P^n(c)) = (0, 2\pi/\sqrt{c})$ and $LSpec^0(\mathbb{C}P^n(c)) = (0, 4\pi/\sqrt{c})$.

We shall show the following:

Proposition 2. *The set $LSpec^\tau(\mathbb{C}P^n(c))$ $(n \geq 2)$ is unbounded for $0 < \tau < 1$.*

Proof. Set $\tau_m = \tau(2m+1, 1) = (1/2)(3m+1)(3m+2)(3m^2+3m+1)^{-3/2}$ for a positive integer m. In the following for fixed τ, we choose m satisfying $\tau_m < \tau$. The equation $\tau_m(4\kappa^2+c)^{3/2} = 3\sqrt{3}c\tau\kappa$ has two positive solutions. We denote by κ_m the larger solution of this equation. One can easily see that $\kappa_m^2 > (c/4)(3\tau/(2\tau_m)-1)$, hence that $\lim_{m\to\infty}\kappa_m = \infty$. We are enough to show that the set $\{\mathcal{L}([\gamma_{\kappa_m,\tau}])\}$ is unbounded. By Lemma and Fact 2 we have

$$\begin{aligned}\mathcal{L}([\gamma_{\kappa_m,\tau}]) &= \sqrt{\frac{3c}{2(4\kappa_m^2+c)}}\times\mathcal{L}([\gamma_{\sqrt{2c}/4,\tau_m}])\\ &= \sqrt{\frac{3c}{2(4\kappa_m^2+c)}}\times\frac{2}{3\sqrt{c}}\pi\sqrt{2\{3(2m+1)^2+1\}}\\ &= 4\pi\sqrt{\frac{3m^2+3m+1}{3(4\kappa_m^2+c)}}.\end{aligned}$$

As we have

$$\tau_m = \frac{(3m+1)(3m+2)}{2(3m^2+3m+1)^{3/2}} = \frac{3\sqrt{3}c\kappa_m\tau}{(4\kappa_m^2+c)^{3/2}},$$

we see that

$$\lim_{m\to\infty}\frac{\kappa_m^2}{m} = \lim_{m\to\infty}\frac{\tau_m\kappa_m^2}{m\tau_m} = \frac{3}{4}c\tau \ (<\infty).$$

Therefore we obtain that $\lim_{m\to\infty}\mathcal{L}([\gamma_{\kappa_m,\tau}]) = \infty$, and get the conclusion. □

Next we discuss the discreteness of the length spectrum of circles of complex torsion τ.

Proposition 3. *For $0<\tau<1$ the set $LSpec^\tau(\mathbb{C}P^n(c))$ $(n\geq 2)$ is a discrete subset of the real line. In particular,* $\inf LSpec^\tau(\mathbb{C}P^n(c)) = \min LSpec^\tau(\mathbb{C}P^n(c))$ *is positive.*

Proof. Put $\mu(\kappa) = 3\sqrt{3}c\kappa\tau(4\kappa^2+c)^{-3/2}$. If we suppose $LSpec^\tau(\mathbb{C}P^n(c))$ has an accumulation point $\xi\in\mathbb{R}$ then there exists a sequence $\{\kappa_m\}_{m=1}^\infty$ of positive numbers with

$$\mathcal{L}([\gamma_{\kappa_m,\tau}]) < \infty,\ \xi = \lim_{m\to\infty}\mathcal{L}([\gamma_{\kappa_m,\tau}]).$$

Choose α_m so that $\mu(\kappa_m) = 3\alpha_m - 4\alpha_m^3$ and $0<\alpha_m<1/2$. Since $\sqrt{(1-\alpha_m^2)/(3\alpha_m^2)}$ is rational by Fact 2, we have mutually prime positive integers p_m, q_m $(p_m>q_m)$ with $\sqrt{(1-\alpha_m^2)/(3\alpha_m^2)} = p_m/q_m$.

When the set $\{\kappa_m\}$ is unbounded, by taking a subsequence, we may suppose $\lim_{m\to\infty}\kappa_m = \infty$ so that $\lim_{m\to\infty}\mu(\kappa_m) = 0$. We note that $\lim_{m\to\infty}p_m/q_m = \infty$,

