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ON THE MODIFIED GOERITZ MATRICES OF 2-PERIODIC LINKS

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1. Introduction

An oriented link $\ell = k_1 \cup \cdots \cup k_\mu$ of μ components in S^3 is called a *2-periodic link* if there is a $\mathbb{Z}_2$-action on the pair (S^3, ℓ) such that the fixed point set f of the action is homeomorphic to a 1-sphere in S^3 disjoint from ℓ. It is known that f is unknotted. Hence the quotient map $p: S^3 \to S^3/\mathbb{Z}_2$ is an 2-fold cyclic branched covering branched over $p(f) = f_*$ and $p(\ell) = \ell_*$ is also an oriented link in the orbit space $S^3/\mathbb{Z}_2 \cong S^3$, which is called the *factor link* of ℓ.

In this paper, we express a relationship between the modified Goeritz matrices of a 2-periodic link ℓ and those of its factor link ℓ_* and the link $\ell_* \cup \bar{f}_*$. As an application, we give an alternative proof of the Gordon and Litherland's formular([3]): $\sigma(\ell) - Lk(\ell, \bar{f}) = \sigma(\ell_*) + \sigma(\ell_* \cup \bar{f}_*)$ for the signature $\sigma(\ell)$ of a 2-periodic null homologous oriented link ℓ in a closed 3-manifold M in the case of a 2-periodic oriented link in S^3. We also show that $n(\ell) = n(\ell_*) + n(\ell_* \cup \bar{f}_*) - 1$, where $n(\ell)$ denotes the nullity of an oriented link ℓ and $\bar{f}_*$ denotes the knot f_* with an arbitrary orientation.

2. Preliminaries

Let ℓ be an oriented link in S^3 and let L be its link diagram in the plane $\mathbb{R}^2 \subset \mathbb{R}^3 = S^3 - \{\infty\}$. Colour the regions of $\mathbb{R}^2 - L$ alternately black and white. Denote the white regions by $X_0, X_1, \cdots, X_w$ (We always take the unbounded region to be white and denote it by X_0). Let $C(L)$ be the set of all crossings of L. Assign an incidence number $\eta(c) = \pm 1$ to each crossing $c \in C(L)$ as in Fig. 2.1 and define a crossing $c \in C(L)$ to be of *type I* or *type II* as indicated in Fig. 2.1.

Let $S(L)$ denote the compact surface with boundary L, more precisely, $S(L)$ is built up out of discs and bands. Each disc lies in $S^2 = \mathbb{R}^2 \cup \{\infty\}$ and is a closed black region less a small neighbourhood of each crossing. Each crossing gives a small half-twisted band. Let $\beta_0(L)$ denote the number of the connected components of the surface $S(L)$.

Let $G'(L) = (g_{ij})_{0 \le i,j \le w}$, where $g_{ij} = -\sum_{c \in C_L(X_i, X_j)} \eta(c)$ for $i \ne j$ and $g_{ii} = \sum_{c \in C_L(X_i)} \eta(c)$, where $C_L(X_i) = \{c \in C(L) | c \text{ is incident to } X_i\}$ and $C_L(X_i, X_j) = \{c \in C(L) | c \text{ is incident to both } X_i \text{ and } X_j\}$.

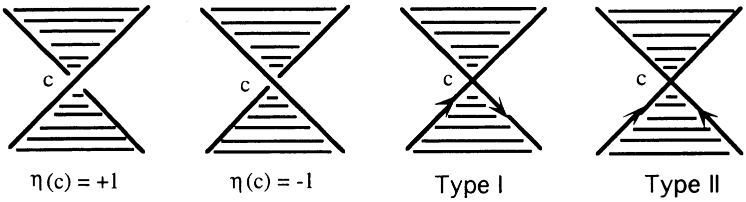

Fig. 2.1.

The principal minor $G(L) = (g_{ij})_{1 \leq i,j \leq w}$ of $G'(L)$ is called the *Goeritz matrix* of ℓ associated to the diagram L([1],[2]).

