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ASYMPTOTIC SOLUTIONS AND EXACT SOLUTIONS FOR EXCEPTIONAL CASES OF SOME CHARACTERISTIC CAUCHY PROBLEMS

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1. Introduction

M.S.Baouendi and C.Goulaouic([1]) considered a Fuchsian partial differential operator with weight $m-k$

$$P = t^k \partial_t^m + \sum_{l=1}^{k} b_{m-l}(x) t^{k-l} \partial_t^{m-l} + \sum_{j+|\alpha|\le m, j<m} t^{\max\{j-m+k+1,0\}} c_{j,\alpha}(t,x) \partial_t^j \partial_x^\alpha,$$

where m is a positive integer, k is a non-negative integer, $b_{m-l}(x)$ are holomorphic functions in a neighborhood of $x=0\in \boldsymbol{C}^n$, and $c_{j,\alpha}(t,x)$ are holomorphic functions in a neighborhood of $(t,x)=(0,0)\in \boldsymbol{C}\times\boldsymbol{C}^n$. In the category of holomorphic functions, they showed the unique solvability of the characteristic Cauchy problem

$$\text{(CP)} \quad \begin{cases} Pu = f(x,t), \\ \partial_t^j u|_{t=0} = g_j(x) \quad (j=0,1,\ldots,\omega(P)-1) \quad (\omega(P) := m-k). \end{cases}$$

under the condition

$$\text{(A)} \qquad \mathcal{C}^{(P)}(0;\lambda) \neq 0 \quad \text{for} \quad \lambda \in \omega(P) + \boldsymbol{N} := \{\omega(P), \omega(P)+1, \ldots\},$$

where $\mathcal{C}^{(P)}(x;\lambda) := (\lambda)_m + \sum_{l=1}^{k} b_{m-l}(x)(\lambda)_{m-l}$ with $(\lambda)_j := \prod_{l=0}^{j-1}(\lambda-l)$. If the condition (A) is not satisfied, then the Cauchy problem does not necessarily have a holomorphic solution for every holomorphic Cauchy data. They also gave a similar result in the category of functions that are of C^∞ class in t and holomorphic in x.

The polynomial $\mathcal{C}^{(P)}(x;\lambda)$ of λ is called the *indicial polynomial* of P, and a root of $\mathcal{C}^{(P)}(x;\lambda)=0$ is called a *characteristic index* of P at x. A characteristic index λ is said to be *exceptional*, if $\lambda\in\omega(P)+\boldsymbol{N}$. The case when (A) is not satisfied, that is, when some characteristic indices at $x=0$ are exceptional, is called the *exceptional case*.

H. Tahara ([5],[6] etc.) considered the characteristic Cauchy problems for Fuchsian *hyperbolic* equations in the category of C^∞ functions on real domains under the same condition (A). The author ([3]) considered a class of operators wider than that of Fuchsian operators, and showed the unique solvability of the characteristic Cauchy problems in the category of functions that are of C^∞ class in t and holomorphic in x, under the same condition (A) for a similarly defined $\mathcal{C}^{(P)}(x;\lambda)$.

In this article, we consider the exceptional case for a class of operators wider than that of Fuchsian operators, and construct asymptotic solutions. For Fuchsian operators, we can easily get "exact" solutions from these asymptotic solutions, using already known results.

In Section 2, we give the statement of the main theorems. After giving some preliminary propositions in Section 3, we prove the main theorems in Section 4.

NOTATIONS.

(i) The set of all integers (resp. nonnegative integers) is denoted by $\boldsymbol{Z}$ (resp. $\boldsymbol{N}$). Put $l+\boldsymbol{N} := \{j \in \boldsymbol{Z} : j \geq l\}$ for $l \in \boldsymbol{Z}$.

(ii) The real part of a complex number z is denoted by Rez, and the imaginary part is denoted by Imz.

(iii) Put $\vartheta := t\partial_t$.

(iv) For a domain Ω in $\boldsymbol{C}^n$, we denote by $\mathcal{O}(\Omega)$ the set of all holomorphic functions on Ω.

(v) For a complete locally convex topological vector space E, put

$$C^N_{flat}([0,T];E) := \left\{ f(t) \in C^N([0,T];E) : \left.\frac{d^j f}{dt^j}\right|_{t=0} = 0 \text{ for } 0 \leq j < N \right\} \quad (N \in \boldsymbol{N}),$$

$$C^{-1}_{flat}([0,T];E) := C^{-1}([0,T];E) := \{f(t) \in C^0((0,T];E) : tf \in C^0([0,T];E)\}.$$

(vi) $\mathcal{R}(\boldsymbol{C}^\times)$ denotes the universal covering of $\boldsymbol{C}^\times := \boldsymbol{C} \setminus \{0\}$.

(vii) For a commutative ring R, the ring of polynomials in λ with the coefficients belonging to R is denoted by $R[\lambda]$. The degree of $F \in R[\lambda]$ is denoted by $\deg_\lambda F$. Also, the ring of formal power series of t with the coefficients belonging to R is denoted by $R[[t]]$.

2. Statement of Main Result

Let Ω be a bounded domain in $\boldsymbol{C}^n$ that contains the origin 0. Consider a linear partial differential operator

$$P = \sum_{j+|\alpha|\leq m} a_{j,\alpha}(t,x)\partial_t^j\partial_x^\alpha, \qquad a_{m,0}(t,x) \equiv t^\kappa \tag{2.1}$$

of order $m \geq 1$, where $\kappa \in \boldsymbol{N}$. Let $T > 0$ and assume that

$$a_{j,\alpha} \in C^{\infty}([0,T]; \mathcal{O}(\Omega)) \quad (j + |\alpha| \leq m).$$

Let $r(j,\alpha)$ be the *vanishing order* of $a_{j,\alpha}$ on the initial surface $\Sigma := \{(0,x) : x \in \Omega\}$, that is

$$r(j,\alpha) := \sup\{r \in \boldsymbol{N} : t^{-r} a_{j,\alpha} \in C^{\infty}([0,T]; \mathcal{O}(\Omega))\}. \tag{2.2}$$

DEFINITION 2.1. (1) Put $\omega(j,\alpha) := j - r(j,\alpha)$, which is considered as a weight of each differential monomial $a_{j,\alpha}(t,x)\partial_t^j \partial_x^\alpha$. Put $\omega(P) := \max_{j+|\alpha|\leq m} \omega(j,\alpha)$, which is called the *weight* of P.
(2) Put $b_{j,\alpha}(x) := \left\{a_{j,\alpha}(t,x)t^{-j+\omega(P)}\right\}\big|_{t=0}$. Especially, we write $b_j(x) := b_{j,0}(x)$.