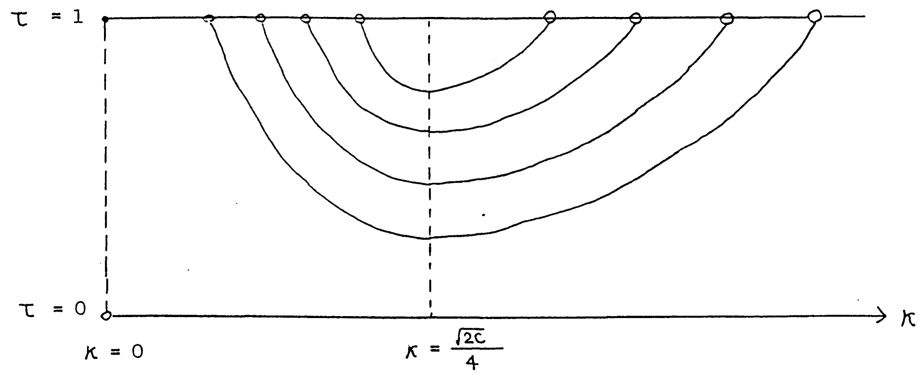

Fig. 3.1.

because $\lim_{m\to\infty}\alpha_m = 0$. In this case as we have

$$\mu(\kappa_m) = \left\{9\left(\frac{p_m}{q_m}\right)^2 - 1\right\}\left\{3\left(\frac{p_m}{q_m}\right)^2 + 1\right\}^{-3/2},$$

we find that

$$\lim_{m\to\infty}\frac{p_m}{\kappa_m^2 q_m} = \lim_{m\to\infty}\frac{\mu(\kappa_m)(p_m/q_m)}{\mu(\kappa_m)\kappa_m^2} = \frac{8}{3c\tau} < \infty.$$

We therefore get

$$\begin{aligned}\mathcal{L}([\gamma_{\kappa_m,\tau}]) &= \sqrt{\frac{3c}{2(4\kappa_m^2+c)}} \times \mathcal{L}([\gamma_{\sqrt{2c}/4,\mu(\kappa_m)}]) \\ &\geq \sqrt{\frac{3c}{2(4\kappa_m^2+c)}} \times \frac{2}{3\sqrt{c}}\pi \times \sqrt{2(3p_m^2+q_m^2)} \longrightarrow \infty.\end{aligned}$$

This contradicts the assumption $\xi < \infty$.

When $\{\kappa_m\}$ is a bounded set, then $\sqrt{3c/(2(4\kappa_m^2+c))}$ is bounded from below by a positive constant. Since $\xi < \infty$, we find that the set $\{\mathcal{L}([\gamma_{\sqrt{2c}/4,\mu(\kappa_m)}])\}$ is also bounded. Therefore the set $\{\kappa_m\}$ is a finite set, because the set

$$\{(p,q) \mid p \text{ and } q \text{ are mutually prime positive numbers, } p > q, \sqrt{3p^2+q^2} \leq L\}$$

is a finite set for each L. This is a contradiction. Thus we obtain that $LSpec^\tau(\mathbb{C}P^n(c))$ is a discrete set. $\square$

Our proof of Proposition 3 stands to show that the multiplicity of the length spectrum $\mathcal{L}^\tau$ $(0 < \tau < 1)$ of $\mathbb{C}P^n(c)$ is finite at each point $\lambda \in \mathbb{R}$. But we do not know whether it is uniformly bounded or not.

Before closing this section we explain our results from a geometric point of view. On the moduli space $Cir(\mathbb{C}P^n(c))$ we have a natural foliation $\{\mathcal{F}_\mu\}_{0\leq\mu\leq1}$ with respect to the length spectrum of circles. Each leaf is of the following form:

$$\begin{aligned}
\mathcal{F}_0 &= \{[\gamma_{\kappa,0}] \mid \kappa > 0\},\\
\mathcal{F}_\mu &= \{[\gamma_{\kappa,\tau}] \mid 3\sqrt{3}c\kappa\tau(4\kappa^2+c)^{-3/2} = \mu\}, \text{ if } 0<\mu<1,\\
\mathcal{F}_1 &= \{[\gamma_{\kappa,1}] \mid \kappa \geq 0\},
\end{aligned}$$

This foliation is transverse to the canonical foliation $\{Cir_\kappa(\mathbb{C}P^n(c))\}_{0\leq\kappa<\infty}$.

Also each leaf $\mathcal{F}_\mu$ $(0 < \mu < 1)$ is transverse to the canonical foliation $\{Cir^\tau(\mathbb{C}P^n(c))\}_{0<\tau<1}$ except at the point $[\gamma_{\sqrt{2c}/4,\mu}]$.

4. The full length spectrum

We devote this secton to study the full length spectrum $LSpec(\mathbb{C}P^n(c))$ of circles in a complex projective space $\mathbb{C}P^n(c)$ $(n \geq 2)$.