Let $C_{II}(L) = \{c_1, c_2, \cdots, c_p\}$ denote the set of all crossings of type II in L and let $A(L) = diag(-\eta(c_1), -\eta(c_2), \cdots, -\eta(c_p))$ be the $p \times p$ diagonal matrix. Then Traldi([5]) defined the *modified Goeritz matrix* $H(L)$ of ℓ associated to L by $H(L) = G(L) \oplus A(L) \oplus B(L)$, where $B(L)$ denotes the $(\beta_0(L) - 1) \times (\beta_0(L) - 1)$ zero matrix.

Two integral matrices H_1 and H_2 are said to be *equivalent*, denoted by $H_1 \approx H_2$, if they can be transformed into each other by a finite number of the following two types of transformations and their inverses:

(I) $H \to UHU^t$, where U is a unimodular matrix of integers,

(II) $H \to H \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

If L_1 and L_2 are link diagrams of ambient isotopic links, then $H(L_1)$ and $H(L_2)$ are equivalent([5]).

The signature $\sigma(\ell)$ and the nullity $n(\ell)$ of an oriented link ℓ in S^3 are given by the formulars:$\sigma(\ell) = \sigma(H(L))$, $n(\ell) = n(H(L)) + 1$, where $\sigma(H(L))$ and $n(H(L))$ are the signature and nullity of the matrix $H(L)$ respectively([4], [5]). The absolute value of the determinant of the modified Goeritz matrix $H(L)$ associated to a diagram L of a link ℓ is clearly an invariant of the link type ℓ, denoted by $|det(\ell)|$.

3. The modified Goeritz matrices of 2-periodic links

Let $\ell = k_1 \cup \cdots \cup k_\mu$ be a 2-periodic oriented link of μ components in S^3. Then we may assume that the homeomorphism of the pair (S^3, ℓ) induced by the periodic $\mathbb{Z}_2$-action is the standard rotation ϕ of $\mathbb{R}^3$ through π about the z-axis and hence the fixed point set f is the z-axis $\cup \infty$. We choose the standard orientation on the z-axis and denote it by $\bar{f}$. Define $Lk(\ell, \bar{f}) = \sum_{i=1}^{\mu} link(k_i, \bar{f})$, where $link(k_i, \bar{f})$ denotes the linking number of k_i and $\bar{f}$.

Applying an isotopy deformation if necessary, we may assume that ℓ is represented by a 2-periodic oriented diagram L in an annulus in $\mathbb{R}^2$, which is divided into 2 pieces L_1 and L_2 such that $\varphi(L_1) = L_2, \varphi(L_2) = L_1$, where φ is the rotation of $\mathbb{R}^2$ through π about the origin. Let $a_1, a_2, \cdots, a_r, a_{r+1}, \cdots, a_m$ denote the intersec-

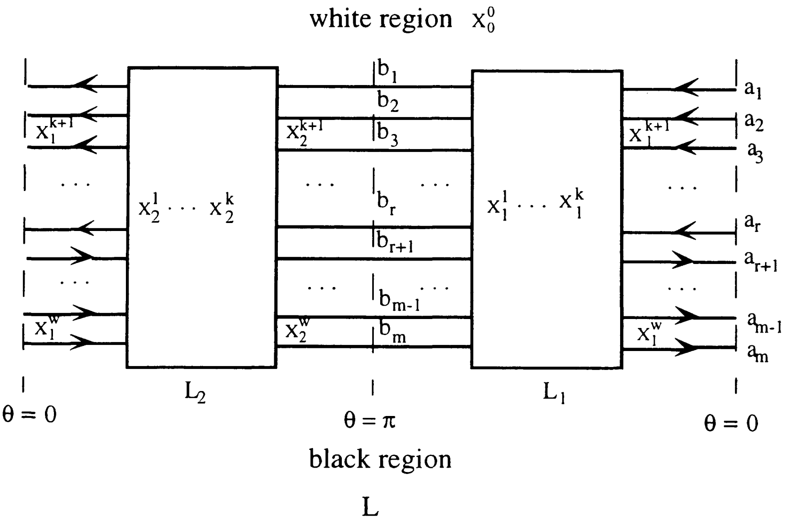

Fig. 3.1.

tion points of L with the line $\theta = 0$ and let $\varphi(a_i) = b_i, i = 1, 2, \cdots, m$, as shown in Fig. 3.1. Note that $Lk(\ell, \bar{f}) = 2r - m$.