In this article, we assume the following two conditions.

(B1) $\quad b_{j,\alpha}(x) \equiv 0 \quad \text{if} \quad \alpha \neq 0.$

(B2) $\quad \omega(P) \geq 0.$

DEFINITION 2.2. Put

$$\mathcal{C}^{(P)}(x;\lambda) := \sum_{j=0}^{m} b_j(x)(\lambda)_j \in \mathcal{O}(\Omega)[\lambda]. \tag{2.3}$$

This is a polynomial in λ, and is called the *indicial polynomial* of P. A root of $\mathcal{C}^{(P)}(x;\lambda) = 0$ is called a *characteristic index* of P at x. A characteristic index λ is said to be *exceptional*, if $\lambda \in \omega(P) + \boldsymbol{N}$. Note that if $j < \omega(P)$ then $b_j(x) \equiv 0$, and hence $\mathcal{C}^{(P)}(x;\lambda) \equiv 0$ for $\lambda = 0, 1, \ldots, \omega(P) - 1$.

We also assume the following "non-degeneracy" condition for $\mathcal{C}^{(P)}$.

(B3) $\quad b_d(0) \neq 0, \text{ where } d := \max\{j : b_j(x) \not\equiv 0 \text{ on } \Omega\} = \deg_\lambda \mathcal{C}^{(P)}.$

For example, Fuchsian partial differential operators satisfy the assumptions (B1)–(B3) with $\omega(P) = m - k$, $d = m$, and $b_m(x) \equiv 1$. The operators considered in [3] and [4] also satisfy these three assumptions.

As for formal solutions when the condition (A) is satisfied, it is easy to prove the following.

Proposition 2.3. *Assume the conditions* (B1)–(B3). *For a subdomain* Ω_0 *of* Ω *including* 0, *the following two conditions are equivalent.*

(a) $\mathcal{C}^{(P)}(x;\lambda) \neq 0$ *on* Ω_0 *for* $\lambda \in \omega(P) + \boldsymbol{N}$.

(b) *For every* $f(x,t) \in \mathcal{O}(\Omega_0)[[t]]$ *and every* $g_j(x) \in \mathcal{O}(\Omega_0)$ $(0 \leq j \leq \omega(P) - 1)$, *there exists a unique formal solution* $u(t,x) \in \mathcal{O}(\Omega_0)[[t]]$ *of the characteristic Cauchy problem*

$$\text{(CP)} \quad \begin{cases} Pu = f(x,t), \\ \partial_t^j u|_{t=0} = g_j(x) \quad (j = 0, 1, \ldots, \omega(P) - 1). \end{cases}$$

REMARK 2.4. The condition (a) may look like a collection of infinite number of conditions. If (A) is satisfied, however, then we can always take Ω_0 for which the condition (a) holds, since $\mathcal{C}^{(P)}(x;\lambda)$ is a polynomial of λ whose top order coefficient does not vanish at $x = 0$.

In order to consider the *exceptional case*, we assume the following condition on $\mathcal{C}^{(P)}(x;\lambda)$, which means that the exceptional characteristic indices at $x = 0$ can be extended holomorphically as characteristic roots.

$$\text{(E)} \qquad \mathcal{C}^{(P)}(x;\lambda) = \prod_{j=1}^{r}(\lambda - \lambda_j(x)) \cdot \mathcal{D}^{(P)}(x;\lambda),$$

where $r \in \boldsymbol{N}$ and

a) $\lambda_j \in \mathcal{O}(\Omega)$, $\lambda_j(0) \in \omega(P) + \boldsymbol{N}$ $\quad (1 \leq j \leq r)$,

b) $\mathcal{D}^{(P)}(0, \lambda) \neq 0$ for every $\lambda \in \omega(P) + \boldsymbol{N}$.

In [2], the author considered the restricted case when λ_j are all constants. In this restricted case, we can give a formal solution u of (CP) in the form

$$u = \sum_{j=0}^{\omega(P)-1} \frac{g_j(x)}{j!} t^j + \sum_{j=0}^{\infty} t^{\omega(P)+j} \sum_{l=0}^{r} u_{j,l}(x)(\log t)^l,$$
$$u_{j,l} \in \mathcal{O}(\Omega_0) \quad (0 \leq j; 0 \leq l \leq r).$$

In general case, we can not expect to have a formal solution of this form. We shall give formal solutions in a more complicated form.

In order to see what kind of functions we need, let us consider the simplest example.

EXAMPLE 2.5. $P := t\partial_t - \lambda(x)$, where $\lambda(x) \in \mathcal{O}(\Omega)$ and $\lambda(0) = p \in \boldsymbol{N}$. Note that $\mathcal{C}^{(P)}(x;\lambda) = \lambda - \lambda(x)$. By freezing x, we can easily solve the equation $Pu = t^p$ in $\mathcal{R}(\boldsymbol{C}^{\times})$:

$$u = \begin{cases} \dfrac{-1}{\lambda(x)-p} t^p + C_x t^{\lambda(x)} & (\text{if } \lambda(x) \neq p), \\ (\log t) t^p \quad + C_x t^{\lambda(x)} & (\text{if } \lambda(x) = p), \end{cases}$$

where C_x is an arbitrary constant that may depend on x. This is not a good representation since we want solutions that is holomorphic with respect to x. A good one is

$$u = \begin{cases} \dfrac{t^{\lambda(x)-p}-1}{\lambda(x)-p} t^p + C(x) t^{\lambda(x)} & (\text{if } \lambda(x) \neq p), \\ (\log t) t^p \quad + C(x) t^{\lambda(x)} & (\text{if } \lambda(x) = p), \end{cases}$$

where $C(x) \in \mathcal{O}(\Omega)$ is arbitrary.

This example suggests that we need a family of functions such as

$$u(t,x) = \begin{cases} \dfrac{t^{\mu(x)}-1}{\mu(x)} & \text{if } \mu(x) \neq 0 \\ \log t & \text{if } \mu(x) = 0 \end{cases}, \tag{2.4}$$

where $\mu \in \mathcal{O}(\Omega)$. Thus, we give the following definitions.

