Kumano-go, N.
Osaka J. Math.
35 (1998), 357–380

A PROOF OF H. KUMANO-GO–TANIGUCHI THEOREM FOR MULTI-PRODUCTS OF FOURIER INTEGRAL OPERATORS

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(Received February 25, 1997)

0. Introduction

We denote the Fourier integral operators on $\boldsymbol{R}^n$ with phase function ϕ_j and symbol $p_j \in \boldsymbol{S}^{m_j}_\rho$ by $I(\phi_j, p_j)$, $j = 1, 2, \ldots, L+1$. If all the canonical maps ω_j associated with phase functions ϕ_j are sufficiently close to the identity, the composite canonical map $\omega_{L+1}\omega_L \cdots \omega_1$ is also near the identity. Moreover, we have

$$I(\phi, q) = I(\phi_{L+1}, p_{L+1}) I(\phi_L, p_L) \cdots I(\phi_1, p_1), \tag{0.1}$$

for some phase function ϕ and some symbol $q \in \boldsymbol{S}_\rho^{\sum_{j=1}^{L+1} m_j}$ (cf. L. Hörmander [6]). Here the correspondence of the symbols $(p_{L+1}, p_L, \ldots, p_1) \to q$ is multi-linear. In [9], [10] and [12], H. Kumano-go–Taniguchi theorem gives the following estimate for the symbol q; that is, for any non-negative integers l, l', there exist a positive constant $C_{l,l'}$ and positive integers l_1, l'_1 such that

$$|q|_{l,l'}^{\left(\sum_{j=1}^{L+1} m_j\right)} \le (C_{l,l'})^L \prod_{j=1}^{L+1} |p_j|_{l_1,l'_1}^{(m_j)}, \tag{0.2}$$

where $|\cdot|_{l,l'}^{(m)}$ denotes the semi-norm of $\boldsymbol{S}^m_\rho$.

This estimate is useful in the calculus of Fourier integral operators. In [9], [10] and [12], this estimate was applied to construct a fundamental solution for hyperbolic systems. Slight modification of this estimate was applied to construct a fundamental solution for Schrödinger equations (cf. D. Fujiwara [1]–[4], H. Kitada and H. Kumano-go [8], N. Kumano-go [11]). However, in their proofs, they used the inverse of the Fourier integral operators whose symbols are equal to 1. Therefore, the canonical maps associated with phase functions ϕ_j must be very close to the identity. Recently, in [5], D. Fujiwara, N. Kumano-go and K. Taniguchi have given a more direct proof and relaxed the condition for the canonical maps associated with phase functions ϕ_j in the case for Schrödinger equations. However they are not successful in the original case for hyperbolic systems. The aim of this paper is to give a proof similar to theirs and to relax the condition for the canonical maps associated with phase functions ϕ_j in the original case for hyperbolic systems.

ACKNOWLEDGEMENTS. The author expresses his sincere gratitude to Professor Kiyoomi Kataoka and Professor Hikosaburo Komatsu for helpful discussions and continuous encouragement. Special thanks go to Professor Daisuke Fujiwara and Professor Kazuo Taniguchi for stimulating discussions. He also would like to thank the referee whose helpful criticism of the manuscript resulted in a number of improvements.

1. Statement of results

In order to state our main theorems, we recall some definitions for Fourier integral operator in H. Kumano-go and K. Taniguchi [9], [10] and [12].

DEFINITION 1.1. Let $m \in \boldsymbol{R}$ and $1/2 \leq \rho \leq 1$. We say that a C^∞-function $p(x,\xi)$ on $\boldsymbol{R}^n_x \times \boldsymbol{R}^n_\xi$ belongs to the class of symbols $\boldsymbol{S}^m_\rho$, if, for any α, β, there exists a positive constant $C_{\alpha,\beta}$ such that

$$|\partial_x^\beta \partial_\xi^\alpha p(x,\xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m+(1-\rho)|\beta|-\rho|\alpha|}, \tag{1.1}$$

where $\langle \xi \rangle = \sqrt{1+|\xi|^2}$.

REMARK. For $p \in \boldsymbol{S}^m_\rho$, we define semi-norms $|p|^{(m)}_{l,l'}$, $l, l' = 0, 1, 2, \ldots$ by

$$|p|^{(m)}_{l,l'} = \max_{|\alpha| \leq l, |\beta| \leq l'} \sup_{(x,\xi)} \frac{|\partial_x^\beta \partial_\xi^\alpha p(x,\xi)|}{\langle \xi \rangle^{m+(1-\rho)|\beta|-\rho|\alpha|}}. \tag{1.2}$$

Then $\boldsymbol{S}^m_\rho$ is a Fréchet space with these semi-norms.

DEFINITION 1.2. Let $\{\kappa_l\}_{l=0}^\infty$ be an increasing sequence of positive constants and $t > 0$. We say that a real-valued C^∞-function $\phi(x,\xi)$ on $\boldsymbol{R}^n_x \times \boldsymbol{R}^n_\xi$ belongs to the class of phase functions $\boldsymbol{P}_\rho(t, \{\kappa_l\}_{l=0}^\infty)$, if $\phi(x,\xi)$ satisfies the following:

$$|\partial_x^\beta \partial_\xi^\alpha \phi(x,\xi)| \leq \kappa_{|\alpha+\beta|} t \langle \xi \rangle^{1-|\alpha|} \quad (|\alpha+\beta| \leq 1), \tag{1.3}$$

$$|\partial_x^\beta \partial_\xi^\alpha \phi(x,\xi)| \leq \kappa_{|\alpha+\beta|} t \langle \xi \rangle^{2\rho-1+(1-\rho)|\beta|-\rho|\alpha|} \quad (|\alpha+\beta| \geq 2). \tag{1.4}$$

REMARK. Usually, "phase function" refers to $(x-y)\xi + \phi(x,\xi)$ in (1.5). However, in the present paper, our phase functions will always be of the form $(x-y)\xi + \phi(x,\xi)$. Thus, in the present paper, we call $\phi(x,\xi)$ "phase function".

DEFINITION 1.3. Let $\phi \in \boldsymbol{P}_\rho(t, \{\kappa_l\}_{l=0}^\infty)$ and $p \in \boldsymbol{S}^m_\rho$. We define the Fourier integral operator $I(\phi, p)$ with phase function ϕ and symbol p by

$$I(\phi,p)u(x) = \int_{\boldsymbol{R}^{2n}} e^{i\{(x-y)\xi+\phi(x,\xi)\}} p(x,\xi) u(y) dy đ\xi \quad (đ\xi = (2\pi)^{-n} d\xi), \tag{1.5}$$

for $u \in \mathcal{S}$, where $\mathcal{S}$ denotes the Schwartz class of rapidly decreasing C^∞-functions on $\boldsymbol{R}^n$, and i denotes $\sqrt{-1}$.

The integrals of the right hand side do not necessarily converge absolutely. We understand integrals of this type as oscillatory integrals (cf. H. Kumano-go [9]).

Let $I(\phi_j, p_j)$, $j = 1, 2, \ldots, L+1$ be Fourier integral operators. Then, the composite of these Fourier integral operators is given by

$$
\begin{aligned}
(1.6) \qquad & I(\phi_{L+1}, p_{L+1}) I(\phi_L, p_L) \cdots I(\phi_1, p_1) u(x_{L+1}) \\
&= \int_{\boldsymbol{R}^{2n}} e^{i(x_{L+1} - x_0)\xi_0} K(x_{L+1}, \xi_0) u(x_0) dx_0 đ\xi_0,
\end{aligned}
$$

where

$$
(1.7) \qquad K(x_{L+1}, \xi_0) = \int_{\boldsymbol{R}^{2nL}} e^{i\Phi} \prod_{j=1}^{L+1} p_j(x_j, \xi_{j-1}) \prod_{j=1}^{L} dx_j đ\xi_j,
$$

and

$$
(1.8) \qquad \Phi = \sum_{j=1}^{L} (x_{j+1} - x_j)(\xi_j - \xi_0) + \sum_{j=1}^{L+1} \phi_j(x_j, \xi_{j-1}).
$$

In order to discuss the oscillatory integrals in (1.7) more generally, we will consider oscillatory integrals in the following form:

$$
\begin{aligned}
(1.9) \qquad & \mathbb{I}(\Phi, p)(x_{L+1}, \xi_0) \\
&= \int_{\boldsymbol{R}^{2nL}} e^{i\Phi} p(x_{L+1}, \xi_L, x_L, \ldots, \xi_1, x_1, \xi_0) \prod_{j=1}^{L} dx_j đ\xi_j,
\end{aligned}
$$

which is defined by the multiple symbol $p = p(x_{L+1}, \xi_L, x_L, \ldots, \xi_1, x_1, \xi_0)$ in $\boldsymbol{S}_\rho^{\widetilde{m}_{L+1}}$. Here, $\boldsymbol{S}_\rho^{\widetilde{m}_{L+1}}$ is as follows.

DEFINITION 1.4. Let $\widetilde{m}_{L+1} = (m_{L+1}, m_L, \ldots, m_1) \in \boldsymbol{R}^{L+1}$ and $1/2 \le \rho \le 1$. We say that a C^∞-function $p = p(x_{L+1}, \xi_L, x_L, \ldots, \xi_1, x_1, \xi_0)$ on $\boldsymbol{R}^{2n(L+1)}$ belongs to the class of multiple symbols $\boldsymbol{S}_\rho^{\widetilde{m}_{L+1}}$, if, for any $\widetilde{\alpha} = (\alpha_L, \alpha_{L-1}, \ldots, \alpha_0)$ and $\widetilde{\beta} = (\beta_{L+1}, \beta_L, \ldots, \beta_1)$, there exists a positive constant $C_{\widetilde{\alpha},\widetilde{\beta}}$ such that

$$
\begin{aligned}
(1.10) \qquad & \left| \left(\prod_{j=1}^{L+1} \partial_{x_j}^{\beta_j} \partial_{\xi_{j-1}}^{\alpha_{j-1}} \right) p(x_{L+1}, \xi_L, x_L, \ldots, \xi_1, x_1, \xi_0) \right| \\
&\le C_{\widetilde{\alpha},\widetilde{\beta}} \prod_{j=1}^{L+1} \langle \xi_{j-1} \rangle^{m_j + (1-\rho)|\beta_j| - \rho|\alpha_{j-1}|}.
\end{aligned}
$$

Remark.