Colour the regions of $\mathbb{R}^2 - L$ alternately black and white. Without the loss of generality we may assume that the surface $S(L)$ is connected and the orientation of ℓ is as indicated in Fig. 3.1. If not, by applying ambient isotopy deformations in $\mathbb{R}^3 - f$, i.e., the Reidemeister moves in $\mathbb{R}^2 - \{0\}$ (hence $Lk(\ell, \bar{f})$ is not changed), L can be deformed to L' so that L' is also a 2-periodic link diagram of ℓ, which has the required orientation and $S(L')$ is connected. Now let $\varphi_* : \mathbb{R}^2 \to \mathbb{R}^2/\varphi(\cong \mathbb{R}^2)$ be the quotient map and let $\varphi_*(L) = L_*$. Then L_* is a link diagram of the factor link ℓ_* of ℓ.

In the case of $Lk(\ell, \bar{f}) \equiv 1 \pmod 2$ we denote the white regions as follows. We denote the unbounded white region by X_0^0. Notice that the bounded region containing the origin is then a black region. Let $X_1^1, X_1^2, \cdots, X_1^k$ denote the white regions in $L_1 \subset L$ which do not intersect the line $\theta = 0$, and let $X_1^{k+1}, X_1^{k+2}, \cdots, X_1^w (w = k + (m-1)/2)$ denote the white regions in L which intersect the line $\theta = 0$. For each $j = 1, 2, \cdots, w$, let $X_2^j = \varphi(X_1^j)$. Note that $\varphi(X_0^0) = X_0^0$ (see Fig. 3.1). For $p \neq q$ or $i \neq j$, let $g_{pq}^{ij} = -\sum_{c \in C_L(X_p^i, X_q^j)} \eta(c)$. For $p = q$ and $i = j$, let $g_{pp}^{ii} = \sum_{c \in C_L(X_p^i)} \eta(c)$. Denote $M = (g_{11}^{ij})_{1 \leq i,j \leq k}$, $N = (g_{11}^{ij})_{k+1 \leq i,j \leq w}$, $P = (g_{11}^{ij})_{1 \leq i \leq k, k+1 \leq j \leq w}$, $Q = (g_{12}^{ij})_{1 \leq i \leq k, k+1 \leq j \leq w}$, $R = (g_{12}^{ij})_{k+1 \leq i,j \leq w}$, $A(L_1) = diag(-\eta(c_1), -\eta(c_2), \cdots, -\eta(c_s))$, where $c_i \in C_{II}(L_1)$, and I_k the $k \times k (k \geq 1)$ identity matrix.

In these notations we have the following Lemma 3.1 and Lemma 3.2.

Lemma 3.1. *Let ℓ be an oriented 2-periodic link with $Lk(\ell, \bar{f}) \equiv 1 \pmod 2$ and let L be the 2-periodic diagram of ℓ as shown in Fig. 3.1 and let $\varphi_*(L) = L_*$. Then*
(1)

$$H(L) = \begin{pmatrix} M & P & O & Q \\ P^t & N & Q^t & R \\ O & Q & M & P \\ Q^t & R & P^t & N \end{pmatrix} \oplus A(L_1) \oplus A(L_1).$$

(2)

$$TH(L)T^{-1} = U\left[H(L_*) \oplus \begin{pmatrix} M & P-Q \\ P^t - Q^t & N-R \end{pmatrix} \oplus A(L_1)\right] U^t,$$

where $T = I_w \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \oplus I_s \oplus I_s$ and U is a unimodular integral matrix.

