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NOTE ON COMPLEX K-GROUPS OF COMPACT LIE GROUPS WITH FUNDAMENTAL GROUP OF PRIME ORDER

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1. Let G be a compact connected simply-connected Lie group and Γ a central subgroup of prime order p. Held and Suter [1] and Hodgkin [3] present two kinds of methods which can be used to determine the structure of $K^*(G/\Gamma)$, where K denotes the complex K-functor.

The purpose of this note is to describe a simple method for the computation of $K^*(G/\Gamma)$, which uses equivariant complex K-theory. Special cases of this method have been applied for calculating, e.g., $K^*(PE_6)$ in [5] and $K^*(PSp(n))$ in [6]. We shall also use the structure theorem on $K^*(G)$ [2] and in addition that on $K^*(L^n(p))$ [4], where $L^n(p)$ denotes the lens space. As a result we can get the structure of $K^*(G/\Gamma)$ as an algebra. Using the notations explained later our result can be stated as follows.

Theorem ([1], [3]). *Let G, Γ and p be as above. Then*

$$K^*(G/\Gamma) = \Lambda_R(\beta(\zeta_2), \cdots, \beta(\zeta_r), \beta(\rho_{r+1}), \cdots, \beta(\rho_\ell), \beta(\kappa))/((V-1)\beta(\kappa))$$

where $\ell = \operatorname{rank} G$ *and* $R = \mathbf{Z}[V]/(p^s(V-1), \cdots, p^s(V-1)^{p-1}, V^p - 1, (V-1)^{s(p-1)+1})$.

The rest of this note is devoted to our proof of the theorem together with the explanation of the symbols for the generators.

2. Let $\rho_1, \cdots, \rho_\ell$ be the fundamental irreducible representations of G with $\ell = \operatorname{rank} G$. Let V denote the canonical non-trivial complex one-dimensional representation of Γ. And let V^k denote the k-fold tensor product of V and qV^k the direct sum of q copies of V^k. Since G admits at least one faithful representation, all the ρ_i's are not trivial on Γ. So by Schur's lemma we may assume that the restrictions of the first $\rho_1, \cdots, \rho_r$ to Γ can be written as

$$\rho_i|\Gamma = p^{s_i} n_i V^{k_i}$$

with $1 \leq k_i \leq p-1$, $(p, n_i) = 1$ and $s_1 \leq s_i$ for all i and the rest are trivial on Γ.

Set $s = s_1$ and denote by V itself the line bundle $G \times_\Gamma V \to G/\Gamma$. Then the order of $V - 1$ in $\tilde{K}(G/\Gamma)$ is a power of p since V is induced from the canonical line bundle over a certain lens space $L^m(p)$. So using ρ_1 we see that $p^s(V-1) = 0$ in $\tilde{K}(G/\Gamma)$, so that there exists a stable isomorphism

$$C : G \times p^s V \cong G \times \mathbf{C}^{p^s} \tag{2.1}$$

of Γ-vector bundles over G.

Let $B(qV)$ and $S(qV)$ be the unit ball and sphere in qV respectively and let $\Sigma^{qV} = B(qV)/S(qV)$ in which the pinched $S(qV)$ serves as base point. Let $L^m(p) = S((m+1)V)/\Gamma$ as usual and write again V for the line bundle $S((m+1)V) \times_\Gamma V \to L^m(p)$. Set $m = t(p-1) + r$ with $0 \leq r < p-1$. By [4] we then have

$$\tilde{K}(L^m(p)) = \mathbf{Z}_{p^{t+1}}\{V-1, \cdots, (V-1)^r\} \oplus \mathbf{Z}_{p^t}\{(V-1)^{r+1}, \cdots, (V-1)^{p-1}\}. \tag{2.2}$$

And its ring structure is given by the relations $V^p = 1$ and $(V-1)^{m+1} = 0$.

Let K_Γ denote the equivariant complex K-functor associated with Γ. Then for a free Γ-space X we have a canonical isomorphism $K_\Gamma(X) \cong K(X/\Gamma)$ which will be identified below. Let us put $n = s(p-1)$. Then by (2.2) we see that $p^s(V-1) = 0$ in $\tilde{K}(L^n(p))$ and hence we have a stable isomorphism

$$T : S((n+1)V) \times \mathbf{C}^{p^s} \cong S((n+1)V) \times p^s V$$

of Γ-vector bundles over $S((n+1)V)$. This gives rise to an element τ of $\tilde{K}_\Gamma(\Sigma^{(n+1)V})$ in a canonical manner such that its restriction to the origin of $B((n+1)V)$ is $p^s(1-V)$ in $R(\Gamma)$, the complex representation ring of Γ.