For given $\mu(0 < \mu < 1)$ we denote by k_μ and K_μ $(k_\mu < K_\mu)$ the positive solutions for the equation $(4\kappa^2+c)^{3/2}\mu = 3\sqrt{3}c\kappa$. When $k_\mu < \kappa < K_\mu$ we see that $\tau(\kappa) = (4\kappa^2+c)^{3/2}\mu/(3\sqrt{3}c\kappa)$ satisfies $0 < \tau(\kappa) < 1$, hence $[\gamma_{\kappa,\tau(\kappa)}] \in Cir(\mathbb{C}P^n(c))$. Since $\Phi_\kappa([\gamma_{\kappa,\tau(\kappa)}]) = [\gamma_{\sqrt{2c}/4,\mu}]$, if $\mathcal{L}([\gamma_{\sqrt{2c}/4,\mu}]) < \infty$ then the open inteval

$$\begin{aligned}
I_\mu &= \left(\sqrt{\frac{3c}{2(4K_\mu^2+c)}} \times \mathcal{L}([\gamma_{\sqrt{2c}/4,\mu}]), \sqrt{\frac{3c}{2(4k_\mu^2+c)}} \times \mathcal{L}([\gamma_{\sqrt{2c}/4,\mu}])\right)\\
&= \{\mathcal{L}([\gamma_{\kappa,\tau(\kappa)}]) \mid k_\mu < \kappa < K_\mu\}
\end{aligned}$$

is contained in $LSpec(\mathbb{C}P^n(c))$. When $\mu = \tau(p,q)$, we can solve the equation and get $k_\mu^2 = cq^2/(9p^2-q^2)$ and $K_\mu^2 = c(3p-q)^2/\{8q(3p+q)\}$. Since $LSpec^0(\mathbb{C}P^n(c)) = (0, 4\pi/\sqrt{c})$ and $LSpec^1(\mathbb{C}P^n(c)) = (0, 2\pi/\sqrt{c})$, we find that

$$LSpec(\mathbb{C}P^n(c)) = \left(0, \frac{4\pi}{\sqrt{c}}\right) \cup \bigcup\{I_{\tau(p,q)} \mid p > q, p \text{ and } q \text{ are mutually prime positive integers}\}.$$

Here

$$I_{\tau(p,q)} = \begin{cases} \left(\dfrac{4\pi}{3\sqrt{c}}\sqrt{2q(3p+q)}, \dfrac{4\pi}{3\sqrt{c}}\sqrt{9p^2-q^2}\right), & \text{if } pq \text{ is even},\\[2ex] \left(\dfrac{2\pi}{3\sqrt{c}}\sqrt{2q(3p+q)}, \dfrac{2\pi}{3\sqrt{c}}\sqrt{9p^2-q^2}\right), & \text{if } pq \text{ is odd}.\end{cases}$$

For example we have

$$I_{\tau(2,1)} = \left(\frac{4}{3}\sqrt{\frac{14}{c}}\pi, \frac{4}{3}\sqrt{\frac{35}{c}}\pi\right), I_{\tau(3,1)} = \left(\frac{4}{3}\sqrt{\frac{5}{c}}\pi, \frac{8}{3}\sqrt{\frac{5}{c}}\pi\right),$$

$$I_{\tau(5,1)} = \left(\frac{8}{3}\sqrt{\frac{2}{c}}\pi, \frac{8}{3}\sqrt{\frac{14}{c}}\pi\right), I_{\tau(7,1)} = \left(\frac{4}{3}\sqrt{\frac{11}{c}}\pi, \frac{4}{3}\sqrt{\frac{110}{c}}\pi\right).$$

Since we can easily find $I_{\tau(3,1)} \cap (0, 4\pi/\sqrt{c}) \neq \emptyset$ and $I_{\tau(2m-1,1)} \cap I_{\tau(2m+1,1)} \neq \emptyset$ for $m \geq 1$, we obtain the following set theoretical property on the full length spectrum.

Proposition 4. *The full length spectrum $LSpec(\mathbb{C}P^n(c))$ $(n \geq 2)$ coincides with the positive real half line $(0, \infty)$.*

Next we shall mention the multiplicity of the full length spectrum $\mathcal{L}$ of $\mathbb{C}P^n(c)$.

