Benvenuti, S and Bove, A
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ON A CLASS OF HYPERBOLIC SYSTEMS WITH MULTIPLE CHARACTERISTICS

STEFANO BENVENUTI and ANTONIO BOVE

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1. Introduction

In this paper we study necessary conditions for the well-posednes of the Cauchy problem for a class of first order differential systems with multiple characteristics.

For a scalar operator it is well known that the correctness of the Cauchy problem, in the case of multiple characteristics, yields some conditions on the lower order terms of the operator, except for a few remarkable cases (see e.g. [7]). Hence it is very natural to expect Levi conditions in the case of systems, even though the nature of the symplectic invariants involved seems fairly misterious.

Pushing the parallel with the scalar case further, much more is known for systems with double characteristics than for systems with characteristics of higher multiplicity. Namely in [9] T. Nishitani proved that if the Cauchy problem is well-posed for the $N \times N$ operator $L = L_1 + L_0$ where $h = \det L_1$ has double characteristics and is not effectively hyperbolic, then if rank $L_1 = N - 1$ at a multiple point one must impose some Levi conditions on the lower order terms.

On the other hand in [10], [8] and in [2] it has been proved that if h is hyperbolic, the set of all double points, Σ, is a smooth manifold, rank Hess $L_1 =$ codim Σ and either h is effectively hyperbolic or rank $L_1 \leq N - 2$ at a point $\rho \in \Sigma$, then L is strongly hyperbolic, i.e. the Cauchy problem is well-posed with no conditions on the lower order terms.

Turning to the case of multiplicity higher than two, in [10] it has been proved that if L_1 has real analytic coefficients and ρ is a characteristic point of $h = \det L_1$ of order r, L is strongly hyperbolic implies that $d^j\ {}^{\mathrm{co}}L_1(\rho) = 0$, $j < r - 2$.

The case we are concerned with in the present paper deals with characteristics of multiplicity 3, whereas the rank of L_1 at a triple point is $N - 1$. Due to the series of the above mentioned results, one expects Levi conditions in this case. If (the localization of) h can be factorized as a product of a linear form times a hyperbolic quadratic form (see below for a more precise statement of the assumptions), we know exactly the necessary conditions for the Cauchy problem for h to be well-posed. It turns out that in the vector case too we can isolate a scalar quantity, playing the role of a sort of subprincipal symbol (see e.g. Theorem 4.1 below), on which the Levi conditions are to be imposed. This fact is essentially due to the rank assumption for L_1 at triple points.

The proof of the theorem is carried out according to the ideas of Hörmander [4] and using the results of [3].

Let us say a word about our notations. $D_{x_j} = i^{-1}\partial_{x_j}$, $j = 0, \ldots, n$. If $p = p_m + p_{m-1} + \cdots$ is a polyhomogeneous symbol, p_j being homogeneous of degree j, we write H_{p_m} the Hamilton field of P_m defined by $H_{p_m}(x,\xi) = (\partial_\xi p_m, -\partial_x p_m)(x,\xi)$. The subprincipal symbol, which is invariantly defined at double characteristic points, is given by

$$p^s_{m-1}(x,\xi) = p_{m-1}(x,\xi) + \frac{i}{2}\langle\partial_x, \partial_\xi\rangle p_m(x,\xi),$$

where $\langle\partial_x, \partial_\xi\rangle = \sum_{j=0}^n (\partial^2/\partial x_j \partial\xi_j)$. If p is hyperbolic w.r.t. the ξ_0 direction, we denote by Γ_p the hyperbolicity cone of p, i.e. the component of $(0, e_0)$ of $\{(x,\xi) | p_m(x,\xi) \neq 0\}$, and by Γ_p^σ the polar of Γ_p w.r.t. the symplectic form $\sigma = d\xi \wedge dx = d\omega$. If ρ is a double characteristic point of p_m, $F_p(\rho)$ denotes the fundamental matrix of p_m at ρ, i.e. $dH_{p_m}(\rho)$. Furthermore $\mathrm{Tr}^+ F_p(\rho) = \sum_j \mu_j(\rho)$, where $i\mu_j(\rho)$ are the eigenvalues of $F_p(\rho)$ lying on the positive imaginary axis.