Consider the exact sequence for the pair $(B((n+1)V), S((n+1)V))$ in K_Γ-theory together with (2.2) when $m = n$. Then it is seen that τ equals the Thom element of $\tilde{K}_\Gamma(\Sigma^{(n+1)V})$ up to a multiple of unit of $R(\Gamma)$. The discussion proceeds by viewing this unit as 1 for brevity. Consider the exact sequence for the cofibration

$$S((n+1)V) \times G \xrightarrow{i} B((n+1)V) \times G \xrightarrow{j} \Sigma^{(n+1)V} \wedge G_+$$

where G_+ denotes the disjoint union of G and a single point $+$ which is taken to be the base point of G_+. Then $j^* : \tilde{K}^*_\Gamma(\Sigma^{(n+1)V} \wedge G_+) \to K_\Gamma(B((n+1)V) \times G) = K^*(G/\Gamma)$ becomes a zero map because $j^*(\tau) = p^s(1-V) = 0$ by (2.1). Hence we have a short exact sequence

$$0 \to K^*(G/\Gamma) \xrightarrow{I} K^*_\Gamma(S((n+1)V) \times G) \xrightarrow{\delta} K^*(G/\Gamma) \to 0 \tag{2.3}$$

under the identification of the Thom isomorphism $K^*(G/\Gamma) \cong \tilde{K}^*_\Gamma(\Sigma^{(n+1)V} \wedge G_+)$. So what we next have to do is to determine the structure of the middle group of (2.3).

3. We now consider the generators of two groups of (2.3). Let X be a compact free Γ-space. Write V for the line bundle $X \times_\Gamma V \to X/\Gamma$ as in §2. Moreover let $f : X \to GL(n, \mathbf{C})$ be a Γ-map where $GL(n, \mathbf{C})$ is the general linear group with the trivial Γ-action. Then such a map defines a unique element of $K_\Gamma^{-1}(X)$, denoted by $\beta(f)$, as follows. The assignment $(x, v) \mapsto (x, f(x)v)$ with $x \in X, v \in \mathbf{C}^n$ yields an isomorphism $\theta : X \times \mathbf{C}^n \cong X \times \mathbf{C}^n$ of Γ-vector bundles over X. We get a Γ-vector bundle over the reduced suspension $S(X_+)$ of X_+ by clutching two n-dimensional product vector bundles over separate cones of X by θ. The reduced vector bundle of this is just $\beta(f)$.

Obviously ρ_i gives rise to $\beta(\rho_i) \in K^{-1}(G/\Gamma)$ for $r+1 \le i \le \ell$ and $\rho_i \circ \pi$ does $\beta(\rho_i \circ \pi) \in K_\Gamma^{-1}(S((n+1)V) \times G)$, denoted by the same symbol $\beta(\rho_i)$ for brevity, where π denotes the projection $S((n+1)V) \times G \to G$.

Define $\bar\rho_i : S((n+1)V) \times G \to GL(d_i, \mathbf{C})$ with $d_i = \text{degree}\rho_i$ for $1 \le i \le r$ by $\bar\rho_i(x, g)(v) = \pi((p^{s_i - s} n_i T)^{-k_i}(x, \rho_i(g)v))$ with $x \in S((n+1)V)$, $g \in G$, $v \in \mathbf{C}^{d_i}$. Here qT denotes the direct sum of q copies of T and π the projection of $S((n+1)V) \times W$ to the second factor. Then $\bar\rho_i$ becomes a Γ-map, so that this gives rise to $\beta(\bar\rho_i)$ of $K_\Gamma^{-1}(S((n+1)V) \times G)$. Furthermore let us put $f(x,g)(v) = \pi(T^p(x,v))$ with $x \in S((n+1)V)$, $g \in G$, $v \in \mathbf{C}^{p^s}$ and write ν for $\beta(f)$ which $f : S((n+1)V) \times G \to GL(p^s, \mathbf{C})$ defines. Then by definition and by making use of (2.1) we have the following.

$$(3.1)\qquad \begin{aligned} I(\beta(\rho_i)) &= \beta(\rho_i)\ (r+1 \le i \le \ell),\ \delta(\nu) = 1 + V + \cdots + V^{p-1}, \\ \delta(\beta(\bar\rho_i)) &= -n_i(1 + V + \cdots + V^{k_i - 1})\ (1 \le i \le r) \end{aligned}$$