DEFINITION 2.6. (1) Define $F^{(j)}(z_1, \dots, z_j; t)$ inductively as follows.

$$F^{(1)}(z_1; t) := \left\{ \begin{array}{ll} \dfrac{t^{z_1}-1}{z_1} & (z_1 \neq 0) \\ \log t & (z_1 = 0) \end{array} \right\} = \sum_{p=1}^{\infty} \frac{(\log t)^p}{p!} z_1^{p-1}, \tag{2.5}$$

$$\begin{aligned} &F^{(j+1)}(z_1, \dots, z_{j+1}; t) \\ &:= \begin{cases} \dfrac{F^{(j)}(z_1, \dots, z_{j-1}, z_{j+1}; t) - F^{(j)}(z_1, \dots, z_{j-1}, z_j; t)}{z_{j+1}-z_j} & (z_{j+1} \neq z_j) \\ \partial_{z_j} F^{(j)}(z_1, \dots, z_j; t) & (z_{j+1} = z_j) \end{cases} \\ &= \sum_{p=j+1}^{\infty} \frac{(\log t)^p}{p!} \sum_{k_1+\dots+k_{j+1}=p-j-1} z_1^{k_1} \dots z_{j+1}^{k_{j+1}} \quad (j \geq 1). \end{aligned} \tag{2.6}$$

Also put $F^{(0)}(\cdot; t) := 1$.

It is easy to see that $F^{(j)}(z_1, \dots, z_j; t)$ is holomorphic on $\boldsymbol{C}^j \times \mathcal{R}(\boldsymbol{C}^\times)$ and symmetric in $(z_1, \dots, z_j) \in \boldsymbol{C}^j$. Further, $F^{(j)}(0, \dots, 0; t) = (1/j!)(\log t)^j$, and if

$0, z_1, \dots, z_j$ are all distinct, then

$$F^{(j)}(z_1, \dots, z_j; t) = c_0(z_1, \dots, z_j) + \sum_{i=1}^{j} c_i(z_1, \dots, z_j) t^{z_i}$$

for some rational functions $c_i(z_1, \dots, z_j)$ $(i = 0, 1, \dots, j)$ of $(z_1, \dots, z_j)$. The function u given by (2.4) can be written as $F^{(1)}(\mu(x); t)$.
(2) For $\mu_1(x), \dots, \mu_r(x) \in \mathcal{O}(\Omega)$ and $h \in \boldsymbol{N}$, put

$$\begin{aligned}\mathcal{F}_\Omega^{(h)}[\mu_1, \dots, \mu_r] : &= \Big\{ v(t,x) = \sum_{l=0}^{h} \sum_{I=(i_1,\dots,i_l)} v_I(x) F^{(l)}(\mu_{i_1}(x), \dots, \mu_{i_l}(x); t) \\ &\qquad : 1 \le i_1 \le \dots \le i_l \le r, v_I(x) \in \mathcal{O}(\Omega) \Big\}, \\ \mathcal{F}_\Omega^{(\infty)}[\mu_1, \dots, \mu_r] : &= \bigcup_{h=0}^{\infty} \mathcal{F}_\Omega^{(h)}[\mu_1, \dots, \mu_r] \ \subset \mathcal{O}(\Omega \times \mathcal{R}(\boldsymbol{C}^\times)).\end{aligned}$$

Note that $\mathcal{F}_\Omega^{(0)}[\mu_1, \dots, \mu_r] = \mathcal{O}(\Omega)$ (constant functions with respect to t). Further, if $h \le k$, then $\mathcal{F}_\Omega^{(h)}[\mu_1, \dots, \mu_r] \subset \mathcal{F}_\Omega^{(k)}[\mu_1, \dots, \mu_r]$. If $\mu_j(x) \equiv 0$ $(1 \le j \le r)$, then

$$\begin{aligned}\mathcal{F}_\Omega^{(h)}[0, \dots, 0] = \mathcal{F}_\Omega^{(h)}[0] &= \bigoplus_{l=0}^{h} \mathcal{O}(\Omega)(\log t)^l \\ &:= \Big\{ v(t,x) = \sum_{l=0}^{h} v_l(x)(\log t)^l : v_l \in \mathcal{O}(\Omega) \Big\}.\end{aligned}$$

By the following proposition, a formal series of the form

$$\sum_{p=0}^{\infty} t^p v_p(t,x), \quad v_p(t,x) \in \mathcal{F}_{\Omega_0}^{(\infty)}[\mu_1, \dots, \mu_r]$$

can be considered as an asymptotic series that is an extension of a formal power series $\sum_{p=0}^{\infty} t^p v_p(x)$, $v_p(x) \in \mathcal{O}(\Omega_0)$.

Proposition 2.7. *If $s > 0$ and $\mathrm{Re}\mu_l(x) + s > 0$ on Ω $(1 \le l \le r)$, then for every $g(t,x) \in \mathcal{F}_\Omega^{(\infty)}[\mu_1, \dots, \mu_r]$, every $(j, \alpha) \in \boldsymbol{N}^{n+1}$, and every compact subset K of Ω, there holds*

$$\sup_{x \in K} |t^s \vartheta^j \partial_x^\alpha g(t,x)| \longrightarrow 0 \quad (t \to 0).$$

This convergence is considered in an arbitrary finite sector of $t \in \mathcal{R}(\boldsymbol{C}^\times)$.

This proposition follows easily from Proposition 3.5 and 3.6 in the next section. By this proposition, we get the following corollary.

Corollary 2.8. *If* $\mathrm{Re}\mu_l(x) + 1 > 0$ *on* Ω $(1 \le l \le r)$, *then for every* $g(t,x) \in \mathcal{F}_\Omega^{(\infty)}[\mu_1, \dots, \mu_r]$ *and every* $p \in \boldsymbol{N}$, *there holds* $t^p g(t,x) \in C_{flat}^{p-1}([0,T]; \mathcal{O}(\Omega))$.

Now, the following is one of the two main theorems of this article.