(1) For $p \in \boldsymbol{S}_\rho^{\widetilde{m}_{L+1}}$, we define semi-norms $|p|_{l,l'}^{(\widetilde{m}_{L+1})}$, $l, l' = 0, 1, 2, \dots$ by

$$(1.11)\qquad |p|_{l,l'}^{(\widetilde{m}_{L+1})} = \max_{\substack{|\alpha_{j-1}| \le l,\\ |\beta_j| \le l',\\ j=1,2,\cdots,L+1}} \sup_{\boldsymbol{R}^{2n(L+1)}} \frac{\left|\left(\prod_{j=1}^{L+1} \partial_{x_j}^{\beta_j} \partial_{\xi_{j-1}}^{\alpha_{j-1}}\right) p\right|}{\prod_{j=1}^{L+1} \langle \xi_{j-1} \rangle^{m_j + (1-\rho)|\beta_j| - \rho|\alpha_{j-1}|}}.$$

Then $\boldsymbol{S}_\rho^{\widetilde{m}_{L+1}}$ is a Fréchet space with these semi-norms.

(2) For $p_j \in \boldsymbol{S}_\rho^{m_j}$, $j = 1, 2, \dots, L+1$, if we set

$$(1.12)\qquad p = \prod_{j=1}^{L+1} p_j(x_j, \xi_{j-1}),$$

then we have $p \in \boldsymbol{S}_\rho^{\widetilde{m}_{L+1}}$. Furthermore we have

$$(1.13)\qquad |p|_{l,l'}^{(\widetilde{m}_{L+1})} \le \prod_{j=1}^{L+1} |p_j|_{l,l'}^{(m_j)}.$$

Now, our first main theorem is the following:

Theorem 1.5. *Let $\{\kappa_l\}_{l=0}^\infty$ be an increasing sequence of positive constants and $M \ge 0$. Set $T = \min\{1/(7\sqrt{n}\kappa_1), 1/(4n\kappa_2)\}$. Then there exists a positive constant C such that*

$$(1.14)\qquad |\mathbb{I}(\Phi, p)(x_{L+1}, \xi_0)| \le C^L |p|_{l_0, l_0'}^{(\widetilde{m}_{L+1})} \langle \xi_0 \rangle^{\sum_{j=1}^{L+1} m_j},$$

for $\sum_{j=1}^{L+1} t_j \le T$, $\sum_{j=1}^{L+1} |m_j| \le M$, $p \in \boldsymbol{S}_\rho^{\widetilde{m}_{L+1}}$ and $\phi_j \in \boldsymbol{P}_\rho(t_j, \{\kappa_l\}_{l=0}^\infty)$, where $l_0 = n+1$, $l_0' = [2M] + 2n + 1$, and the positive constant C depends only on M, $\{\kappa_l\}_{l=0}^\infty$ and n, not L.

In order to state our second main theorem, we state the following proposition.

Proposition 1.6. *Let $\{\kappa_l\}_{l=0}^\infty$ be an increasing sequence of positive constants.*

(1) *Assume that $\sum_{j=1}^{L+1} t_j \le 1/(4n\kappa_2)$ and $\phi_j \in \boldsymbol{P}_\rho(t_j, \{\kappa_l\}_{l=0}^\infty)$, $j = 1, 2, \dots, L+1$. Then, for $(x, \xi) \in \boldsymbol{R}^{2n}$, the equations*

$$(1.15)\qquad \begin{cases} 0 = -(x_j - x_{j+1}) + \partial_{\xi_j} \phi_{j+1}(x_{j+1}, \xi_j), \\ 0 = -(\xi_j - \xi_{j-1}) + \partial_{x_j} \phi_j(x_j, \xi_{j-1}), \\ j = 1, 2, \dots, L, \quad x_{L+1} = x, \quad \xi_0 = \xi, \end{cases}$$

have a unique solution $\{x_j, \xi_j\}_{j=1}^{L} = \{x_j^*, \xi_j^*\}_{j=1}^{L}(x, \xi)$.

(2) *Let* Φ^* *be the function defined by*

$$\Phi^*(x,\xi) = \sum_{j=1}^{L}(x_{j+1}^* - x_j^*)(\xi_j^* - \xi_0^*) + \sum_{j=1}^{L+1}\phi_j(x_j^*, \xi_{j-1}^*), \tag{1.16}$$

with $x_{L+1}^* = x$ *and* $\xi_0^* = \xi$.

Then there exists an increasing sequence of positive constants $\{\kappa_l'\}_{l=0}^{\infty}$ *such that*

$$\Phi^* \in \boldsymbol{P}_\rho\left(\sum_{j=1}^{L+1} t_j, \{\kappa_l'\}_{l=0}^{\infty}\right), \tag{1.17}$$

for $\sum_{j=1}^{L+1} t_j \le 1/(4n\kappa_2)$ *and* $\phi_j \in \boldsymbol{P}_\rho(t_j, \{\kappa_l\}_{l=0}^{\infty})$, *where the increasing sequence of positive constants* $\{\kappa_l'\}_{l=0}^{\infty}$ *depends only on* $\{\kappa_l\}_{l=0}^{\infty}$ *and* n, *not* L.

Our second main theorem is the following:

Theorem 1.7. *Let* $\{\kappa_l\}_{l=0}^{\infty}$ *be an increasing sequence of positive constants and* $M \ge 0$. *Set* $T = \min\{1/(7\sqrt{n}\kappa_1), 1/(4n\kappa_2)\}$.

(1) *For* $\sum_{j=1}^{L+1} t_j \le T$, $p \in \boldsymbol{S}_\rho^{\widetilde{m}_{L+1}}$ *and* $\phi_j \in \boldsymbol{P}_\rho(t_j, \{\kappa_l\}_{l=0}^{\infty})$, *set*

$$q(x_{L+1}, \xi_0) = e^{-i\Phi^*(x_{L+1},\xi_0)}\mathbb{I}(\Phi, p)(x_{L+1}, \xi_0). \tag{1.18}$$

Then we have $q \in \boldsymbol{S}_\rho^{\sum_{j=1}^{L+1} m_j}$.

(1) *For any non-negative integers* l, l', *there exists a positive constant* $C_{l,l'}$ *such that*

$$|q|_{l,l'}^{(\sum_{j=1}^{L+1} m_j)} \le (C_{l,l'})^L |p|_{l_1,l_1'}^{(\widetilde{m}_{L+1})}, \tag{1.19}$$

for $\sum_{j=1}^{L+1} t_j \le T$, $\sum_{j=1}^{L+1}|m_j| \le M$, $p \in \boldsymbol{S}_\rho^{\widetilde{m}_{L+1}}$ *and* $\phi_j \in \boldsymbol{P}_\rho(t_j, \{\kappa_l\}_{l=0}^{\infty})$, *where* $l_1 = n + 1 + 2l + 2l'$, $l_1' = [2M] + 2n + 1 + 2l + 3l'$, *and the positive constant* $C_{l,l'}$ *depends only on* M, $\{\kappa_l\}_{l=0}^{\infty}$ *and* n, *not* L.

From the theorem above, we can relax the condition for the canonical maps associated with phase functions ϕ_j of H. Kumano-go–Taniguchi theorem in the following form.

Theorem 1.8. *Let* $\{\kappa_l\}_{l=0}^{\infty}$ *be an increasing sequence of positive constants and* $M \ge 0$. *Set* $T = \min\{1/(7\sqrt{n}\kappa_1), 1/(4n\kappa_2)\}$.

(1) *For* $\sum_{j=1}^{L+1} t_j \le T$, $p_j \in \boldsymbol{S}_\rho^{m_j}$ *and* $\phi_j \in \boldsymbol{P}_\rho(t_j, \{\kappa_l\}_{l=0}^{\infty})$, *there exists a symbol* $q \in \boldsymbol{S}_\rho^{\sum_{j=1}^{L+1} m_j}$ *such that*

$$(1.20)\qquad I(\Phi^*,q) = I(\phi_{L+1},p_{L+1})I(\phi_L,p_L)\dots I(\phi_1,p_1).$$

(1) *For any non-negative integers l, l', there exists a positive constant $C_{l,l'}$ such that*

$$(1.21)\qquad |q|_{l,l'}^{(\sum_{j=1}^{L+1} m_j)} \le (C_{l,l'})^L \prod_{j=1}^{L+1} |p_j|_{l_1,l_1'}^{(m_j)},$$

for $\sum_{j=1}^{L+1} t_j \le T$, $\sum_{j=1}^{L+1} |m_j| \le M$, $p_j \in \boldsymbol{S}_\rho^{m_j}$ and $\phi_j \in \boldsymbol{P}_\rho(t_j, \{\kappa_l\}_{l=0}^\infty)$, where $l_1 = n+1+2l+2l'$, $l_1' = [2M]+2n+1+2l+3l'$, and the positive constant $C_{l,l'}$ depends only on M, $\{\kappa_l\}_{l=0}^\infty$ and n, not L.

REMARK. The condition $\sum_{j=1}^{L+1} t_j \le T$ implies how close to the identity the canonical maps associated with phase functions ϕ_j need to be. In our proof, the right hand side T of this inequality depends only on κ_1, κ_2 and n. However, in the original proof, T depends on κ_1, κ_2, …, κ_k and n, with some large integer $k > 2$ depending on n. Moreover, T must be chosen very small. Therefore, the canonical maps with phase functions ϕ_j must be very close to the identity.

2. Some Lemmas

In this section, we state two important lemmas needed later. First lemma is found in H. Kumano-go and K. Taniguchi [9], [10].

Lemma 2.1. *Let $A = (a_{jk})$ be an $L \times L$ real matrix. If there exists a positive constant $0 \le c < 1$ such that*

$$(2.1)\qquad \sum_{k=1}^{L} |a_{jk}| \le c,$$

for any $j = 1,2,\dots,L$, then we have

$$(2.2)\qquad (1-c)^L \le \det(I_L - A) \le (1+c)^L,$$

where I_L denotes the $L \times L$ unit matrix.

Proof. By induction. See Proposition 5.3 in Chapter 10 §5 of H. Kumano-go [9]. □

Second lemma is slight modification of Proposition 3.3 in D. Fujiwara, N. Kumano-go and K. Taniguchi [5].

Let N and L be positive integers and $x \in \boldsymbol{R}^N$. For $j = 1,2,\dots,L+1$, let P_j be the first-order partial differential operator with smooth coefficients given by

$$P_j = \sum_{\beta_j \le \gamma_j, |\beta_j| \le 1} a_{j,\beta_j}(x) \partial_x^{\beta_j}, \tag{2.3}$$

where $\gamma_j \in \{0,1\}^N \subset \boldsymbol{N}_0^N$ and $a_{j,\beta_j}(x) \in C^\infty(\boldsymbol{R}^N)$. Furthermore, we assume the following properties:

1° There exists a positive integer Γ independent of N and of L such that

$$|\gamma_j| \le \Gamma, \tag{2.4}$$

for $j = 1, 2, \ldots, L+1$.