Proof. (1) For $p, q = 0, 1, 2$, let $G_{pq} = (g_{pq}^{ij})_{1 \le i,j \le w}$. Then $G'(L) = (G_{pq})_{0 \le p,q \le 2}$. It is easy to see that the Goeritz matrix $G(L)$ of ℓ associated to L is the matrix of the form: for an integral matrix X,

$$G(L) = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} M & P & O & Q \\ P^t & N & X & R \\ O & X^t & M & P \\ Q^t & R^t & P^t & N \end{pmatrix}.$$

For $k+1 \le i \le w$ and $1 \le j \le w$, X_1^i and X_2^j are incident if and only if $\varphi(X_1^i) = X_2^i$ and $\varphi(X_2^j) = X_1^j$ are incident, and their corresponding crossing types are the same. Thus $g_{12}^{ij} = g_{21}^{ij}$ for $k+1 \le i \le w, 1 \le j \le w$. Hence $X = (g_{12}^{ij})_{k+1 \le i \le w, 1 \le j \le k} = (g_{21}^{ij})_{k+1 \le i \le w, 1 \le j \le k} = Q^t$ and $R^t = (g_{21}^{ij})_{k+1 \le i,j \le w} = (g_{12}^{ij})_{k+1 \le i,j \le w} = R$. It is obvious that $A(L) = A(L_1) \oplus A(L_1)$.

(2) Note that the colouring of the diagram L induces the colouring of the diagram L_* of ℓ_*. Let $X^0 = \varphi_*(X_0^0), X^j = \varphi_*(X_1^j)$ for each $j = 1, \cdots, w$. Then $\{X^j | j = 0, 1, \cdots, w\}$ is the set of all white regions of L_*. Hence $G(L_*) = (g_{ij})_{1 \le i,j \le w}$, where $g_{ij} = -\sum_{c \in C_{L_*}(X^i, X^j)} \eta(c) (i \ne j), g_{ii} = \sum_{c \in C_{L_*}(X^i)} \eta(c)$. If $1 \le i \le k$, then X_1^i intersect neither the line $\theta = 0$ nor the line $\theta = \pi$. So $(g_{ij})_{1 \le i,j \le k} = (g_{11}^{ij})_{1 \le i,j \le k} = M$. Notice that for $k+1 \le j \le w$, the region X^j of L_* is $\varphi_*((X_1^j \cup X_2^j) \cap L_1)$. So $(g_{ij})_{1 \le i \le k, k+1 \le j \le w} = (g_{11}^{ij} + g_{12}^{ij})_{1 \le i \le k, k+1 \le j \le w} = P + Q$. Let $\bar{g}_{pq}^{ij} = -\sum_{c \in C_{L_1}(X_p^i, X_q^j)} \eta(c)$, and $\bar{\bar{g}}_{pq}^{ij} = -\sum_{c \in C_{L_2}(X_p^i, X_q^j)} \eta(c)$. Then $(g_{ij})_{k+1 \le i,j \le w} = (\bar{g}_{11}^{ij} + \bar{g}_{22}^{ij} + \bar{g}_{12}^{ij})_{k+1 \le i,j \le w} = (\bar{g}_{11}^{ij} + \bar{\bar{g}}_{11}^{ij})_{k+1 \le i,j \le w} + (g_{12}^{ij})_{k+1 \le i,j \le w} = N + R$. Since $A(L_*) = A(L_1)$, $H(L_*) = \begin{pmatrix} M & P+Q \\ P^t + Q^t & N+R \end{pmatrix} \oplus A(L_1)$ and so the result follows by easy calculations. □

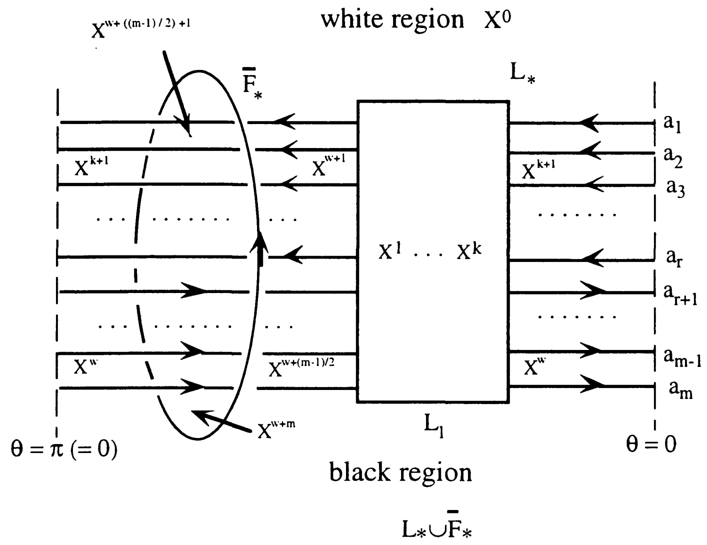

Fig. 3.2.