Theorem 2.9. *Assume the conditions* (B1)–(B3) *and* (E). *Put* $\mu_j(x) := \lambda_j(x) - \lambda_j(0)$ $(j = 1, 2, \dots, r)$, *and put* $\mu_0(x) \equiv 0$. *Take a subdomain* Ω_0 *of* Ω *including* 0 *and satisfying the following three conditions.*
(a) $\mathcal{D}^{(P)}(x; \mu_j(x) + q) \neq 0$ *on* Ω_0 *for every* $q \in \omega(P) + \boldsymbol{N}$ *and* $j = 0, 1, 2, \dots, r$.
(b) *If* $j \neq l$, *then* $\mu_j(x) - \mu_l(x) \notin \boldsymbol{Z} \setminus \{0\}$ *on* Ω_0.
(c) $\mathrm{Re}\mu_l(x) + 1 > 0$ *on* Ω_0 $(1 \le l \le r)$.
Then, for every $f = \sum_{j=0}^{\infty} f_j(x) t^j \in \mathcal{O}(\Omega_0)[[t]]$ *and every* $g_j \in \mathcal{O}(\Omega_0)$ $(0 \le j \le \omega(P) - 1)$, *there exists a formal solution of the Cauchy problem* (CP) *in the form*

$$u(t,x) = \sum_{j=0}^{\omega(P)-1} \frac{g_j(x)}{j!} t^j + \sum_{p=0}^{\infty} t^{\omega(P)+p} v_p(t,x), \tag{2.7}$$

where $v_p \in \mathcal{F}_{\Omega_0}^{(r+mp)}[\mu_1, \dots, \mu_r]$. *In other words, for every* $N \in \boldsymbol{N}$, *there holds*

$$P\left(\sum_{j=0}^{\omega(P)-1} \frac{g_j(x)}{j!} t^j + \sum_{p=0}^{N} t^{\omega(P)+p} v_p(t,x) \right) - \sum_{j=0}^{N} f_j(x) t^j \in C_{flat}^N([0,T]; \mathcal{O}(\Omega_0)).$$

If λ_j *are all constants in addition, then we can take* $v_p \in \mathcal{F}_{\Omega_0}^{(r)}[0]$ *for every* $p \in \boldsymbol{N}$, *that is* $v_p(t,x) = \sum_{l=0}^{r} v_{p,l}(x)(\log t)^l$, *where* $v_{p,l} \in \mathcal{O}(\Omega_0)$ $(p \ge 0; 0 \le l \le r)$.

REMARK 2.10. (1) We can always take Ω_0 satisfying (a)–(c), since $\mu_j(0) = 0$ $(1 \le j \le r)$ and since $\mathcal{D}^{(P)}$ is a polynomial of λ whose top order coefficient does not vanish at $x = 0$.
(2) The condition (c) is needed for $\partial_t^j u - g_j(x) = o(1)$ $(t \to 0)$. The condition (c) and Corollary 2.8 implies that $t^{\omega(P)+p} v_p \in C_{flat}^{\omega(P)+p-1}([0,T]; \mathcal{O}(\Omega_0))$ $(p \in \boldsymbol{N})$.

EXAMPLE 2.11. Consider $P = t\partial_t - t\partial_x - x - t$. There exists a solution

$$u = \left\{ \begin{array}{ll} \dfrac{t^{t+x} - 1}{t+x} & (\text{if } x \neq -t) \\ \log t & (\text{if } x = -t) \end{array} \right\} = F^{(1)}(t + x; t)$$

of $Pu = 1$. This solution can be rewritten as

$$u = \sum_{p=0}^{\infty} t^p F^{(p+1)}(x, \ldots, x; t),$$

which is a solution in the form (2.7) with $\omega(P) = 0$ and $v_p(t,x) = F^{(p+1)}(x, \ldots, x; t)$. As a matter of fact, for every constant α,

$$u = \sum_{p=0}^{\infty} t^p \big\{(1 + \alpha x) F^{(p+1)}(x, \ldots, x; t) + \alpha F^{(p)}(x, \ldots, x; t)\big\}$$

is also a formal solution in the form (2.7). Thus, the formal solution in Theorem 2.9 is not unique in general.

Finally, we consider exact solutions.

Theorem 2.12. *In addition to the assumptions of Theorem* 2.9, *assume that there exist* $T_0 > 0$ *and a domain* Ω_0' *of* $\boldsymbol{C}^n$ *including* 0 *for which the following well-posedness of the flat Cauchy problem holds.*

(F) *For every* $f \in C^\infty_{flat}([0,T]; \mathcal{O}(\Omega_0))$, *there exists a unique* $u \in C^\infty_{flat}([0,T_0]; \mathcal{O}(\Omega_0'))$ *such that* $Pu = f$ *on* $[0,T_0] \times \Omega_0'$.

Then, for every $f \in C^\infty([0,T]; \mathcal{O}(\Omega))$ *and every* $g_j \in \mathcal{O}(\Omega)$ $(0 \le j \le \omega(P) - 1)$, *there exists an exact solution* $u \in C^{\omega(P)-1}([0,T_0]; \mathcal{O}(\Omega_0'))$ *of* (CP) *whose asymptotic expansion is the formal solution given in Theorem* 2.9, *in the sense that for every* $N \in \boldsymbol{N}$, *there holds*

$$u(t,x) - \left(\sum_{j=0}^{\omega(P)-1} \frac{g_j(x)}{j!} t^j + \sum_{p=0}^{N} t^{\omega(P)+p} v_p(t,x) \right) \in C^{\omega(P)+N}_{flat}([0,T_0]; \mathcal{O}(\Omega_0')).$$

Note that the Fuchsian operators and the operators considered in [3] satisfy the condition (F).

We can also consider another version of the existence of exact solutions by considering a sector $S_\theta(T) := \{t \in \mathcal{R}(\boldsymbol{C}^\times) : |t| < T, |\arg t| < \theta\}$ instead of $(0,T)$. We, however, omit the detail.

Remark 2.13. We can obtain similar theorems for $C^\infty(U)$ instead of $\mathcal{O}(\Omega)$. We omit the detail since the proof is almost the same. Note that Fuchsian hyperbolic operators considered by H.Tahara satisfy the assumption corresponding to (F) in the case of $C^\infty(U)$ and hence we can also obtain exact solutions.

3. Preliminaries

In this section, we study some basic properties of the functions $F^{(j)}(z_1, \dots, z_j; t)$ and of the function spaces $\mathcal{F}_\Omega^{(h)}[\mu_1, \dots, \mu_r]$.

We write $z[j] := (z_1, \dots, z_j)$, which are considered as variables of $\boldsymbol{C}^j$, and write $\alpha[j] := (\alpha_1, \dots, \alpha_j) \in \boldsymbol{N}^j$.