2° There exists a positive integer K independent of N and of L such that

$$\sharp\left\{ j = 1, 2, \ldots, k; \quad \partial_x^{\beta_{k+1}} a_{j,\beta_j}(x) \not\equiv 0 \right\} \le K, \tag{2.5}$$

for $k = 1, 2, \cdots, L$, $\beta_j \le \gamma_j$, $|\beta_j| \le 1$, $j = 1, 2, \ldots, k$ and $0 \ne \beta_{k+1} \le \gamma_{k+1}$.

Then we get the following lemma:

Lemma 2.2.

(1) *The product of operators $P_{L+1} P_L \cdots P_1$ is of the form*

$$\begin{aligned} &P_{L+1} P_L \cdots P_1 \\ &= \sum'_{\{\beta_j\}_{j=1}^{L+1}} \sum''_{\{\alpha_j\}_{j=0}^{L+1}} C(\{\beta_j\}_{j=1}^{L+1}, \{\alpha_j\}_{j=0}^{L+1}) \left(\prod_{j=1}^{L+1} \partial_x^{\alpha_j} a_{j,\beta_j}(x) \right) \partial_x^{\alpha_0}, \end{aligned} \tag{2.6}$$

where $\sum'_{\{\beta_j\}_{j=1}^{L+1}}$ is the summation with respect to $\{\beta_j\}_{j=1}^{L+1}$ such that $\beta_j \le \gamma_j$ and $|\beta_j| \le 1$ for $j = 1, 2, \ldots, L+1$, $\sum''_{\{\alpha_j\}_{j=0}^{L+1}}$ is the summation with respect to $\{\alpha_j\}_{j=0}^{L+1}$ such that $\sum_{j=0}^{L+1} \alpha_j = \sum_{j=1}^{L+1} \beta_j$ and $\alpha_{L+1} = 0$, and $C(\{\beta_j\}_{j=1}^{L+1}, \{\alpha_j\}_{j=0}^{L+1})$ is a non-negative integer.

(2) *Furthermore, there exists a positive integer C independent of N and of L such that*

$$\sum'_{\{\beta_j\}_{j=1}^{L+1}} \sum''_{\{\alpha_j\}_{j=0}^{L+1}} C(\{\beta_j\}_{j=1}^{L+1}, \{\alpha_j\}_{j=0}^{L+1}) \le C^{L+1}. \tag{2.7}$$

We can choose $C \le (1 + \Gamma(K+1))$.

Proof. By induction. Proposition 3.3 in D. Fujiwara, N. Kumano-go and K. Taniguchi [5]. □

3. Proof of Theorem 1.5

In this section, we prove Theorem 1.5.

Proof of Theorem 1.5.

1°. From (1.8), for $j = 1, 2, \dots, L$, we have

$$\partial_{\xi_j}\Phi = -(x_j - x_{j+1}) + \partial_{\xi_j}\phi_{j+1}(x_{j+1}, \xi_j),$$
$$\partial_{x_j}\Phi = -(\xi_j - \xi_{j-1}) + \partial_{x_j}\phi_j(x_j, \xi_{j-1}). \tag{3.1}$$

Set

$$M_j = \frac{1 - i\langle\xi_j\rangle^{1/2}(\partial_{\xi_j}\Phi)\langle\xi_j\rangle^{1/2}\partial_{\xi_j}}{1 + |\langle\xi_j\rangle^{1/2}(\partial_{\xi_j}\Phi)|^2},$$
$$N_j = \frac{1 - i\langle\xi_{j-1}\rangle^{-1/2}(\partial_{x_j}\Phi)\langle\xi_{j-1}\rangle^{-1/2}\partial_{x_j}}{1 + |\langle\xi_{j-1}\rangle^{-1/2}(\partial_{x_j}\Phi)|^2}. \tag{3.2}$$

We denote the adjoint operators of M_j and of N_j respectively by M_j^* and by N_j^*. Then we can write

$$M_j^* = a_j^1(x_{j+1}, \xi_j, x_j)\partial_{\xi_j} + a_j^0(x_{j+1}, \xi_j, x_j),$$
$$N_j^* = b_j^1(\xi_j, x_j, \xi_{j-1})\partial_{x_j} + b_j^0(\xi_j, x_j, \xi_{j-1}), \tag{3.3}$$

where

$$a_j^1(x_{j+1}, \xi_j, x_j) = \frac{i\langle\xi_j\rangle^{1/2}(\partial_{\xi_j}\Phi)}{1 + |\langle\xi_j\rangle^{1/2}(\partial_{\xi_j}\Phi)|^2}\langle\xi_j\rangle^{1/2},$$
$$a_j^0(x_{j+1}, \xi_j, x_j) = \frac{1}{1 + |\langle\xi_j\rangle^{1/2}(\partial_{\xi_j}\Phi)|^2} + \partial_{\xi_j}\left(\frac{i\langle\xi_j\rangle^{1/2}(\partial_{\xi_j}\Phi)}{1 + |\langle\xi_j\rangle^{1/2}(\partial_{\xi_j}\Phi)|^2}\langle\xi_j\rangle^{1/2}\right), \tag{3.4}$$

and

$$b_j^1(\xi_j, x_j, \xi_{j-1}) = \frac{i\langle\xi_{j-1}\rangle^{-1/2}(\partial_{x_j}\Phi)}{1 + |\langle\xi_{j-1}\rangle^{-1/2}(\partial_{x_j}\Phi)|^2}\langle\xi_{j-1}\rangle^{-1/2},$$
$$b_j^0(\xi_j, x_j, \xi_{j-1}) = \frac{1}{1 + |\langle\xi_{j-1}\rangle^{-1/2}(\partial_{x_j}\Phi)|^2} + \partial_{x_j}\left(\frac{i\langle\xi_{j-1}\rangle^{-1/2}(\partial_{x_j}\Phi)}{1 + |\langle\xi_{j-1}\rangle^{-1/2}(\partial_{x_j}\Phi)|^2}\langle\xi_{j-1}\rangle^{-1/2}\right). \tag{3.5}$$

2°. We note the formula $\langle\xi + \eta\rangle \le |\eta| + \langle\xi\rangle$. Then, when $|\xi_j - \xi_{j-1}| \le (1/2)\langle\xi_{j-1}\rangle$, we have

$$2^{-1}\langle\xi_{j-1}\rangle \le \langle\xi_j\rangle \le 2\langle\xi_{j-1}\rangle. \tag{3.6}$$

And when $|\xi_j - \xi_{j-1}| > (1/2)\langle \xi_{j-1}\rangle$, we have

$$\begin{aligned}(3.7)\qquad |\partial_{x_j}\Phi| &\ge |\xi_j - \xi_{j-1}| - \sqrt{n}\kappa_1 t_j\langle \xi_{j-1}\rangle \\ &\ge (1-2\sqrt{n}\kappa_1 t_j)|\xi_j - \xi_{j-1}| \\ &\ge (1-2\sqrt{n}\kappa_1 T)3^{-1}\langle \xi_j\rangle.\end{aligned}$$

Using (3.6) and (3.7), we get the following estimates for derivatives of b_j^1 and b_j^0: For any α_j, β_j, α_{j-1}, there exists a positive constant $C_{\alpha_j,\beta_j,\alpha_{j-1}}$ independent of j such that

$$\begin{aligned}(3.8)\qquad |\partial_{\xi_j}^{\alpha_j}\partial_{x_j}^{\beta_j}\partial_{\xi_{j-1}}^{\alpha_{j-1}} b_j^1(\xi_j, x_j, \xi_{j-1})| &\le C_{\alpha_j,\beta_j,\alpha_{j-1}}\frac{1}{(1+\langle \xi_{j-1}\rangle^{-1}|\partial_{x_j}\Phi|^2)^{1/2}} \\ &\quad\times \langle \xi_j\rangle^{-|\alpha_j|/2}\langle \xi_{j-1}\rangle^{-1/2+|\beta_j|/2-|\alpha_{j-1}|/2}, \\ |\partial_{\xi_j}^{\alpha_j}\partial_{x_j}^{\beta_j}\partial_{\xi_{j-1}}^{\alpha_{j-1}} b_j^0(\xi_j, x_j, \xi_{j-1})| &\le C_{\alpha_j,\beta_j,\alpha_{j-1}}\frac{1}{(1+\langle \xi_{j-1}\rangle^{-1}|\partial_{x_j}\Phi|^2)^{1/2}} \\ &\quad\times \langle \xi_j\rangle^{-|\alpha_j|/2}\langle \xi_{j-1}\rangle^{|\beta_j|/2-|\alpha_{j-1}|/2}.\end{aligned}$$

Furthermore, we get the following estimates for derivatives of a_j^1 and a_j^0: For any α_j, there exists a positive constant C_{α_j} independent of j such that

$$\begin{aligned}(3.9)\qquad |\partial_{\xi_j}^{\alpha_j} a_j^1(x_{j+1}, \xi_j, x_j)| &\le C_{\alpha_j}\frac{1}{(1+\langle \xi_j\rangle|\partial_{\xi_j}\Phi|^2)^{1/2}}\langle \xi_j\rangle^{1/2-|\alpha_j|/2}, \\ |\partial_{\xi_j}^{\alpha_j} a_j^0(x_{j+1}, \xi_j, x_j)| &\le C_{\alpha_j}\frac{1}{(1+\langle \xi_j\rangle|\partial_{\xi_j}\Phi|^2)^{1/2}}\langle \xi_j\rangle^{-|\alpha_j|/2}.\end{aligned}$$

3°. We take $\chi \in C_0^\infty(\boldsymbol{R}^n)$ such that

$$(3.10)\qquad 0 \le \chi \le 1 \quad\text{and}\quad \chi(x) = \begin{cases} 1 & (|x| \le 1/3) \\ 0 & (|x| \ge 1/2)\end{cases}.$$

For simplicity, when $k > k'$, we set $\prod_{j=k}^{k'} \cdots = 1$.