Now we consider the modified Goeritz matrix of the oriented link $\ell_* \cup \bar{f}_*$ in $S^3/\mathbb{Z}_2 \cong S^3$, where $\bar{f}_* = p(\bar{f})$. From Fig. 3.1 we can obtain the diagram $L_* \cup \bar{F}_*$ in Fig. 3.2 as a diagram of $\ell_* \cup \bar{f}_*$ and denote the white regions of the coloured diagram $L_* \cup \bar{F}_*$ by $X^0, X^1, \cdots, X^{w+m}$ as indicated in Fig. 3.2. Then we obtain the following

Lemma 3.2. *Let $L_* \cup \bar{F}_*$ be the diagram of $\ell_* \cup \bar{f}_*$ shown in Fig. 3.2. Then*

$$VH(L_* \cup \bar{F}_*)V^t = \begin{pmatrix} M & P-Q \\ P^t - Q^t & N-R \end{pmatrix} \oplus A(L_1) \oplus \begin{pmatrix} N_2 & I_{(m-1)/2} \\ I_{(m-1)/2} & O \end{pmatrix}$$
$$\oplus E \oplus (2),$$

where V is an unimodular integral matrix, N_2 is an integral matrix, $E = -I_r \oplus I_{m-r-1}$ if r is even, and $E = -I_{r+1} \oplus I_{m-r}$ if r is odd.

Proof. Let $G'(L_* \cup \bar{F}_*) = (g_{ij})_{0 \le i,j \le w+m}$, where $g_{ij} = -\sum_{c \in C_{L_* \cup \bar{F}_*}(X^i, X^j)} \eta(c)$. We may identify X^i in $L_* \cup \bar{F}_*$ with X_1^i in L for $i = 1, \cdots, w$. Let $\bar{g}_{pq}^{ij} = -\sum_{c \in C_{L_1}(X_p^i, X_q^j)} \eta(c)$ for $p, q = 0, 1, 2$ and $L_1 \subset L$. Denote $E_1 = (g_{0j})_{1 \le j \le k} = (g_{01}^{0j})_{1 \le j \le k}$, $E_2 = (g_{0j})_{k+1 \le j \le w} = (\bar{g}_{01}^{0j})_{k+1 \le j \le w}$, $E_3 = (g_{0w+j})_{1 \le j \le (m-1)/2} = (\bar{g}_{02}^{0k+j})_{1 \le j \le (m-1)/2}$, and $E_4 = (g_{0j})_{w+(m-1)/2+1 \le j \le w+m} = (-2 \ \ 0 \ \ \cdots \ \ 0)$. Notice that $E_2 + E_3 = (g_{01}^{0j})_{k+1 \le j \le w}$.

For $1 \le i, j \le k$, $(g_{ij}) = (g_{11}^{ij}) = M$. For $1 \le i \le k$ and $k+1 \le j \le w$, $(g_{ij}) = (g_{11}^{ij}) = P$. For $1 \le i \le k$ and $1 \le j \le (m-1)/2$, since $g_{iw+j} = g_{12}^{ik+j}$,

$(g_{iw+j}) = (g_{12}^{ik+j}) = (g_{12}^{st})_{1\leq s\leq k,k+1\leq t\leq w} = Q$.

For $1 \leq i \leq k$ and $w+(m-1)/2+1 \leq j \leq w+m$, X^i and X^j are not incident for each pair of i and j. Thus $(g_{ij}) = O$, the $k \times (m-1)/2$ zero matrix.