The following lemma is trivial by the definition and the symmetry.

Lemma 3.1.

$$\partial_{z_l}(F^{(j)}(z[j]; t)) = F^{(j+1)}(z[j], z_l; t), \quad (1 \leq l \leq j)$$

By a repeated use of this lemma, we can easily show the following.

Proposition 3.2.

$$\partial_{z[j]}^{\alpha[j]}(F^{(j)}(z[j]; t)) = \alpha[j]! F^{(j+|\alpha[j]|)}(z[j], \alpha[j] \circledast z[j]; t),$$

where $\alpha[j]! := \alpha_1! \dots \alpha_j!$, $|\alpha[j]| := \alpha_1 + \dots + \alpha_j$, *and*

$$\alpha[j] \circledast z[j] := (\overbrace{z_1, \dots, z_1}^{\alpha_1}, \dots, \overbrace{z_j, \dots, z_j}^{\alpha_j}).$$

The following proposition gives the principal meaning of $F^{(j)}(z[j]; t)$.

Proposition 3.3.

$$(\vartheta - z_{j-r+1}) \dots (\vartheta - z_j) F^{(j)}(z[j]; t) = F^{(j-r)}(z[j-r]; t) \quad (j \geq r \geq 1).$$

Proof. We have

$$\begin{aligned} \vartheta F^{(j)}(z[j]; t) &= \sum_{p \geq j} \frac{1}{p!} p (\log t)^{p-1} \sum_{k_1 + \dots + k_j = p - j} z_1^{k_1} \dots z_j^{k_j} \\ &= \sum_{p \geq j} \frac{1}{(p-1)!} (\log t)^{p-1} \sum_{k_1 + \dots + k_j = p - j} z_1^{k_1} \dots z_j^{k_j} \\ &= \sum_{p \geq j-1} \frac{1}{p!} (\log t)^p \sum_{k_1 + \dots + k_j = p - j + 1} z_1^{k_1} \dots z_j^{k_j}. \end{aligned}$$

Further, we have

$$\sum_{k_1 + \dots + k_j = p - j + 1} z_1^{k_1} \dots z_j^{k_j} = \sum_{k_1 + \dots + k_{j-1} = p - j + 1} z_1^{k_1} \dots z_{j-1}^{k_{j-1}}$$

$$+\Big(\sum_{k_1+\ldots+k_j=p-j} z_1^{k_1}\ldots z_j^{k_j}\Big)z_j,$$

where the last term is 0 if $p = j-1$. Hence, $\vartheta F^{(j)}(z[j];t) = F^{(j-1)}(z[j-1];t) + z_j F^{(j)}(z[j];t)$, that is,

$$(\vartheta - z_j)F^{(j)}(z[j];t) = F^{(j-1)}(z[j-1];t). \tag{3.1}$$

Note that this holds also for $j = 1$. By an iterative use of this formula, we can get the desired result. □

Proposition 3.4. *Let $A(x;\lambda) \in \mathcal{O}(\Omega)[\lambda]$, $j \in \boldsymbol{N}$, and $\mu_l(x) \in \mathcal{O}(\Omega)$ $(1 \le l \le j)$. Put $\mu_0(x) \equiv 0$.*

(1) *There exist $A_l^{(j)}(x) \in \mathcal{O}(\Omega)$ $(l = 1,\ldots,j)$ such that*

$$A(x;\vartheta)F^{(j)}(\mu_1(x),\ldots,\mu_j(x);t) = A(x;\mu_j(x))F^{(j)}(\mu_1(x),\ldots,\mu_j(x);t) + \sum_{l=1}^{j} A_l^{(j)}(x)F^{(j-l)}(\mu_1(x),\ldots,\mu_{j-l}(x);t).$$

$$\Big(\textit{If } j = 0, \textit{ then } A(x;\vartheta)F^{(0)}(\cdot;t) = A(x;0)F^{(0)}(\cdot;t) = A(x;0).\Big)$$

If $l > \deg_\lambda A$, then we can take $A_l^{(j)}(x) \equiv 0$.

(2) *If $A(x;\mu_l(x)) \neq 0$ on Ω $(0 \le l \le j)$, then the equation*

$$A(x;\vartheta)u = F^{(j)}(\mu_1(x),\ldots,\mu_j(x);t)$$

has a solution u of the form

$$u = \sum_{l=0}^{j} u_l(x)F^{(j-l)}(\mu_1(x),\ldots,\mu_{j-l}(x);t),$$

where $u_l(x) \in \mathcal{O}(\Omega)$.

Proof. (1) We give a proof by the induction on $\deg_\lambda A$. If $\deg_\lambda A = 0$, the conclusion is trivial.

Assume that the conclusion is valid up to $\deg_\lambda A = m-1$ $(m \ge 1)$. Let $\deg_\lambda A = m$. We can write as $A(x;\lambda) = \widetilde{A}(x;\lambda)\lambda + B(x)$, where $\deg_\lambda \widetilde{A} = m-1$. Hence, by (3.1) we have

$$A(x;\vartheta)F^{(j)}(\mu_1(x),\ldots,\mu_j(x);t)$$
$$= \widetilde{A}(x;\vartheta)\big\{\mu_j(x)F^{(j)}(\mu_1(x),\ldots,\mu_j(x);t) + F^{(j-1)}(\mu_1(x),\ldots,\mu_{j-1}(x);t)\big\}$$

$$
\begin{aligned}
&+ B(x)F^{(j)}(\mu_1(x),\dots,\mu_j(x);t)\\
&= \mu_j(x)\Big\{\widetilde{A}(x;\mu_j(x))F^{(j)}(\mu_1(x),\dots,\mu_j(x);t)\\
&\qquad + \sum_{l=1}^{j}\widetilde{A}_l^{(j)}(x)F^{(j-l)}(\mu_1(x),\dots,\mu_{j-l}(x);t)\Big\}\\
&\quad + \widetilde{A}(x;\mu_{j-1}(x))F^{(j-1)}(\mu_1(x),\dots,\mu_{j-1}(x);t)\\
&\quad + \sum_{l=1}^{j-1}\widetilde{A}_l^{(j-1)}(x)F^{(j-1-l)}(\mu_1(x),\dots,\mu_{j-1-l}(x);t)\\
&\quad + B(x)F^{(j)}(\mu_1(x),\dots,\mu_j(x);t)\\
&= A(x;\mu_j(x))F^{(j)}(\mu_1(x),\dots,\mu_j(x);t)\\
&\quad + \sum_{l=1}^{j}A_l^{(j)}(x)F^{(j-l)}(\mu_1(x),\dots,\mu_{j-l}(x);t),
\end{aligned}
$$