For $R = 0, 1, 2, \ldots, L$ and $0 = j_0 < j_1 < \ldots < j_R < j_{R+1} = L+1$, let

$$\begin{aligned}(3.11)\qquad \chi_{j_0,j_1,\ldots,j_R} &= \prod_{r=1}^{R+1}\prod_{j=j_{r-1}+1}^{j_r-1}\chi\Big((\xi_j - \xi_{j_{r-1}})/\langle \xi_{j_{r-1}}\rangle\Big) \\ &\quad\times\prod_{r=1}^{R}\Big(1-\chi\Big((\xi_{j_r} - \xi_{j_{r-1}})/\langle \xi_{j_{r-1}}\rangle\Big)\Big).\end{aligned}$$

We divide $\mathbb{I}(\Phi, p)$ into 2^L terms as follows:

$$(3.12)\qquad \mathbb{I}(\Phi, p) = \sum_{R=0}^{L}\ \sum_{0=j_0<j_1<\ldots<j_R<j_{R+1}=L+1}\mathbb{I}(\Phi, \chi_{j_0,j_1,\ldots,j_R}p).$$

4°. We consider $\mathbb{I}(\Phi, \chi_{j_0,j_1,\dots,j_R}p)$. Set $J = [2M] + 2n + 1$. Integrating by parts, we have

$$\mathbb{I}(\Phi, \chi_{j_0,j_1,\dots,j_R}p) = \mathbb{I}(\Phi, p^{\circ}_{j_0,j_1,\dots,j_R}), \tag{3.13}$$

where

$$\begin{aligned} p^{\circ}_{j_0,j_1,\dots,j_R} &= (M_L^*)^{n+1}(M_{L-1}^*)^{n+1}\cdots(M_1^*)^{n+1} \\ &\quad \circ (N_L^*)^J(N_{L-1}^*)^J\cdots(N_1^*)^J\chi_{j_0,j_1,\dots,j_R}p. \end{aligned} \tag{3.14}$$

Therefore, by Lemma 2.2, there exists a positive constant C_1 such that

$$\begin{aligned} &|p^{\circ}_{j_0,j_1,\dots,j_R}| \le (C_1)^L |p|^{(\widetilde{m}_{L+1})}_{n+1,J}\langle\xi_0\rangle^{m_1} \\ &\times \prod_{r=1}^{R+1}\prod_{j=j_{r-1}+1}^{j_r-1}\left\{\frac{1}{(1+\langle\xi_j\rangle|\partial_{\xi_j}\Phi|^2)^{(n+1)/2}}\cdot\frac{1}{(1+\langle\xi_{j-1}\rangle^{-1}|\partial_{x_j}\Phi|^2)^{J/2}}\langle\xi_j\rangle^{m_{j+1}}\right\} \\ &\times\prod_{r=1}^{R}\left\{\frac{1}{(1+\langle\xi_{j_r}\rangle|\partial_{\xi_{j_r}}\Phi|^2)^{(n+1)/2}}\cdot\frac{1}{(1+\langle\xi_{j_r-1}\rangle^{-1}|\partial_{x_{j_r}}\Phi|^2)^{J/2}}\langle\xi_{j_r}\rangle^{m_{j_r+1}}\right\}. \end{aligned} \tag{3.15}$$

5°. For $r = 1, 2, \dots, R+1$ and $j = j_{r-1}+1, j_{r-1}+2, \dots, j_r - 1$, we note that

$$|\xi_j - \xi_{j_{r-1}}| \le \frac{1}{2}\langle\xi_{j_{r-1}}\rangle, \tag{3.16}$$

on the support of $p^{\circ}_{j_0,j_1,\dots,j_R}$. Using the formula $\langle\xi+\eta\rangle \le |\eta| + \langle\xi\rangle$, we have

$$2^{-1}\langle\xi_{j_{r-1}}\rangle \le \langle\xi_j\rangle \le 2\langle\xi_{j_{r-1}}\rangle, \tag{3.17}$$

for $r = 1, 2, \dots, R+1$ and $j = j_{r-1}+1, j_{r-1}+2, \dots, j_r - 1$ on the support of $p^{\circ}_{j_0,j_1,\dots,j_R}$. Therefore, there exists a positive constant C_2 such that

$$\begin{aligned} &|p^{\circ}_{j_0,j_1,\dots,j_R}| \le (C_2)^L |p|^{(\widetilde{m}_{L+1})}_{n+1,J}\langle\xi_0\rangle^{\sum_{j=1}^{L+1}m_j} \\ &\times\prod_{j=j_R+1}^{L}\left\{\frac{1}{(1+\langle\xi_{j_R}\rangle|\partial_{\xi_j}\Phi|^2)^{(n+1)/2}}\cdot\frac{1}{(1+\langle\xi_{j_R}\rangle^{-1}|\partial_{x_j}\Phi|^2)^{J/2}}\right\} \\ &\times\prod_{r=1}^{R}\prod_{j=j_{r-1}+1}^{j_r-1}\left\{\frac{1}{(1+\langle\xi_{j_{r-1}}\rangle|\partial_{\xi_j}\Phi|^2)^{(n+1)/2}}\cdot\frac{1}{(1+\langle\xi_{j_{r-1}}\rangle^{-1}|\partial_{x_j}\Phi|^2)^{(J-2M)/4}}\right\} \\ &\times\prod_{r=1}^{R}\frac{1}{(1+\langle\xi_{j_r}\rangle|\partial_{\xi_{j_r}}\Phi|^2)^{(n+1)/2}}\cdot\prod_{r=1}^{R}\langle\xi_{j_r}\rangle^{-(J-2M)/4} \\ &\times\prod_{r=1}^{R}\frac{\langle\xi_{j_r}\rangle^{\sum_{j=j_r+1}^{j_{r+1}}m_j+(J-2M)/4}\lambda_r(\xi_0)}{\prod_{j=j_{r-1}+1}^{j_r}(1+\langle\xi_{j_{r-1}}\rangle^{-1}|\partial_{x_j}\Phi|^2)^{M+(J-2M)/4}}, \end{aligned} \tag{3.18}$$

where $\lambda_1(\xi_0) = \langle\xi_0\rangle^{-\sum_{j=j_1+1}^{L+1} m_j}$ and $\lambda_r(\xi_0) = 1$ for $r \neq 1$.

6°. For $r = 1, 2, \dots, R$, we note that

$$|\xi_{j_r} - \xi_{j_{r-1}}| \geq \frac{1}{3}\langle\xi_{j_{r-1}}\rangle, \tag{3.19}$$

on the support of $p^{\circ}_{j_0,j_1,\dots,j_R}$. Using the formula $\langle\xi+\eta\rangle \leq |\eta| + \langle\xi\rangle$, we have

$$|\xi_{j_r} - \xi_{j_{r-1}}| \geq \frac{1}{4}\langle\xi_{j_r}\rangle, \tag{3.20}$$

for $r = 1, 2, \dots, R$ on the support of $p^{\circ}_{j_0,j_1,\dots,j_R}$. Furthermore, noting (3.17) and (3.19), we have

$$\begin{aligned}
&\prod_{j=j_{r-1}+1}^{j_r} (1 + \langle\xi_{j_{r-1}}\rangle^{-1}|\partial_{x_j}\Phi|^2)^{1/2} \\
&\geq 2^{-(j_r-j_{r-1})/2} \prod_{j=j_{r-1}+1}^{j_r} (1 + \langle\xi_{j_{r-1}}\rangle^{-1/2}|\partial_{x_j}\Phi|) \\
&\geq 2^{-(j_r-j_{r-1})/2}\langle\xi_{j_{r-1}}\rangle^{-1/2} \sum_{j=j_{r-1}+1}^{j_r} |\partial_{x_j}\Phi| \\
&\geq 2^{-(j_r-j_{r-1})/2}\langle\xi_{j_{r-1}}\rangle^{-1/2} \sum_{j=j_{r-1}+1}^{j_r} \Big(|\xi_j - \xi_{j-1}| - \sqrt{n}\kappa_1 t_j\langle\xi_{j-1}\rangle\Big) \\
&\geq 2^{-(j_r-j_{r-1})/2}\langle\xi_{j_{r-1}}\rangle^{-1/2} \sum_{j=j_{r-1}+1}^{j_r} \Big(|\xi_j - \xi_{j-1}| - 2\sqrt{n}\kappa_1 t_j\langle\xi_{j_{r-1}}\rangle\Big) \\
&\geq 2^{-(j_r-j_{r-1})/2}3^{-1/2}(1 - 6\sqrt{n}\kappa_1 T)|\xi_{j_r} - \xi_{j_{r-1}}|^{1/2},
\end{aligned} \tag{3.21}$$

for $r = 1, 2, \dots, R$ on the support of $p^{\circ}_{j_0,j_1,\dots,j_R}$.

Therefore, there exists a positive constant C_3 such that

$$\begin{aligned}
&|p^{\circ}_{j_0,j_1,\dots,j_R}| \leq (C_3)^L |p|^{(\widetilde{m}_{L+1})}_{n+1,J}\langle\xi_0\rangle^{\sum_{j=1}^{L+1} m_j} \\
&\times \prod_{j=j_R+1}^{L} \left\{\frac{1}{(1+\langle\xi_{j_R}\rangle|\partial_{\xi_j}\Phi|^2)^{(n+1)/2}} \cdot \frac{1}{(1+\langle\xi_{j_R}\rangle^{-1}|\partial_{x_j}\Phi|^2)^{J/2}}\right\} \\
&\times \prod_{r=1}^{R}\prod_{j=j_{r-1}+1}^{j_r-1} \left\{\frac{1}{(1+\langle\xi_{j_{r-1}}\rangle|\partial_{\xi_j}\Phi|^2)^{(n+1)/2}} \cdot \frac{1}{(1+\langle\xi_{j_{r-1}}\rangle^{-1}|\partial_{x_j}\Phi|^2)^{(J-2M)/4}}\right\} \\
&\times \prod_{r=1}^{R} \frac{1}{(1+\langle\xi_{j_r}\rangle|\partial_{\xi_{j_r}}\Phi|^2)^{(n+1)/2}} \cdot \prod_{r=1}^{R}\langle\xi_{j_r}\rangle^{-(J-2M)/4}.
\end{aligned} \tag{3.22}$$

7°. For $r=1,2,\dots,R+1$ and $j=j_{r-1}+1, j_{r-1}+2,\dots,j_r-1$, let

$$z_j=\partial_{\xi_j}\Phi=-(x_j-x_{j+1})+\partial_{\xi_j}\phi_{j+1}(x_{j+1},\xi_j),\qquad (3.23)$$
$$\zeta_j=\partial_{x_j}\Phi=-(\xi_j-\xi_{j-1})+\partial_{x_j}\phi_j(x_j,\xi_{j-1}).$$

For simplicity, we set $k=j_{r-1}+1$, $k'=j_r-1$ and

$$\widetilde{x}_{k,k'}=(x_k,x_{k+1},\dots,x_{k'}),\quad \widetilde{\xi}_{k,k'}=(\xi_k,\xi_{k+1},\dots,\xi_{k'}),\qquad (3.24)$$
$$\widetilde{z}_{k,k'}=(z_k,z_{k+1},\dots,z_{k'}),\quad \widetilde{\zeta}_{k,k'}=(\zeta_k,\zeta_{k+1},\dots,\zeta_{k'}).$$