If $k+1 \leq i \leq w$ and $w+(m-1)/2+1 \leq j \leq w+m$, then the regions X^i and X^j are incident exactly at one crossing only for $j = w+(m-1)/2+i$ and $j = w+(m-1)/2+i+1$ whose incidence number is -1 for $j = w+(m-1)/2+i$ and 1 for $j = w+(m-1)/2+i+1$. So $(g_{ij}) = J$, where J is the $(m-1)/2 \times ((m-1)/2+1)$ matrix of the form:for $m = 1$, $J = \emptyset$ and for $m > 1$,

$$J = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{pmatrix}$$

Similarly for $w+1 \leq i \leq w+(m-1)/2$ and $w+(m-1)/2+1 \leq j \leq w+m$, $(g_{ij}) = J$.

Now let $N_1 = (g_{ij})_{k+1\leq i,j\leq w}, N_2 = (g_{ij})_{w+1\leq i,j\leq w+(m-1)/2}$, $R_1 = (g_{ij})_{k+1\leq i\leq w, w+1\leq j\leq w+(m-1)/2}, K = (g_{ij})_{w+(m-1)/2+1\leq i,j\leq w+m}$. Then

$$G'(L_* \cup \bar{F}_*) = (g_{ij})_{0\leq i,j\leq w+m} = \begin{pmatrix} -a & E_1 & E_2 & E_3 & E_4 \\ E_1^t & M & P & Q & O \\ E_2^t & P^t & N_1 & R_1 & J \\ E_3^t & Q^t & R_1^t & N_2 & J \\ E_4^t & O & J^t & J^t & K \end{pmatrix},$$

where a is the sum of all entries of the row matrices E_1, E_2, E_3, and E_4.

For $w+(m-1)/2+1 \leq i,j \leq w+m$, $g_{ij} = 0 (i \neq j)$ and $g_{ii} = -d_i$, where d_i is the sum of all the i-th rows of E_4^t and $2J^t$. But $d_{w+m} = -2$ and the other d_i's are all zero. Hence $K = \begin{pmatrix} O & O \\ O & 2 \end{pmatrix}$.

Now by deleting the first row and the first column of $G'(L_* \cup \bar{F}_*)$, we obtain the Goeritz matrix $G(L_* \cup \bar{F}_*)$ of $\ell_* \cup \bar{f}_*$ associated to $L_* \cup \bar{F}_*$.

Let E be the diagonal matrix whose diagonal entry corresponding to each type II crossing c in the diagram $L_* \cup \bar{F}_*$ generated by intersecting L_* with $\bar{F}_*$ is $-\eta(c)$. If r is even, then the number of these type II crossings with incidence number $+1$ is equal to r and the number of these type II crossings with incidence number -1 is equal to $m-r-1$ and hence $E = -I_r \oplus I_{m-r-1}$. Similarly for r odd, we have $E = -I_{r+1} \oplus I_{m-r}$. Thus $A(L_* \cup \bar{F}_*) = A(L_1) \oplus E$. Since $S(L_*)$ is connected, $S(L_* \cup \bar{F})$ is also connected. Hence $B(L_* \cup \bar{F})$ is the empty matrix. Therefore $H(L_* \cup \bar{F}_*) = G(L_* \cup \bar{F}_*) \oplus A(L_* \cup \bar{F}_*) = G(L_* \cup \bar{F}_*) \oplus A(L_1) \oplus E$.

It is not difficult to see that $R_1 + R_1^t = R, N_1 + N_2 = N$ and there is a unimodular integral matrix V such that $VH(L_* \cup \bar{F}_*)V^t =$

$$\begin{pmatrix} M & P-Q \\ P^t-Q^t & N-R \end{pmatrix} \oplus A(L_1) \oplus \begin{pmatrix} N_2 & I_{(m-1)/2} \\ I_{(m-1)/2} & O \end{pmatrix} \oplus E \oplus (2). \qquad \square$$

Theorem 3.3. *Let ℓ be an oriented 2-periodic link in S^3 with the fixed point set $\bar{f}$ and let ℓ_* be the factor link of ℓ. Then there exist 2-periodic diagrams L and $L_* \cup \bar{F}_*$ of ℓ and $\ell_* \cup \bar{f}_*$ satisfying the following:*
(1) $Lk(\ell, \bar{f}) \equiv 1 \pmod 2$.