where $\widetilde{A}_l^{(j)}, \widetilde{A}_l^{(j-1)} \in \mathcal{O}(\Omega)$ and $A_l^{(j)}(x) = \mu_j(x)\widetilde{A}_l^{(j)}(x) + \delta_{l,1}\widetilde{A}(x;\mu_{j-1}(x)) + \widetilde{A}_{l-1}^{(j-1)}(x)$ $(\widetilde{A}_0^{(j-1)}(x) \equiv 0)$. If $l > \deg_\lambda A - 1$, then $\widetilde{A}_l^{(j)}(x) \equiv \widetilde{A}_l^{(j-1)}(x) \equiv 0$ by the induction hypothesis. Hence, if $l > \deg_\lambda A (\geq 1)$ then $A_l^{(j)}(x) \equiv 0$.

(2) We give a proof by the induction on j. If $j = 0$, then $(1/A(x;0))F^{(0)}(\cdot;t) = 1/A(x;0)$ is a solution of $A(x;\vartheta)u = F^{(0)}(\cdot;t) = 1$.

Assume that the conclusion is valid up to $j-1$. By (1), we have

$$
\begin{aligned}
&A(x;\vartheta)\left(\frac{1}{A(x;\mu_j(x))}F^{(j)}(\mu_1(x),\dots,\mu_j(x);t)\right)\\
&\qquad = F^{(j)}(\mu_1(x),\dots,\mu_j(x);t) + \sum_{l=1}^{j}B_l(x)F^{(j-l)}(\mu_1(x),\dots,\mu_{j-l}(x);t),
\end{aligned}
$$

where $B_l \in \mathcal{O}(\Omega)$ $(1 \leq l \leq j)$. Since we can solve

$$
A(x;\vartheta)v_l = B_l(x)F^{(j-l)}(\mu_1(x),\dots,\mu_{j-l}(x);t) \quad (1 \leq l \leq j)
$$

by the induction hypothesis, we can get the desired solution u. □

Proposition 2.7 is a simple corollary of the following two propositions.

Proposition 3.5. *Let $s > 0$ and put $\Omega_s[j] := \{z[j] \in \boldsymbol{C}^j : \mathrm{Re} z_l + s > 0\ (1 \leq l \leq j)\}$. For every compact subset $K[j]$ of $\Omega_s[j]$, there holds*

$$
\sup_{z[j]\in K[j]} |t^s F^{(j)}(z[j];t)| \longrightarrow 0 \quad (t \to 0).
$$

This convergence is considered in an arbitrary finite sector of $t \in \mathcal{R}(\boldsymbol{C}^\times)$.

Proof. Let $j = 1$. We can write as

$$F^{(1)}(z_1; t) = \int_0^1 t^{z_1\sigma} \log t d\sigma,$$

and there exists $\epsilon > 0$ such that $\mathrm{Re}(z_1\sigma + s) \geq \epsilon$ for $z_1 \in K[1]$ and $\sigma \in [0, 1]$. If $t = |t|e^{i\theta}$ and $|\theta| \leq M$, then $|t^{z_1\sigma+s}| \leq |t|^{\mathrm{Re}(z_1\sigma+s)} e^{M|\mathrm{Im}(z_1\sigma+s)|}$ and $|\log t| = |\log|t| + i\theta| \leq |\log|t|| + M$. Hence, we have

$$\sup_{z_1 \in K[1]} |t^s F^{(1)}(z_1; t)| \leq C|t|^\epsilon (\log|t| + M) \to 0 \quad (t \to 0).$$

Assume that the conclusion is valid up to j. By Cauchy's integral formula, we have

$$\sup_{z[j] \in K[j]} \left|\partial_{z_j} \{t^s F^{(j)}(z[j]; t)\}\right| \to 0 \quad (t \to 0)$$

for every compact subset $K[j]$ of $\Omega_s[j]$. Hence, from

$$F^{(j+1)}(z[j+1]; t) = \int_0^1 \partial_z \{F^{(j)}(z[j-1], z; t)\}|_{z=z_{j+1}\sigma+z_j(1-\sigma)} d\sigma,$$

we have

$$\sup_{z[j+1] \in K[j+1]} |t^s F^{(j+1)}(z[j+1]; t)| \to 0 \qquad (t \to 0)$$

for every compact subset $K[j+1]$ of $\Omega_s[j+1]$. □

Proposition 3.6. *Let $\mu_1, \dots, \mu_r \in \mathcal{O}(\Omega)$ and let $h \in \boldsymbol{N}$.*

(1) *If $g(t, x) \in \mathcal{F}_\Omega^{(h)}[\mu_1, \dots, \mu_r]$, then*

$$\vartheta g(t, x) \in \mathcal{F}_\Omega^{(h)}[\mu_1, \dots, \mu_r].$$

(2) *If $g(t, x) \in \mathcal{F}_\Omega^{(h)}[\mu_1, \dots, \mu_r]$ and $v(x) \in \mathcal{O}(\Omega)$, then*

$$v(x)g(t, x) \in \mathcal{F}_\Omega^{(h)}[\mu_1, \dots, \mu_r].$$

(3) *If $g(t, x) \in \mathcal{F}_\Omega^{(h)}[\mu_1, \dots, \mu_r]$, then*

$$\partial_{x_i} g(t, x) \in \mathcal{F}_\Omega^{(h+1)}[\mu_1, \dots, \mu_r] \quad (1 \leq i \leq n).$$

If $\mu_1(x), \dots, \mu_r(x)$ are all constants, in addition, then

$$\partial_{x_i} g(t,x) \in \mathcal{F}_\Omega^{(h)}[\mu_1,\dots,\mu_r] \quad (1 \le i \le n).$$

Proof. (1) follows from Proposition 3.4 (1). (2) follows from the definition. (3) follows from Lemma 3.1 and the definition. □

Now, we consider the key equation

$$\mathcal{C}(x;\vartheta)v = g(t,x), \tag{3.2}$$

where $\mathcal{C}(x;\lambda) \in \mathcal{O}(\Omega)[\lambda]$ and $g \in \mathcal{F}_\Omega^{(h)}[\mu_1,\dots,\mu_r]$.