Then we have

$$\frac{\partial(\widetilde{z}_{k,k'},\widetilde{\zeta}_{k,k'})}{\partial(\widetilde{x}_{k,k'},\widetilde{\xi}_{k,k'})}=-\begin{pmatrix}\Delta_{k'-k+1} & 0\\ 0 & {}^t\Delta_{k'-k+1}\end{pmatrix}+\begin{pmatrix}\Lambda^1_{k,k'} & \Lambda^2_{k,k'}\\ \Lambda^3_{k,k'} & \Lambda^4_{k,k'}\end{pmatrix},\qquad (3.25)$$

where $\Delta_{k'-k+1}$, $\Lambda^1_{k,k'}$, $\Lambda^2_{k,k'}$, $\Lambda^3_{k,k'}$ and $\Lambda^4_{k,k'}$ are $(n(k'-k+1))\times(n(k'-k+1))$ matrices defined by

$$\Delta_{k'-k+1}=\begin{pmatrix} I_n & -I_n & 0 & \dots & 0\\ 0 & I_n & -I_n & \ddots & \vdots\\ 0 & 0 & I_n & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & -I_n\\ 0 & \dots & 0 & 0 & I_n\end{pmatrix},\qquad (3.26)$$

$$\Lambda^1_{k,k'}=\begin{pmatrix} 0 & \partial_{x_{k+1}}\partial_{\xi_k}\phi_{k+1} & 0 & \dots & 0\\ 0 & 0 & \partial_{x_{k+2}}\partial_{\xi_{k+1}}\phi_{k+2} & \ddots & \vdots\\ 0 & 0 & 0 & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & \partial_{x_{k'}}\partial_{\xi_{k'-1}}\phi_{k'}\\ 0 & \dots & 0 & 0 & 0\end{pmatrix},\qquad (3.27)$$

$$\Lambda^2_{k,k'}=\begin{pmatrix} \partial^2_{\xi_k}\phi_{k+1} & 0 & 0 & \dots & 0\\ 0 & \partial^2_{\xi_{k+1}}\phi_{k+2} & 0 & \ddots & \vdots\\ 0 & 0 & \partial^2_{\xi_{k+2}}\phi_{k+3} & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & \dots & 0 & 0 & \partial^2_{\xi_{k'}}\phi_{k'+1}\end{pmatrix},\qquad (3.28)$$

$$(3.29)\qquad \Lambda^3_{k,k'} = \begin{pmatrix} \partial^2_{x_k}\phi_k & 0 & 0 & \dots & 0 \\ 0 & \partial^2_{x_{k+1}}\phi_{k+1} & 0 & \ddots & \vdots \\ 0 & 0 & \partial^2_{x_{k+2}}\phi_{k+2} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0 & \partial^2_{x_{k'}}\phi_{k'} \end{pmatrix},$$

and

$$(3.30)\ \Lambda^4_{k,k'} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \partial_{\xi_k}\partial_{x_{k+1}}\phi_{k+1} & 0 & 0 & \ddots & \vdots \\ 0 & \partial_{\xi_{k+1}}\partial_{x_{k+2}}\phi_{k+2} & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \partial_{\xi_{k'-1}}\partial_{x_{k'}}\phi_{k'} & 0 \end{pmatrix}.$$

Furthermore, we can write

$$(3.31)\qquad \det\frac{\partial(\widetilde{z}_{k,k'},\widetilde{\zeta}_{k,k'})}{\partial(\widetilde{x}_{k,k'},\widetilde{\xi}_{k,k'})} = (-1)^{2n(k'-k+1)}\det\begin{pmatrix} \Delta_{k'-k+1} & 0 \\ 0 & {}^t\Delta_{k'-k+1} \end{pmatrix} \times \det\left\{ I_{2n(k'-k+1)} - \begin{pmatrix} \Lambda^5_{k,k'} & \Lambda^6_{k,k'} \\ \Lambda^7_{k,k'} & \Lambda^8_{k,k'} \end{pmatrix} \right\},$$

where

$$(3.32)\qquad \begin{aligned} \Lambda^5_{k,k'} &= (\Delta_{k'-k+1})^{-1}\Lambda^1_{k,k'}, \\ \Lambda^6_{k,k'} &= \langle\xi_{j_{r-1}}\rangle\cdot(\Delta_{k'-k+1})^{-1}\Lambda^2_{k,k'}, \\ \Lambda^7_{k,k'} &= \langle\xi_{j_{r-1}}\rangle^{-1}\cdot({}^t\Delta_{k'-k+1})^{-1}\Lambda^3_{k,k'}, \\ \Lambda^8_{k,k'} &= ({}^t\Delta_{k'-k+1})^{-1}\Lambda^4_{k,k'}. \end{aligned}$$

Hence, by Lemma 2.1 and (3.17), we have

$$(3.33)\qquad \begin{aligned} &(1-3n\kappa_2T)^{2n(j_r-j_{r-1}-1)} \\ &\le \det\frac{\partial(\widetilde{z}_{j_{r-1}+1,j_r-1},\widetilde{\zeta}_{j_{r-1}+1,j_r-1})}{\partial(\widetilde{x}_{j_{r-1}+1,j_r-1},\widetilde{\xi}_{j_{r-1}+1,j_r-1})} \\ &\le (1+3n\kappa_2T)^{2n(j_r-j_{r-1}-1)}, \end{aligned}$$

for $r = 1, 2, \dots, R+1$ on the support of $p^\circ_{j_0,j_1,\dots,j_R}$.

Therefore, there exists a positive constant C_4 such that

$$\left|p^{\circ}_{j_0,j_1,\ldots,j_R}\prod_{r=1}^{R+1}\det\frac{\partial(\widetilde{x}_{j_{r-1}+1,j_r-1},\widetilde{\xi}_{j_{r-1}+1,j_r-1})}{\partial(\widetilde{z}_{j_{r-1}+1,j_r-1},\widetilde{\zeta}_{j_{r-1}+1,j_r-1})}\right|\le (C_4)^L|p|^{(\widetilde{m}_{L+1})}_{n+1,J}\langle\xi_0\rangle^{\sum_{j=1}^{L+1}m_j}$$

$$\times\prod_{j=j_R+1}^{L}\left\{\frac{\langle\xi_{j_R}\rangle^{n/2}}{(1+\langle\xi_{j_R}\rangle|z_j|^2)^{(n+1)/2}}\cdot\frac{\langle\xi_{j_R}\rangle^{-n/2}}{(1+\langle\xi_{j_R}\rangle^{-1}|\zeta_j|^2)^{J/2}}\right\}$$

$$\times\prod_{r=1}^{R}\prod_{j=j_{r-1}+1}^{j_r-1}\left\{\frac{\langle\xi_{j_{r-1}}\rangle^{n/2}}{(1+\langle\xi_{j_{r-1}}\rangle|z_j|^2)^{(n+1)/2}}\cdot\frac{\langle\xi_{j_{r-1}}\rangle^{-n/2}}{(1+\langle\xi_{j_{r-1}}\rangle^{-1}|\zeta_j|^2)^{(J-2M)/4}}\right\}$$

$$\times\prod_{r=1}^{R}\frac{\langle\xi_{j_r}\rangle^{n/2}}{(1+\langle\xi_{j_r}\rangle|x_{j_r}-x_{j_r+1}-\partial_{\xi_{j_r}}\phi_{j_r+1}(x_{j_r+1},\xi_{j_r})|^2)^{(n+1)/2}}$$

$$\times\prod_{r=1}^{R}\langle\xi_{j_r}\rangle^{-n/2-(J-2M)/4}. \tag{3.34}$$

8°. We change the variables:

$$(\widetilde{x}_{j_{r-1}+1,j_r-1},\widetilde{\xi}_{j_{r-1}+1,j_r-1})\Longrightarrow(\widetilde{z}_{j_{r-1}+1,j_r-1},\widetilde{\zeta}_{j_{r-1}+1,j_r-1}),$$

for $r=1,2,\ldots,R+1$.

Now, for $r=1,2,\ldots,R$, x_{j_r+1} is a function depending only on $x_{j_{r+1}}$, $\widetilde{z}_{j_r+1,j_{r+1}-1}$, $\widetilde{\zeta}_{j_r+1,j_{r+1}-1}$ and ξ_{j_r}, not x_{j_r}.

Keeping this in mind, we integrate in the following order. First we integrate by x_{j_1}, x_{j_2}, ..., x_{j_R}. Secondly we integrate by $\widetilde{z}_{j_{r-1}+1,j_r-1}$, $\widetilde{\zeta}_{j_{r-1}+1,j_r-1}$, $r=1,2,\ldots,R+1$. Thirdly we integrate by ξ_{j_R}, $\xi_{j_{R-1}}$,...,ξ_{j_1}. Then there exists a positive constant C_5 such that

$$|\mathbb{I}(\Phi,p^{\circ}_{j_0,j_1,\ldots,j_R})|\le(C_5)^L|p|^{(\widetilde{m}_{L+1})}_{n+1,J}\langle\xi_0\rangle^{\sum_{j=1}^{L+1}m_j}. \tag{3.35}$$

Therefore, we have

$$\begin{aligned}|\mathbb{I}(\Phi,p)|&\le\sum_{R=0}^{L}\sum_{0=j_0<j_1<\ldots<j_R<j_{R+1}=L+1}|\mathbb{I}(\Phi,p^{\circ}_{j_0,j_1,\ldots,j_R})|\\&\le(2C_5)^L|p|^{(\widetilde{m}_{L+1})}_{n+1,J}\langle\xi_0\rangle^{\sum_{j=1}^{L+1}m_j}.\end{aligned} \tag{3.36}$$

□

Now, for $R=0,1,2,\ldots,L$ and $0=j_0<j_1<\ldots<j_R<j_{R+1}=L+1$, set

$$(3.37)\qquad E_{j_0,j_1,\dots,j_R} = \left\{ \begin{array}{l} (x_{L+1},\xi_L,x_L,\xi_{L-1},\dots,x_1,\xi_0); \\ |\xi_j - \xi_{j_{r-1}}| \le \dfrac{1}{2}\langle\xi_{j_{r-1}}\rangle \\ (1 \le r \le R+1, \quad j_{r-1}+1 \le j \le j_r - 1) \\ |\xi_{j_r} - \xi_{j_{r-1}}| > \dfrac{1}{2}\langle\xi_{j_{r-1}}\rangle \\ (1 \le r \le R) \end{array} \right\}.$$

Looking over the proof of Theorem 1.5 once again, we can get the following corollary.