From (3.1) and the fact that $(p, k_1) = 1$, $(p, n_1) = 1$ it follows that there exist two polynomials $a(X)$, $b(X) \in \mathbf{Z}[X]$ such that if we put

$$\gamma = a(V)\beta(\bar\rho_1) + b(V)\nu \in K_\Gamma^{-1}(S((n+1)V) \times G)$$

then

$$(3.2)\qquad \delta(\gamma) = 1.$$

By (3.1) and (3.2) we see that there exist more elements $\beta(\zeta_i)$ $(2 \le i \le r)$ and $\beta(\kappa)$ of $K^{-1}(G/\Gamma)$ such that

$$(3.3)\qquad \begin{aligned} I(\beta(\zeta_i)) &= \beta(\bar\rho_i) + n_i(1 + V + \cdots + V^{k_i - 1})\gamma \quad (2 \le i \le r), \\ I(\beta(\kappa)) &= (1 + V + \cdots + V^{p-1})\beta(\bar\rho_1) + n_1 k_1 \nu. \end{aligned}$$

4. Let $1 \le k \le n+1$. Denote by the same symbol the images of $\beta(\bar\rho_i)$'s and $\beta(\rho_j)$'s of $K_\Gamma^{-1}(S((n+1)V) \times G)$ by $(i \times 1)^* : K_\Gamma^{-1}(S((n+1)V) \times G) \to K_\Gamma^{-1}(S(kV) \times$

G) where i denotes an inclusion $S(kV) \subset S((n+1)V)$. And write ν for the image of the Thom element of $K^{-1}(S^{2k-1}) \cong \mathbf{Z}$ by the transfer $K^{-1}(S^{2k-1}) \to K_\Gamma^{-1}(S(kV))$.

Let $k-1 = t(p-1)+r$ with $0 \leq r < p-1$ and set

$$R_k = \mathbf{Z}[V]/\left(p^{t+1}(V-1), \cdots, p^{t+1}(V-1)^r,\right.$$
$$\left. p^t(V-1)^{r+1}, \cdots, p^t(V-1)^{p-1}, V^p-1, (V-1)^k\right).$$

We will show that

$$(4.1) \quad K_\Gamma^*(S(kV) \times G) = \Lambda_{R_k}(\nu, \beta(\bar\rho_1), \cdots, \beta(\bar\rho_r), \beta(\rho_{r+1}), \cdots, \beta(\rho_\ell))/((V-1)\nu)$$

by induction on k.

Choose a circle subgroup S^1 of G which contains Γ and view this S^1 as $S(V)$. Then using the multiplication $S(V) \times G \to G$ yields a homeomorphism $S(V) \times_\Gamma G \approx S^1 \times G$ and so we have $K_\Gamma^*(S(V) \times G) \cong K^*(S^1 \times G)$. According to [2] $K^*(G) = \Lambda(\beta(\rho_1), \cdots, \beta(\rho_\ell))$. From this and the definition of $\beta(\bar\rho_i)$'s and $\beta(\rho_j)$'s it can be easily checked that $K_\Gamma^*(S(V) \times G) = \Lambda(\nu, \beta(\bar\rho_1), \cdots, \beta(\bar\rho_r), \beta(\rho_{r+1}), \cdots, \beta(\rho_\ell))$.

Next consider the exact sequence for the pair $(S((k+1)V) \times G, S(V) \times G)$ in K_Γ-theory. Because of $S((k+1)V)/S(V) \approx \Sigma^V \wedge S(kV)_+$ we then obtain an exact sequence

$$\cdots \to K_\Gamma^*(S(V) \times G) \xrightarrow{\delta} K_\Gamma^*(S(kV) \times G)$$
$$\xrightarrow{J} K_\Gamma^*(S((k+1)V) \times G) \xrightarrow{I} K_\Gamma^*(S(V) \times G) \to \cdots$$

under the identification of the Thom isomorphism $K_\Gamma^*(S(kV) \times G) \cong \tilde K_\Gamma^*(\Sigma^V \wedge (S(kV) \times G)_+)$. And we see that there hold the equalities $J(1) = 1 - V$, $J(\nu) = \nu$ and $\delta(\nu) = 1 + V + \cdots + V^{p-1}$. By making use of these formulas we can proceed with the induction on k and we have (4.1) consequently.

Proof of Theorem. We have $\delta(\nu\beta(\bar\rho_1)) = \beta(\kappa)$ in addition to (3.1), (3.2) and (3.3). Using these formulas the theorem follows immediately from (4.1) when $k = n+1$ and the exactness of (2.3). □

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