Proposition 3.7. *Let* $\mu_1,\dots,\mu_r \in \mathcal{O}(\Omega)$ *and* $r' \in \boldsymbol{N}$. *Put* $\mu_0(x) \equiv 0$. *Assume that* $\mathcal{C}(x;\lambda)$ *can be decomposed as*

$$\mathcal{C}(x;\lambda) = \prod_{l=1}^{r'} \bigl(\lambda - \mu_{j_l}(x)\bigr) \cdot \mathcal{D}(x;\lambda),$$

where

(a) $\{j_1,\dots,j_{r'}\} \subset \{1,\dots,r\}$,

(b) $\mathcal{D}(x;\lambda) \in \mathcal{O}(\Omega)[\lambda]$,

(c) $\mathcal{D}(x;\mu_l(x)) \neq 0$ *on* Ω $(0 \le l \le r)$.

Then, for $h \in \boldsymbol{N}$ *and for every* $g(t,x) \in \mathcal{F}_\Omega^{(h)}[\mu_1,\dots,\mu_r]$, *there exists a solution* $v(t,x) \in \mathcal{F}_\Omega^{(h+r')}[\mu_1,\dots,\mu_r]$ *of the equation* (3.2).

Proof. By Proposition 3.4 (2), we have a solution $w \in \mathcal{F}_\Omega^{(h)}[\mu_1,\dots,\mu_r]$ of $\mathcal{D}(x;\vartheta)w = g(t,x)$. By Proposition 3.3, we also have a solution $v \in \mathcal{F}_\Omega^{(h+r')}[\mu_1,\dots,\mu_r]$ of

$$\prod_{l=1}^{r'} \bigl(\vartheta - \mu_{j_l}(x)\bigr)v = w(t,x). \qquad \square$$

By a repeated use of Proposition 3.7, we prove Theorem 2.9 in the next section.

4. Proof of Main Theorems

In this section, we prove Theorems 2.9 and 2.12.

Proof of Theorem 2.9. Put $G(t,x) := \sum_{j=0}^{\omega(P)-1}(g_j(x)/j!)t^j$ and $u = G(t,x) + t^{\omega(P)}\widetilde{u}$. If we put $\widetilde{f} := f - P(G)$ and $\widetilde{P}(\widetilde{u}) := P(t^{\omega(P)}\widetilde{u})$, then the equation $Pu = f$ is equivalent to $\widetilde{P}\widetilde{u} = \widetilde{f}$. The operator $\widetilde{P} = P \circ t^{\omega(P)}$ is an operator of the form

(2.1) with $\omega(\widetilde{P}) = 0$, and $\widetilde{P}$ satisfies the conditions (B1)–(B3). Further, $\mathcal{C}^{(\widetilde{P})}(x;\lambda) = \mathcal{C}^{(P)}(x;\lambda+\omega(P))$, and hence $\widetilde{P}$ satisfies the condition (E). Assumptions (a)–(c) also hold for $\widetilde{P}$. Thus, without loss of generality, we may assume that $\omega(P) = 0$.

By the condition (B1), we can expand P formally with respect to t as

$$P = \mathcal{C}^{(P)}(x;\vartheta) + \sum_{l=1}^{\infty} t^l B_l(x,\partial_x;\vartheta),$$

where $B_l(x,\partial_x;\vartheta) = \sum_{j+|\alpha|\le m} b_{l,j,\alpha}(x)\partial_x^\alpha \vartheta^j$ with $b_{l,j,\alpha}(x) \in \mathcal{O}(\Omega)$. By Proposition 3.6, we have

$$g(t,x) \in \mathcal{F}^{(h)}_{\Omega_0}[\mu_1,\ldots,\mu_r] \Longrightarrow B_l(x,\partial_x;\vartheta)g(t,x) \in \mathcal{F}^{(h+m)}_{\Omega_0}[\mu_1,\ldots,\mu_r].$$

Substituting $f = \sum_{p=0}^{\infty} t^p f_p(x)$ and $u = \sum_{p=0}^{\infty} t^p v_p(t,x)$ into $Pu = f$, we get infinite number of equations

$$(\mathrm{R})_p \qquad \mathcal{C}^{(P)}(x;\vartheta+p)v_p(t,x) = f_p(x) - \sum_{l=1}^{p} B_l(x,\partial_x;\vartheta+p-l)v_{p-l}(t,x)$$

$(p = 0,1,\ldots)$. (Note that $\vartheta(t^p v) = t^p(\vartheta+p)v$.)

For $p \in \boldsymbol{N}$, put $r_p := \#\{j \in \{1,\ldots,r\} : \lambda_j(0) = p\}$, where $\#A$ denotes the cardinal of a set A, and put $R_p := \sum_{l=0}^{p} r_l$. Note that $R_p \le r$ for every $p \in \boldsymbol{N}$.

First, consider the equation $(\mathrm{R})_0$. We have $f_0(x) \in \mathcal{O}(\Omega_0) = \mathcal{F}^{(0)}_{\Omega_0}[\mu_1,\ldots,\mu_r]$. We can apply Proposition 3.7 on Ω_0 to $\mathcal{C}(x;\lambda) := \mathcal{C}^{(P)}(x;\lambda)$, by taking $r' := r_0$, considering $j_1,\ldots,j_{r'}$ as those j that satisfy $\lambda_j(0) = 0$, and putting $\mathcal{D}(x;\lambda) := \prod_{j:\lambda_j(0)\ne 0}\big(\lambda - \mu_j(x) - \lambda_j(0)\big)\cdot \mathcal{D}^{(P)}(x;\lambda)$. Hence, we can get a solution $v_0(t,x) \in \mathcal{F}^{(r_0)}_{\Omega_0}[\mu_1,\ldots,\mu_r]$ of $(\mathrm{R})_0$.