$$S\left[H(L) \oplus \begin{pmatrix} I_a & O \\ O & -I_b \end{pmatrix} \oplus (2)\right] S^{-1} \approx \begin{pmatrix} H(L_*) & O \\ O & H(L_* \cup \bar{F}_*) \end{pmatrix}.$$

(2) $Lk(\ell, \bar{f}) \equiv 0 \pmod 2$.
Let $\ell \circ u$ denote the splittable 2-periodic link consisting of ℓ and the unknot u and let h^- denote the left handed Hopf link. Then

$$S\left[H(L \cup U) \oplus \begin{pmatrix} I_a & O \\ O & -I_{b+1} \end{pmatrix} \oplus (2)\right] S^{-1} \approx \begin{pmatrix} H(L_*) & O \\ O & H((L_* \cup \bar{F}_*) \sharp D^-) \end{pmatrix} \oplus (0),$$

where S is an invertible rational matrix, $L_* = \varphi_*(L)$, $L \cup U$ and D^- are diagrams of $\ell \circ u$ and h^- respectively, and $a - b + 1 = -Lk(\ell, \bar{f})$.

Proof. (1) It follows from Lemma 3.1 and 3.2 that

$$TH(L)T^{-1} \oplus \begin{pmatrix} N_2 & I_{(m-1)/2} \\ I_{(m-1)/2} & O \end{pmatrix} \oplus E \oplus (2) = X[H(L_*) \oplus H(L_* \cup \bar{F}_*)]X^t,$$

where X is a unimodular integral matrix. Note that

$$\begin{pmatrix} N_2 & I_{(m-1)/2} \\ I_{(m-1)/2} & O \end{pmatrix} \oplus E \oplus (2) = Y\left[\begin{pmatrix} I_a & O \\ O & -I_b \end{pmatrix} \oplus (2)\right] Y^{-1},$$

$a - b = m - 2r - 1 = -Lk(\ell, \bar{f}) - 1$, where Y is an invertible rational matrix. This leads to the result.

(2) Let $L \cup U$ be the 2-periodic diagram of $\ell \circ u$ as shown in Fig. 3.3, where U denotes the diagram of the unknot u. Note that $Lk(\ell \cup u, \bar{f}) = Lk(\ell, \bar{f}) - 1 \equiv 1 \pmod 2$. By (1), we obtain

$$S\left[H(L \cup U) \oplus \begin{pmatrix} I_a & O \\ O & -I_b \end{pmatrix} \oplus (2)\right] S^{-1} = \begin{pmatrix} H(L_* \cup U_*) & O \\ O & H((L_* \cup U_*) \cup \bar{F}_*) \end{pmatrix}$$

and $a - (b+1) + 1 = Lk(\ell \circ u, \bar{f}) = -Lk(\ell, \bar{f}) - 1$. Since $L_* \cup U_*$ and $(L_* \cup U_*) \cup \bar{F}_*$ are ambient isotopic to a splittable link diagram $L_* \circ U_*$ and the connected sum $(L_* \cup$

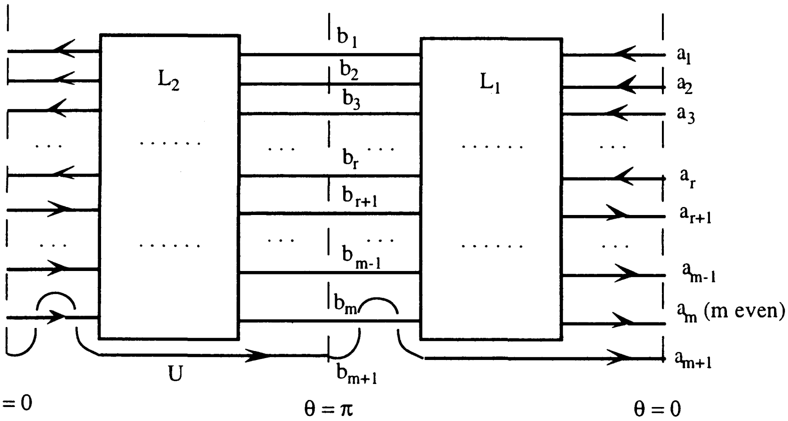

Fig. 3.3.