Proposition 5. *The multiplicity of length spectrum $\mathcal{L}$ of $\mathbb{C}P^n(c)$ is finite at each point $\lambda \in \mathbb{R}$.*

Proof. If $\lambda \in I_{\tau(p,q)}$ we have that $\lambda > 2\pi/(3\sqrt{c})\sqrt{2q(3p+q)}$. As $p > q \geq 1$, we have $6p < 2q(3p+q) < (9/(4\pi^2))c\lambda^2$. Since the number of pairs (p,q) of mutually prime positive numbers with $q < p < 3/(8\pi^2)c\lambda^2$ is finite, we get that the number of pairs (p,q) with $\lambda \in I_{\tau(p,q)}$ and with $p > q$ is finite for each $\lambda \in (0,\infty)$. We hence obtain the conclusion. □

Finally we shall show that there are simple spectrum for $\mathcal{L}$. Since the length spectrum of geodesics is $LSpec_0(\mathbb{C}P^n(c)) = \{2\pi/\sqrt{c}\}$ and the length spectrum of holomorphic circles and totally real circles are $LSpec^1(\mathbb{C}P^n(c)) = (0, 2\pi/\sqrt{c})$ and $LSpec^0(\mathbb{C}P^n(c)) = (0, 4\pi/\sqrt{c})$, it is trivial that $\mathcal{L}$ is not simple on $(0, 2\pi/\sqrt{c}]$. We now study the strict extent of $I_{\tau(p,q)}$. It is clear that $I_{\tau(2,1)} \subset ((4/3)\sqrt{5/c}\ \pi, \infty)$. When $p \geq 3$, as $2q(3p+q) \geq 2(3\cdot 3+1)$, we also have $I_{\tau(p,q)} \subset ((4/3)\sqrt{5/c}\ \pi, \infty)$. Therefore $\mathcal{L}$ is simple at $\lambda \in (2\pi/\sqrt{c}, (4/3)\sqrt{5/c}\ \pi]$. On the other hand by the proof of Proposition 4, we have $\bigcup_{m=1}^{\infty} I_{\tau(2m+1,1)} = ((4/3)\sqrt{5/c}\ \pi, \infty)$. Similarly we find $\bigcup_{m=1}^{\infty} I_{\tau(2m,1)} = ((4/3)\sqrt{14/c}\ \pi, \infty)$. Hence we get that the multiplicity of $\mathcal{L}$ at $\lambda \in ((4/3)\sqrt{14/c}\ \pi, \infty)$ is at least 2. Since

$$\left(\frac{4}{3}\sqrt{\frac{5}{c}}\pi, \frac{4}{\sqrt{c}}\pi\right] = LSpec^0(\mathbb{C}P^n(c)) \cap I_{\tau(3,1)},$$

$$\left(\frac{4}{\sqrt{c}}\pi, \frac{8}{3}\sqrt{\frac{5}{c}}\pi\right) \subset I_{\tau(3,1)} \cap I_{\tau(5,1)},$$

$$\left[\frac{8}{3}\sqrt{\frac{5}{c}}\pi, \frac{4}{3}\sqrt{\frac{14}{c}}\pi\right] \subset I_{\tau(5,1)} \cap I_{\tau(7,1)},$$

we obtain the following:

Proposition 6. *In a complex projective space $\mathbb{C}P^n(c)$ $(n \geq 2)$ the length spec-*

trum $\mathcal{L}$ at λ is simple if and only if $\lambda \in ((2/\sqrt{c})\pi, (4/3)\sqrt{5/c}\ \pi]$.

In the last stage we shall mention briefly on the length spectrum of general Kähler manifold M. When M admits $\mathbb{C}P^1(c_1) \times \mathbb{C}P^1(c_2)$ as a totally geodesic complex submanifold, the length spectrum $LSpec^1_\kappa(M) = \mathcal{L}(Cir^1_\kappa(M)) \cap \mathbb{R}$ contains

$$\left\{ \frac{2\pi}{\sqrt{\kappa^2 + c_1}}, \frac{2\pi}{\sqrt{\kappa^2 + c_2}} \right\}$$
$$\bigcup \left\{ 2\pi \times \text{L.C.M.} \left(\frac{1}{\sqrt{\kappa^2 + c_1\beta}}, \frac{1}{\sqrt{\kappa^2 + c_2(1-\beta)}} \right) \middle| \sqrt{\frac{\kappa^2 + c_2(1-\beta)}{\kappa^2 + c_1\beta}} \text{ is rational}, 0 < \beta < 1 \right\}$$

(see [4]). This suggests us the length spectrum of holomorphic circles characterizes some geometries of Kähler manifolds.

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T. Adachi
Department of Mathematics
Nagoya Institute of Technology
Nagoya, 466-8555 Japan
e-mail: adachi@math.kyy.nitech.ac.jp

S. Maeda
Department of Mathematics
Shimane University
Matsue, 690-8504 Japan
e-mail: smaeda@math.shimane-u.ac.jp