Next, assume that $v_p(t,x) \in \mathcal{F}^{(R_p+mp)}_{\Omega_0}[\mu_1,\ldots,\mu_r]$ are solutions of $(\mathrm{R})_p$ $(0 \le p \le q-1)$. We have

$$f_q(x) - \sum_{l=1}^{q} B_l(x,\partial_x;\vartheta+q-l)v_{q-l}(t,x) \in \mathcal{F}^{(R_{q-1}+m(q-1)+m)}_{\Omega_0}[\mu_1,\ldots,\mu_r].$$

We can apply Proposition 3.7 on Ω_0 to $\mathcal{C}(x;\lambda) := \mathcal{C}^{(P)}(x;\lambda+q)$, by taking $r' := r_q$, considering $j_1,\ldots,j_{r'}$ as those j that satisfy $\lambda_j(0) = q$, and putting $\mathcal{D}(x;\lambda) := \prod_{j:\lambda_j(0)\ne q}\big(\lambda - \mu_j(x) - \lambda_j(0) + q\big)\cdot \mathcal{D}^{(P)}(x;\lambda+q)$. Hence, we can get a solution $v_q(t,x) \in \mathcal{F}^{(R_{q-1}+m(q-1)+m+r_q)}_{\Omega_0}[\mu_1,\ldots,\mu_r]$ of $(\mathrm{R})_q$.

Thus, we get a formal solution $u = \sum_{p=0}^{\infty} t^p v_p(t,x)$ of $Pu = f$, where $v_p \in \mathcal{F}^{(R_p+mp)}_{\Omega_0}[\mu_1,\ldots,\mu_r] \subset \mathcal{F}^{(r+mp)}_{\Omega_0}[\mu_1,\ldots,\mu_r]$.

If λ_j are all constants, then $\mu_j(x) \equiv 0$ $(0 \le j \le r)$, and we have

$$g(t,x) \in \mathcal{F}^{(h)}_{\Omega}[0] \Longrightarrow B_l(x,\partial_x;\vartheta)g(t,x) \in \mathcal{F}^{(h)}_{\Omega}[0]$$

by the last comment in Proposition 3.6 (3). Hence, we can take $v_p \in \mathcal{F}_{\Omega_0}^{(R_p)}[0] \subset \mathcal{F}_{\Omega_0}^{(r)}[0]$ in the argument above. Since $\mathcal{F}_{\Omega_0}^{(r)}[0] = \oplus_{l=0}^{r} \mathcal{O}(\Omega_0)(\log t)^l$ as is stated in Definition 2.6, we get the desired results. □

Proof of Theorem 2.12. We have only to construct $v \in C^{\omega(P)-1}([0,T]; \mathcal{O}(\Omega_0))$ such that $f - Pv \in C_{flat}^{\infty}([0,T]; \mathcal{O}(\Omega_0))$ and that the asymptotic expansion of v is the formal solution given in Theorem 2.9. In fact, if we can construct such v, then by the assumption (F), we can take $w \in C_{flat}^{\infty}([0,T_0]; \mathcal{O}(\Omega_0'))$ such that $Pw = f - Pv$. Thus, we get the desired exact solution $u := v + w$ of (CP).

Take $\psi(t) \in C^{\infty}[0,\infty)$ such that $\psi(t) = 1$ for $[0,1/2]$ and $\psi(t) = 0$ for $[1,\infty)$. For a formal series $u = \sum_{j=0}^{\omega(P)-1} (g_j(x)/j!)t^j + \sum_{p=0}^{\infty} t^{\omega(P)+p} v_p(t,x) =: \sum_{l=0}^{\infty} u_l(t,x)t^l$ given in Theorem 2.9, we consider

$$v := \sum_{l=0}^{\infty} u_l(t,x)t^l \psi(t/\epsilon_l) \tag{4.2}$$

for suitably chosen $\epsilon_l > 0$. Note that $u_l \in \mathcal{F}_{\Omega_0}^{(\infty)}[\mu_1, \dots, \mu_r]$ and that $u_l t^l \psi(t/\epsilon_l) \in C_{flat}^{l-1}([0,T]; \mathcal{O}(\Omega))$ ($l \in \boldsymbol{N}$) by Corollary 2.8. Take an increasing sequence $\{U_n\}_{n \in \boldsymbol{N}}$ of subdomains of Ω_0 such that $K_n := \overline{U_n}$ are compact subsets of Ω_0 and $\cup_n U_n = \Omega_0$. Put $||w||_n := \sup_{z \in K_n} |w(z)|$. For every p, l and n, the function $t^{p+1}\partial_t^p u_l(t,x)$ is bounded on $(0,T] \times K_n$, by Proposition 2.7 and the condition (c) in Theorem 2.9. Since for every h and q, there holds $\sup_{0 \le t \le T} |t^h \partial_t^q \{\psi(t/\epsilon)\}| \le C\epsilon^{h-q}$ with some constant C independent of ϵ, we can easily show that for every l and k with $0 \le k \le l-1$, there exists a constant $C_{l,k}$ such that

$$\sup_{0 \le t \le T} ||\partial_t^k \big(u_l(t,\cdot)t^l \psi(t/\epsilon)\big)||_l \le C_{l,k}\epsilon^{l-k-1} \quad \text{for every } \epsilon > 0.$$

Hence, for every $l \ge 2$, we can take $\epsilon_l > 0$ such that

$$\sum_{k=0}^{l-2} \sup_{0 \le t \le T} ||\partial_t^k \big(u_l(t,\cdot)t^l \psi(t/\epsilon_l)\big)||_l \le \left(\frac{1}{2}\right)^l. \tag{4.3}$$

Put

$$v_N := \sum_{l=0}^{N} u_l(t,x)t^l \psi(t/\epsilon_l), \ r_N := \sum_{l=N+1}^{\infty} u_l(t,x)t^l \psi(t/\epsilon_l) \quad (N \ge \omega(P)+1).$$

By the estimate (4.3), r_N converges in $C_{flat}^{N-1}([0,T]; \mathcal{O}(U_{N+1}))$ for every N. By the construction of u_l in the proof of Theorem 2.9, there holds

$$f - Pv_N \in C_{flat}^{N-\omega(P)}([0,T]; \mathcal{O}(\Omega_0)).$$

Hence, $v = v_N + r_N$ satisfies

$$v \in C^{\omega(P)-1}([0,T];\mathcal{O}(U_{N+1})), \quad f - Pv \in C_{flat}^{N-\omega(P)-1}([0,T];\mathcal{O}(U_{N+1}))$$

for every N. Thus, we get

$$v \in C^{\omega(P)-1}([0,T];\mathcal{O}(\Omega_0)), \quad f - Pv \in C_{flat}^{\infty}([0,T];\mathcal{O}(\Omega_0)). \qquad \square$$

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