Corollary 3.1. *Let $\{\kappa_l\}_{l=0}^{\infty}$ be an increasing sequence of positive constants and $M \ge 0$. Set $T = \min\{1/(7\sqrt{n}\kappa_1), 1/(4n\kappa_2)\}$. Then there exist positive constants C' and C'' independent of L satisfying the following:*

(1) *Let $\sum_{j=1}^{L+1} t_j \le T$, $\sum_{j=1}^{L+1}|m_j| \le M$, $p \in \boldsymbol{S}_\rho^{\widetilde{m}_{L+1}}$ and $\phi_j \in \boldsymbol{P}_\rho(t_j,\{\kappa_l\}_{l=0}^{\infty})$. If E_0 contains the support of p, we have*

$$(3.38)\qquad |\mathbb{I}(\Phi,p)(x_{L+1},\xi_0)| \le (C')^L |p|_{n+1,n+1}^{(\widetilde{m}_{L+1})} \langle\xi_0\rangle^{\sum_{j=1}^{L+1} m_j}.$$

(2) *Let $\sum_{j=1}^{L+1} t_j \le T$, $\sum_{j=1}^{L+1}|m_j| \le M$, $p \in \boldsymbol{S}_\rho^{\widetilde{m}_{L+1}}$ and $\phi_j \in \boldsymbol{P}_\rho(t_j,\{\kappa_l\}_{l=0}^{\infty})$. Let $R = 1,2,\dots,L$ and $0 = j_0 < j_1 < \dots < j_R < j_{R+1} = L+1$. If $E_{j_0,j_1,\dots,j_R}$ contains the support of p, we have*

$$(3.39)\qquad |\mathbb{I}(\Phi,p)(x_{L+1},\xi_0)| \le (C'')^L |p|_{l_0,l_0'}^{(\widetilde{m}_{L+1})} \langle\xi_0\rangle^{-(M-\sum_{j=1}^{L+1}\max\{0,m_j\})},$$

where $l_0 = n+1$ and $l_0' = [2M]+2n+1$.

4. Proof of Proposition 1.6

In this section, we prove Proposition 1.6.

Proof of Proposition 1.6.

1°. First we assume that the solution $\{x_j^*,\xi_j^*\}_{j=1}^{L}$ of (1.15) exists. Then we have

$$(4.1)\qquad \begin{aligned} |\xi_j^* - \xi_{j-1}^*| &\le \sqrt{n}\kappa_1 t_j \langle\xi_{j-1}^*\rangle \\ &\le \sqrt{n}\kappa_1 t_j \left\{ \sum_{k=1}^{L} |\xi_k^* - \xi_{k-1}^*| + \langle\xi_0\rangle \right\}, \end{aligned}$$

for $j = 1,2,\dots,L$. Hence we get

$$(4.2)\qquad \sum_{j=1}^{L}|\xi_j^*-\xi_{j-1}^*|\le\frac{\sqrt{n}\kappa_1\sum_{j=1}^{L}t_j}{1-\sqrt{n}\kappa_1\sum_{j=1}^{L}t_j}\langle\xi_0\rangle\le\frac{1}{2}\langle\xi_0\rangle.$$

Therefore, the solution $\{x_j^*,\xi_j^*\}_{j=1}^{L}$ of (1.15) satisfies

$$(4.3)\qquad |\xi_j^*-\xi_0|\le\frac{1}{2}\langle\xi_0\rangle,$$

for $j=1,2,\dots,L$.

2°. For $(\widetilde{x}_{1,L},\widetilde{\xi}_{1,L})\in \boldsymbol{R}^{2nL}$, we introduce the norms $||^t(\widetilde{x}_{1,L},\widetilde{\xi}_{1,L})||_{\infty}^{\xi_0}$, $||^t(\widetilde{x}_{1,L},\widetilde{\xi}_{1,L})||_{1}^{\xi_0}$ given by

$$(4.4)\qquad ||^t(\widetilde{x}_{1,L},\widetilde{\xi}_{1,L})||_{\infty}^{\xi_0}=\max_{j=1,2,\dots,L}|x_j|+\langle\xi_0\rangle^{-1}\max_{j=1,2,\dots,L}|\xi_j|,$$

$$||^t(\widetilde{x}_{1,L},\widetilde{\xi}_{1,L})||_{1}^{\xi_0}=\sum_{j=1}^{L}\{|x_j|+\langle\xi_0\rangle^{-1}|\xi_j|\}.$$

Let $\Omega_{\infty}^{\xi_0}$ be the normed space $(\boldsymbol{R}^{2nL},||\cdot||_{\infty}^{\xi_0})$ and let $\Omega_1^{\xi_0}$ be the normed space $(\boldsymbol{R}^{2nL},||\cdot||_1^{\xi_0})$. Let $\Theta_{\infty}^{\xi_0}$ be the closed set of $\Omega_{\infty}^{\xi_0}$ given by

$$(4.5)\qquad \Theta_{\infty}^{\xi_0}=\left\{(\widetilde{x}_{1,L},\widetilde{\xi}_{1,L})\in\Omega_{\infty}^{\xi_0};\ \ |\xi_j-\xi_0|\le\frac{1}{2}\langle\xi_0\rangle,\ \ j=1,2,\dots,L\right\}.$$

Let Δ_L be the matrix obtained by putting $k=1$ and $k'=L$ in (3.26).

For $(\widetilde{x}_{1,L},\widetilde{\xi}_{1,L})\in\Theta_{\infty}^{\xi_0}$, we consider the mapping $\mathcal{F}:(\widetilde{x}_{1,L},\widetilde{\xi}_{1,L})\longmapsto(\widetilde{y}_{1,L},\widetilde{\eta}_{1,L})$ given by

$$(4.6)\qquad {}^t(\widetilde{y}_{1,L},\widetilde{\eta}_{1,L})=\Delta^{-1}\Psi(x_{L+1},\widetilde{x}_{1,L},\widetilde{\xi}_{1,L},\xi_0),$$

where

$$(4.7)\qquad \Delta=\begin{pmatrix}\Delta_L & 0\\ 0 & {}^t\Delta_L\end{pmatrix},$$

and

$$(4.8)\qquad \Psi(x_{L+1},\widetilde{x}_{1,L},\widetilde{\xi}_{1,L},\xi_0)=\begin{pmatrix}0\\ \vdots\\ 0\\ x_{L+1}\\ \xi_0\\ 0\\ \vdots\\ 0\end{pmatrix}+\begin{pmatrix}\partial_{\xi_1}\phi_2(x_2,\xi_1)\\ \partial_{\xi_2}\phi_3(x_3,\xi_2)\\ \vdots\\ \partial_{\xi_L}\phi_{L+1}(x_{L+1},\xi_L)\\ \partial_{x_1}\phi_1(x_1,\xi_0)\\ \partial_{x_2}\phi_2(x_2,\xi_1)\\ \vdots\\ \partial_{x_L}\phi_L(x_L,\xi_{L-1})\end{pmatrix}.$$

From (4.5), we have

$$2^{-1}\langle\xi_0\rangle \le \langle\xi_j\rangle \le 2\langle\xi_0\rangle, \tag{4.9}$$

for $j = 1, 2, \ldots, L$. Furthermore, from (4.6), we have

$$\begin{aligned} |\eta_j - \xi_0| &\le \sum_{k=1}^{j} |\partial_{x_k}\phi_k(x_k, \xi_{k-1})| \le \sum_{j=1}^{L} \sqrt{n}\kappa_1 t_j \langle\xi_{j-1}\rangle \\ &\le 2\sqrt{n}\kappa_1 \sum_{j=1}^{L} t_j \langle\xi_0\rangle \le \frac{1}{2}\langle\xi_0\rangle, \end{aligned} \tag{4.10}$$

for $j = 1, 2, \ldots, L$. Therefore, the map $\mathcal{F} : \Theta_\infty^{\xi_0} \to \Theta_\infty^{\xi_0}$ is well-defined.
3°. Let $\Lambda_{1,L}^1$, $\Lambda_{1,L}^2$, $\Lambda_{1,L}^3$ and $\Lambda_{1,L}^4$ be the matrices obtained by putting $k = 1$ and $k' = L$ in (3.27)–(3.30). Set

$$\Lambda(x_{L+1}, \widetilde{x}_{1,L}, \widetilde{\xi}_{1,L}, \xi_0) = \begin{pmatrix} \Lambda_{1,L}^1 & \Lambda_{1,L}^2 \\ \Lambda_{1,L}^3 & \Lambda_{1,L}^4 \end{pmatrix}. \tag{4.11}$$

For $(\widetilde{x}_{1,L}, \widetilde{\xi}_{1,L}), (\widetilde{x}'_{1,L}, \widetilde{\xi}'_{1,L}) \in \Theta_\infty^{\xi_0}$, let

$$\begin{aligned} {}^t(\widetilde{y}_{1,L}, \widetilde{\eta}_{1,L}) &= \Delta^{-1}\Psi(x_{L+1}, \widetilde{x}_{1,L}, \widetilde{\xi}_{1,L}, \xi_0), \\ {}^t(\widetilde{y}'_{1,L}, \widetilde{\eta}'_{1,L}) &= \Delta^{-1}\Psi(x_{L+1}, \widetilde{x}'_{1,L}, \widetilde{\xi}'_{1,L}, \xi_0). \end{aligned} \tag{4.12}$$

Then we have

$$\begin{aligned} &\|{}^t(\widetilde{y}'_{1,L}, \widetilde{\eta}'_{1,L}) - {}^t(\widetilde{y}_{1,L}, \widetilde{\eta}_{1,L})\|_\infty^{\xi_0} \le \|\Delta^{-1}\|_{\Omega_1^{\xi_0} \to \Omega_\infty^{\xi_0}} \\ &\times \left\| \int_0^1 \Lambda(x_{L+1}, \widetilde{x}_{1,L} + \theta(\widetilde{x}'_{1,L} - \widetilde{x}_{1,L}), \widetilde{\xi}_{1,L} + \theta(\widetilde{\xi}'_{1,L} - \widetilde{\xi}_{1,L}), \xi_0) d\theta \right\|_{\Omega_\infty^{\xi_0} \to \Omega_1^{\xi_0}} \\ &\times \|{}^t(\widetilde{x}'_{1,L}, \widetilde{\xi}'_{1,L}) - {}^t(\widetilde{x}_{1,L}, \widetilde{\xi}_{1,L})\|_\infty^{\xi_0}. \end{aligned} \tag{4.13}$$

Clearly we have

$$\|\Delta^{-1}\|_{\Omega_1^{\xi_0} \to \Omega_\infty^{\xi_0}} \le 1. \tag{4.14}$$