$\bar{F}_*)\sharp D^-$, $H(L_* \cup U_*) \approx H(L_*) \oplus (0)$ and $H((L_* \cup U_*) \cup \bar{F}_*) \approx H((L_* \cup \bar{F}_*) \sharp D^-)$. This implies the result. □

Corollary 3.4. *Let ℓ be a 2-periodic oriented link in S^3 and let ℓ_* be its factor link. Then*
(1) $\sigma(\ell) - Lk(\ell, \bar{f}) = \sigma(\ell_*) + \sigma(\ell_* \cup \bar{f}_*)$.
(2) $n(\ell) = n(\ell_*) + n(\ell_* \cup \bar{f}_*) - 1$, *where $\bar{f}_*$ denotes the knot f_* with an arbitrary orientation.*

Proof. (1) CASE I. $Lk(\ell, \bar{f}) \equiv 1 \pmod 2$. The relation of (1) in Theorem 3.3 gives that $\sigma(\ell) + a - b + 1 = \sigma(\ell_*) + \sigma(\ell_* \cup \bar{f}_*)$. Since $a - b + 1 = -Lk(\ell, \bar{f})$, the result follows.

CASE II. $Lk(\ell, \bar{f}) \equiv 0 \pmod 2$. The relation of (2) in Theorem 3.3 gives that $\sigma(\ell \circ u) + a - b = \sigma(\ell_*) + \sigma((\ell_* \cup \bar{f}_*)\sharp h^-)$. Note that $\sigma(\ell \circ u) = \sigma(\ell), \sigma((\ell_* \cup \bar{f}_*)\sharp h^-) = \sigma(\ell_* \cup \bar{f}_*) + \sigma(h^-) = \sigma(\ell_* \cup \bar{f}_*) - 1$ (see [4, Lemma 7.2, 7.4]). Since $a - b = -Lk(\ell, \bar{f}) - 1$, the result follows.

(2) Since $n(H(L)) = n(\ell) + 1, n(H(L \cup U)) = n(H(L \circ U)) = n(H(L)) + 1, n(H(L_*) \oplus (0)) = n(\ell_*) + 1$, and $n(H((L_* \cup \bar{F}_*)\sharp D^-)) = n(\ell_* \cup \bar{f}_*) + n(h^-) - 1 = n(\ell_* \cup \bar{f}_*) - 1$ (see [4, Lemma 6.3, 6.4]), Theorem 3.3 implies the result.

Now reversing the orientation of the fixed point set f only changes the sign of some diagonal entries of the diagonal matrix $A(L_* \cup \bar{F}_*)$ in $H(L_* \cup \bar{F}_*)$. This implies that the equations do not depend on the choice of the orientation of f. □

REMARK 3.5. Let $k_* \cup f_*$ be an oriented link in S^3, where f_* is the unknot. Then the inverse image of k_* in the 2-fold cyclic cover $M_2(f_*)$ branched over f_* gives a 2-

periodic oriented link k in S^3. Clearly any 2-periodic link in S^3 arises in this way. If k_* is a knot, then by Corollary 3.4(2) $n(k) = n(k_* \cup f_*)$.

It is well known that $|det(G(K))| = order(H_1(M_2(k);\mathbb{Z}))$ for any diagram K of a knot k, where $order(H_1(M_2(k);\mathbb{Z}))$ denotes the order of the first homology group $H_1(M_2(k);\mathbb{Z})$ of the 2-fold cyclic cover $M_2(k)$ branched over k with integer coefficients. Now if k_* is a knot and $Lk(k_*, f_*)$ is an odd integer, then k is also a knot and $n(k) = n(k_* \cup f_*) = 1$. Furthermore, $|det(H(K))| = |det(G(K))|$ for any diagram K of the knot k. So by Theorem 3.3 (1) we obtain that

$$order(H_1(M_2(k);\mathbb{Z})) = \frac{1}{2} order(H_1(M_2(k_*);\mathbb{Z}))|det(k_* \cup f_*)|.$$

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