Noting that

$$\begin{aligned} \langle\xi_j + \theta(\xi'_j - \xi_j)\rangle &\le (1-\theta)|\xi_j - \xi_0| + \theta|\xi'_j - \xi_0| + \langle\xi_0\rangle, \\ \langle\xi_0\rangle &\le (1-\theta)|\xi_j - \xi_0| + \theta|\xi'_j - \xi_0| + \langle\xi_j + \theta(\xi'_j - \xi_j)\rangle, \end{aligned} \tag{4.15}$$

we have

$$2^{-1}\langle\xi_0\rangle \le \langle\xi_j+\theta(\xi'_j-\xi_j)\rangle \le 2\langle\xi_0\rangle, \tag{4.16}$$

for $j=1,2,\dots,L$ and $0\le\theta\le 1$. Hence we get

$$\begin{aligned}&\left\|\int_0^1 \Lambda(x_{L+1},\widetilde{x}_{1,L}+\theta(\widetilde{x}'_{1,L}-\widetilde{x}_{1,L}),\widetilde{\xi}_{1,L}+\theta(\widetilde{\xi}'_{1,L}-\widetilde{\xi}_{1,L}),\xi_0)d\theta\right\|_{\Omega_\infty^{\xi_0}\to\Omega_1^{\xi_0}} \\ &\quad\le 3n\kappa_2\sum_{j=1}^{L+1}t_j<1.\end{aligned} \tag{4.17}$$

By (4.13), (4.14) and (4.17), $\mathcal{F}$ is a contraction. Hence there exists a unique solution $\{x_j^*,\xi_j^*\}_{j=1}^{L}\in\Theta_\infty^{\xi_0}$ such that

$${}^t(\widetilde{x}^*_{1,L},\widetilde{\xi}^*_{1,L})=\Delta^{-1}\Psi(x_{L+1},\widetilde{x}^*_{1,L},\widetilde{\xi}^*_{1,L},\xi_0). \tag{4.18}$$

Therefore, there exists a unique solution $\{x_j^*,\xi_j^*\}_{j=1}^{L}\in\Theta_\infty^{\xi_0}$ such that

$$\begin{cases}0=-(x_j^*-x_{j+1}^*)+\partial_{\xi_j}\phi_{j+1}(x_{j+1}^*,\xi_j^*),\\ 0=-(\xi_j^*-\xi_{j-1}^*)+\partial_{x_j}\phi_j(x_j^*,\xi_{j-1}^*),\\ j=1,2,\dots,L,\quad x_{L+1}^*=x_{L+1},\quad \xi_0^*=\xi_0.\end{cases} \tag{4.19}$$

4°. Clearly, from (4.19), we have

$$\begin{aligned}&|x_j^*-x_{j+1}^*|\le\sqrt{n}\kappa_1t_{j+1},\\ &|\xi_j^*-\xi_{j-1}^*|\le\sqrt{n}\kappa_1t_j\langle\xi_{j-1}^*\rangle\le 2\sqrt{n}\kappa_1t_j\langle\xi_0\rangle,\end{aligned} \tag{4.20}$$

for $j=1,2,\dots,L$. Furthermore, for any α_0, β_{L+1} with $|\alpha_0+\beta_{L+1}|\ge 1$, there exists a positive constant $C_{\alpha_0,\beta_{L+1}}$ such that

$$\begin{aligned}&|\partial_{x_{L+1}}^{\beta_{L+1}}\partial_{\xi_0}^{\alpha_0}(x_j^*-x_{j+1}^*)|\le C_{\alpha_0,\beta_{L+1}}t_{j+1}\langle\xi_0\rangle^{-(1-\rho)+(1-\rho)|\beta_{L+1}|-\rho|\alpha_0|},\\ &|\partial_{x_{L+1}}^{\beta_{L+1}}\partial_{\xi_0}^{\alpha_0}(\xi_j^*-\xi_{j-1}^*)|\le C_{\alpha_0,\beta_{L+1}}t_j\langle\xi_0\rangle^{\rho+(1-\rho)|\beta_{L+1}|-\rho|\alpha_0|},\end{aligned} \tag{4.21}$$

for $j=1,2,\dots,L$. Therefore we get (1.17). □

5. Proof of Theorem 1.7

In this section, we prove Theorem 1.7.

Proof of Theorem 1.7.

1°. For $R = 0, 1, 2, \ldots, L$ and $0 = j_0 < j_1 < \ldots < j_R < j_{R+1} = L + 1$, let

$$q_{j_0, j_1, \ldots, j_R} = e^{-i\Phi^*} \mathbb{I}(\Phi, \chi_{j_0, j_1, \ldots, j_R} p). \tag{5.1}$$

Then we have

$$q = \sum_{R=0}^{L} \sum_{0=j_0<j_1<\ldots<j_R<j_{R+1}=L+1} q_{j_0, j_1, \ldots, j_R}. \tag{5.2}$$

2°. First we consider the case where $R \neq 0$. We can write

$$\begin{aligned} \partial_{\xi_0} q_{j_0, j_1, \ldots, j_R} &= -i(\partial_{\xi_0} \Phi^*) e^{-i\Phi^*} \mathbb{I}(\Phi, \chi_{j_0, j_1, \ldots, j_R} p) \\ &\quad + e^{-i\Phi^*} \mathbb{I}(\Phi, i(\partial_{\xi_0} \Phi) \chi_{j_0, j_1, \ldots, j_R} p) \\ &\quad + e^{-i\Phi^*} \mathbb{I}(\Phi, \partial_{\xi_0} (\chi_{j_0, j_1, \ldots, j_R} p)). \end{aligned} \tag{5.3}$$

Note that

$$\partial_{\xi_0} \Phi = -(x_{L+1} - x_1) + \partial_{\xi_0} \phi_1(x_1, \xi_0), \tag{5.4}$$

and

$$i(x_{L+1} - x_1) e^{i \sum_{j=1}^{L} (x_{j+1} - x_j)(\xi_j - \xi_0)} = \left(\sum_{j=1}^{L} \partial_{\xi_j} \right) e^{i \sum_{j=1}^{L} (x_{j+1} - x_j)(\xi_j - \xi_0)}. \tag{5.5}$$

Integrating by parts, we can write

$$\begin{aligned} \partial_{\xi_0} q_{j_0, j_1, \ldots, j_R} &= -i(\partial_{\xi_0} \Phi^*) e^{-i\Phi^*} \mathbb{I}(\Phi, \chi_{j_0, j_1, \ldots, j_R} p) \\ &\quad + e^{-i\Phi^*} \sum_{j=0}^{L} \mathbb{I}(\Phi, i(\partial_{\xi_j} \phi_{j+1}) \chi_{j_0, j_1, \ldots, j_R} p) \\ &\quad + e^{-i\Phi^*} \sum_{j=0}^{L} \mathbb{I}(\Phi, \partial_{\xi_j} (\chi_{j_0, j_1, \ldots, j_R} p)). \end{aligned} \tag{5.6}$$

Here we note that

$$(\partial_{\xi_j} \phi_{j+1}) \chi_{j_0, j_1, \ldots, j_R} p, \quad \partial_{\xi_j} (\chi_{j_0, j_1, \ldots, j_R} p) \in \boldsymbol{S}_{\rho}^{\widetilde{m}_{L+1}}. \tag{5.7}$$

If we apply Corollary 3.1 (2) to the right hand side of (5.6) with M in Corollary 3.1 (2) replaced by $M + \rho$, then we have the estimate of $\partial_{\xi_0} q_{j_0, j_1, \ldots, j_R}$. Estimates for higher derivatives will be proved in a similar manner. It is enough to take

$l_1 \geq n+1+l$ and $l_1' \geq \left[2M+2\rho l+2l'\right]+2n+1+l'$.

3°. Next we consider the case where $R=0$. We change the variables:

$$y_j = x_j - x_j^*,\qquad \eta_j = \xi_j - \xi_j^*, \tag{5.8}$$

for $j=1,2,\ldots,L$. Then we have

$$q_0 = \int_{\boldsymbol{R}^{2nL}} e^{i\Pi} a(x_{L+1},\eta_L,y_L,\ldots,\eta_1,y_1,\xi_0)\prod_{j=1}^{L} dy_j đ\eta_j, \tag{5.9}$$

where

$$\begin{aligned} &a(x_{L+1},\eta_L,y_L,\ldots,\eta_1,y_1,\xi_0) \\ &= (\chi_0 p)(x_{L+1},\xi_L^*+\eta_L,x_L^*+y_L,\ldots,\xi_1^*+\eta_1,x_1^*+y_1,\xi_0), \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} \Pi &= \Phi - \Phi^* \\ &= -\sum_{j=1}^{L} y_j(\eta_j-\eta_{j-1}) \\ &\quad + \sum_{j=1}^{L} y_j \int_0^1 (1-\theta)(\partial_{x_j}^2\phi_j)(x_j^*+\theta y_j,\xi_{j-1}^*+\theta\eta_{j-1})d\theta\cdot y_j \\ &\quad + \sum_{j=1}^{L} \eta_j \int_0^1 (1-\theta)(\partial_{\xi_j}^2\phi_{j+1})(x_{j+1}^*+\theta y_{j+1},\xi_j^*+\theta\eta_j)d\theta\cdot \eta_j \\ &\quad + \sum_{j=2}^{L} y_j \int_0^1 (1-\theta)(\partial_{\xi_{j-1}}\partial_{x_j}\phi_j)(x_j^*+\theta y_j,\xi_{j-1}^*+\theta\eta_{j-1})d\theta\cdot \eta_{j-1} \\ &\quad + \sum_{j=1}^{L-1} \eta_j \int_0^1 (1-\theta)(\partial_{x_{j+1}}\partial_{\xi_j}\phi_{j+1})(x_{j+1}^*+\theta y_{j+1},\xi_j^*+\theta\eta_j)d\theta\cdot y_{j+1}, \end{aligned} \tag{5.11}$$

with $x_{L+1}^* = x_{L+1}$, $\xi_0^*=\xi_0$, $y_{L+1}=0$ and $\eta_0=0$.

For any α, β, we define $a_{\alpha,\beta}(x_{L+1},\eta_L,y_L,\ldots,\eta_1,y_1,\xi_0)$ such that

$$\partial_{x_{L+1}}^{\beta}\partial_{\xi_0}^{\alpha} q_0 = \int_{\boldsymbol{R}^{2nL}} e^{i\Pi} a_{\alpha,\beta}(x_{L+1},\eta_L,y_L,\ldots,\eta_1,y_1,\xi_0)\prod_{j=1}^{L} dy_j đ\eta_j. \tag{5.12}$$

Note that

$$2^{-1}\langle\xi_0\rangle \le \langle\xi_j^* + \theta\eta_j\rangle \le 2\langle\xi_0\rangle, \tag{5.13}$$

for $j = 1, 2, \ldots, L$ and $0 \le \theta \le 1$ on the support of $a_{\alpha,\beta}$.

For any α_0, β_{L+1} and non-negative integers K, K', there exists a positive constant C_1 such that

$$\begin{aligned}
&\left|\partial_{x_{L+1}}^{\beta_{L+1}}\partial_{\xi_0}^{\alpha_0}\left(\prod_{j=1}^{L}\partial_{y_j}^{\beta_j}\partial_{\eta_j}^{\alpha_j}\right)a_{\alpha,\beta}(x_{L+1},\eta_L,y_L,\ldots,\eta_1,y_1,\xi_0)\right| \\
&\le (C_1)^L|p|^{(\widetilde{m}_{L+1})}_{|\alpha+\beta+\alpha_0+\beta_{L+1}|+K,|\alpha+\beta+\alpha_0+\beta_{L+1}|+K'} \\
&\quad\times \langle\xi_0\rangle^{\sum_{j=1}^{L+1}m_j+(1-\rho)|\beta+\beta_{L+1}|-\rho|\alpha+\alpha_0|+\sum_{j=1}^{L}(|\beta_j|/2-|\alpha_j|/2)} \\
&\quad\times\left(1+\langle\xi_0\rangle^{1/2}\max_{j=1,2,\ldots,L}|y_j|+\langle\xi_0\rangle^{-1/2}\max_{j=1,2,\ldots,L}|\eta_j|\right)^{2|\alpha+\beta|},
\end{aligned}\tag{5.14}$$

for any $|\alpha_j| \le K$ and $|\beta_j| \le K'$, $j = 1, 2, \ldots, L$.

4°. We restore the variables:

$$\begin{aligned} x_j &= y_j + x_j^*, \\ \xi_j &= \eta_j + \xi_j^*, \end{aligned}\tag{5.15}$$

for $j = 1, 2, \ldots, L$. Then we have

$$\int_{\boldsymbol{R}^{2nL}} e^{i\Pi}a_{\alpha,\beta}(x_{L+1},\eta_L,y_L,\ldots,\eta_1,y_1,\xi_0)\prod_{j=1}^{L}dy_j\,đ\eta_j = e^{-i\Phi^*}\mathbb{I}(\Phi,p_{\alpha,\beta}),\tag{5.16}$$

where

$$\begin{aligned}
&p_{\alpha,\beta}(x_{L+1},\xi_L,x_L,\ldots,\xi_1,x_1,\xi_0) \\
&\quad= a_{\alpha,\beta}(x_{L+1},\xi_L-\xi_L^*,x_L-x_L^*,\ldots,\xi_1-\xi_1^*,x_1-x_1^*,\xi_0).
\end{aligned}\tag{5.17}$$

For any non-negative integers K, K', there exists a positive constant C_2 such that

$$\begin{aligned}
&\left|\left(\prod_{j=1}^{L}\partial_{x_j}^{\beta_j}\partial_{\xi_j}^{\alpha_j}\right)p_{\alpha,\beta}(x_{L+1},\xi_L,x_L,\ldots,\xi_1,x_1,\xi_0)\right| \\
&\le (C_2)^L|p|^{(\widetilde{m}_{L+1})}_{|\alpha+\beta|+K,|\alpha+\beta|+K'}\langle\xi_0\rangle^{\sum_{j=1}^{L+1}m_j+(1-\rho)|\beta|-\rho|\alpha|+\sum_{j=1}^{L}(|\beta_j|/2-|\alpha_j|/2)} \\
&\quad\times\left(1+\langle\xi_0\rangle^{1/2}\max_{j=1,2,\ldots,L}|x_j-x_j^*|+\langle\xi_0\rangle^{-1/2}\max_{j=1,2,\ldots,L}|\xi_j-\xi_j^*|\right)^{2|\alpha+\beta|},
\end{aligned}\tag{5.18}$$

for any $|\alpha_j| \leq K$ and $|\beta_j| \leq K'$, $j = 1, 2, \dots, L$.
5°. For $j = 1, 2, \dots, L$, let

$$z_j = \partial_{\xi_j} \Phi, \quad \zeta_j = \partial_{x_j} \Phi. \tag{5.19}$$

Let $\Psi(x_{L+1}, \widetilde{x}_{1,L}, \widetilde{\xi}_{1,L}, \xi_0)$ be the vector in (4.8) and $\Lambda(x_{L+1}, \widetilde{x}_{1,L}, \widetilde{\xi}_{1,L}, \xi_0)$ the matrix in (4.11). Since

$$\begin{aligned} {}^t(\widetilde{z}_{1,L}, \widetilde{\zeta}_{1,L}) &= -\Delta^t(\widetilde{x}_{1,L}, \widetilde{\xi}_{1,L}) + \Psi(x_{L+1}, \widetilde{x}_{1,L}, \widetilde{\xi}_{1,L}, \xi_0), \\ {}^t(0,0) &= -\Delta^t(\widetilde{x}^*_{1,L}, \widetilde{\xi}^*_{1,L}) + \Psi(x_{L+1}, \widetilde{x}^*_{1,L}, \widetilde{\xi}^*_{1,L}, \xi_0), \end{aligned} \tag{5.20}$$

we can write

$${}^t(\widetilde{z}_{1,L}, \widetilde{\zeta}_{1,L}) = -\Delta \left(I_{2nL} - \Delta^{-1} \int_0^1 \Lambda_\theta d\theta \right) {}^t(\widetilde{x}_{1,L} - \widetilde{x}^*_{1,L}, \widetilde{\xi}_{1,L} - \widetilde{\xi}^*_{1,L}), \tag{5.21}$$

where

$$\Lambda_\theta = \Lambda(x_{L+1}, \widetilde{x}^*_{1,L} + \theta(\widetilde{x}_{1,L} - \widetilde{x}^*_{1,L}), \widetilde{\xi}^*_{1,L} + \theta(\widetilde{\xi}_{1,L} - \widetilde{\xi}^*_{1,L}), \xi_0). \tag{5.22}$$

Furthermore, note that

$$||\Delta^{-1}||_{\Omega_1^{\xi_0} \to \Omega_\infty^{\xi_0}} \leq 1, \tag{5.23}$$

and

$$\left\| \int_0^1 \Lambda_\theta d\theta \right\|_{\Omega_\infty^{\xi_0} \to \Omega_1^{\xi_0}} \leq 3n\kappa_2 \sum_{j=1}^{L+1} t_j \leq \frac{3}{4}. \tag{5.24}$$

Hence, we have

$$||{}^t(\widetilde{x}_{1,L} - \widetilde{x}^*_{1,L}, \widetilde{\xi}_{1,L} - \widetilde{\xi}^*_{1,L})||_\infty^{\xi_0} \leq 4||{}^t(\widetilde{z}_{1,L}, \widetilde{\zeta}_{1,L})||_1^{\xi_0}. \tag{5.25}$$

Therefore,

$$\begin{aligned} &\left(1 + \langle\xi_0\rangle^{1/2} \max_{j=1,2,\dots,L} |x_j - x_j^*| + \langle\xi_0\rangle^{-1/2} \max_{j=1,2,\dots,L} |\xi_j - \xi_j^*|\right) \\ &\leq 4\left(1 + \langle\xi_0\rangle^{1/2} \sum_{j=1}^{L} |z_j| + \langle\xi_0\rangle^{-1/2} \sum_{j=1}^{L} |\zeta_j|\right) \\ &\leq 4 \prod_{j=1}^{L} (1 + \langle\xi_0\rangle^{1/2}|z_j|) \cdot \prod_{j=1}^{L} (1 + \langle\xi_0\rangle^{-1/2}|\zeta_j|) \\ &\leq 4 \cdot 2^L \prod_{j=1}^{L} (1 + \langle\xi_0\rangle|z_j|^2)^{1/2} \cdot \prod_{j=1}^{L} (1 + \langle\xi_0\rangle^{-1}|\zeta_j|^2)^{1/2}. \end{aligned} \tag{5.26}$$

6°. Integrating by parts, we have

$$\mathbb{I}(\Phi, p_{\alpha,\beta}) = \mathbb{I}(\Phi, p^{\circ}_{\alpha,\beta}), \tag{5.27}$$

where

$$\begin{aligned} p^{\circ}_{\alpha,\beta} &= (M_L^*)^{2|\alpha+\beta|+n+1}(M_{L-1}^*)^{2|\alpha+\beta|+n+1}\cdots(M_1^*)^{|\alpha+\beta|+n+1} \\ &\quad\circ (N_L^*)^{2|\alpha+\beta|+n+1}(N_{L-1}^*)^{2|\alpha+\beta|+n+1}\cdots(N_1^*)^{2|\alpha+\beta|+n+1}p_{\alpha,\beta}. \end{aligned} \tag{5.28}$$

Hence, there exists a positive constant C_3 such that

$$\begin{aligned} |\partial^{\beta}_{x_{L+1}}\partial^{\alpha}_{\xi_0} q_0| &= |\mathbb{I}(\Phi, p^{\circ}_{\alpha,\beta})| \\ &\le (C_3)^L |p|^{(\widetilde{m}_{L+1})}_{3|\alpha+\beta|+n+1,3|\alpha+\beta|+n+1}\langle\xi_0\rangle^{\sum_{j=1}^{L+1} m_j+(1-\rho)|\beta|-\rho|\alpha|}. \end{aligned} \tag{5.29}$$

7°. Now, we separate $a_{\alpha,\beta}$ in (5.11) depending on the degree of the term:

$$\left(1+\langle\xi_0\rangle^{1/2}\max_{j=1,2,\dots,L}|y_j| + \langle\xi_0\rangle^{-1/2}\max_{j=1,2,\dots,L}|\eta_j|\right),$$

to get a better estimate. Similarly, we can make better estimates for (5.12)–(5.29).

In particular, the new estimates for (5.29) is the following:

$$|\partial^{\beta}_{x_{L+1}}\partial^{\alpha}_{\xi_0} q_0| \le (C_4)^L |p|^{(\widetilde{m}_{L+1})}_{2|\alpha+\beta|+n+1,2|\alpha+\beta|+n+1}\langle\xi_0\rangle^{\sum_{j=1}^{L+1} m_j+(1-\rho)|\beta|-\rho|\alpha|}. \tag{5.30}$$

Therefore, in the case where $R = 0$, it is enough to take $l_1 \ge n+1+2(l+l')$ and $l_1' \ge n+1+2(l+l')$. □

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