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2. Notations and Statement of the Result

Let Ω denote an open subset of $\mathbb{R}^{n+1}$, $0 \in \Omega$; we call $x = (x_0, x') = (x_0, \cdots, x_n)$ the generic point of Ω. Let $L(x, D)$ be a differential operator defined on $C^\infty(\Omega; \mathbb{C}^N)$, where N is a positive integer. We shall consider the Cauchy problem for $L(x, D)$ with respect to the surfaces $x_0 =$ const., which we shall assume to be non–characteristic for L:

$$\begin{cases} L(x,D)u = f \quad, \ x_0 \lesssim 0, \\ u|_{x_0=0} = g(x') \ , \end{cases} \tag{2.1}$$

$f \in \mathcal{D}'(\Omega)$, $g \in \mathcal{D}'(\Omega \cap \{x_0 = 0\})$. We say that the Cauchy problem (2.1) is well–posed in $\Omega \cap \{x_0 \lesssim 0\}$ if

(a) for every $f \in (C_0^\infty(\Omega))^N$ there is a $u \in (\mathcal{E}'(\Omega))^N$ such that $L(x, D)u = f$ in $\Omega \cap \{x_0 \lesssim 0\}$.

(b) for every $u \in (\mathcal{E}'(\Omega))^N$ such that $L(x, D)u = 0$ in $\Omega \cap \{x_0 \lesssim 0\}$, we have $u = 0$ in $\Omega \cap \{x_0 \lesssim 0\}$.

We are interested in necessary conditions for the well–posedness of the Cauchy problem (2.1).

We make the following assumptions:

(H1) Denote by $L_1(x,\xi)$ the principal symbol of L, i.e. the part homogeneous of degree 1 with respect to $\xi \in \mathbb{R}^{n+1} = \mathbb{R}_{\xi_0} \times \mathbb{R}^n_{\xi'}$. Then $L_1(x,\xi)$ is assumed to be hyperbolic in the following sense: denote by $h(x,\xi) = \det L_1(x,\xi)$; h is a polynomial in ξ of degree N. Then

$$h(x,\xi) \text{ is hyperbolic with respect to } \xi_0.$$

(H2) The characteristic roots of the polynomial $\xi_0 \mapsto h(x,\xi_0,\xi')$ have multiplicity of order at most 3.

(H3) Let ρ be a triple characteristic point of $h(x,\xi)$ and denote by $T_\rho(T^*\Omega) \ni \delta z \mapsto h_\rho(\delta z)$ the localization of h at ρ; h_ρ is then a third order polynomial hyperbolic with respect to $(0,e_0)$ on which we shall require the following:
 (i) $h_\rho(\delta z) = (\delta\xi_0 - \ell_1(\delta x, \delta\xi'))Q_2(\delta z)$, ℓ_1 being a real linear form in $(\delta x, \delta\xi')$ and
 (ii) $Q_2(\delta z)$ is a real hyperbolic quadratic form such that:
  (a) $\ker F^2_{Q_2}(\rho) \cap \operatorname{ran} F^2_{Q_2}(\rho) = \{0\}$.
  (b) If

$$V^+ = \bigoplus_{\substack{i\lambda \in \operatorname{sp}(F_{Q_2}(\rho)) \\ \lambda > 0}} \ker(F_{Q_2}(\rho) - i\lambda I),$$

  then $\forall v \neq 0$, $v \in V^+$, $(1/i)\sigma(v,\bar v) > 0$.
  (c) $\operatorname{sp}(F_{Q_2}(\rho)) \subset i\mathbb{R}$.

(H4) Let ρ be a triple point of $h(x,\xi)$ then

$$\operatorname{rank} L_1(\rho) = N - 1.$$

Since L is a differential operator then

$$L(x,D) = L_1(x,D) + B(x), \tag{2.2}$$

where $B \in C^\infty(\Omega; M_N(\mathbb{C}))$ and

$$L_1(x,\xi) = \sum_{j=0}^{n} A_j(x)\xi_j, \tag{2.3}$$

where $A_j \in C^\infty(\Omega; M_N(\mathbb{C}))$ and A_0 is non–singular, since the surfaces $x_0 = \text{const.}$ are non characteristic.

Put

$$L^s(x,\xi) = B(x) + \frac{i}{2}\sum_{j=0}^{n} \frac{\partial L_1}{\partial x_j \partial \xi_j}(x,\xi) \tag{2.4}$$

and define

$$\mathcal{L}(x,\xi) = L^s(x,\xi)\,{}^{\mathrm{co}}L_1(x,\xi) - \frac{i}{2}\{L_1,\,{}^{\mathrm{co}}L_1\}(x,\xi), \tag{2.5}$$

where

$$\{L_1,\,{}^{\mathrm{co}}L_1\}(x,\xi) = \left(\sum_{j=0}^{n} \frac{\partial L_1}{\partial \xi_j}\frac{\partial\,{}^{\mathrm{co}}L_1}{\partial x_j} - \frac{\partial L_1}{\partial x_j}\frac{\partial\,{}^{\mathrm{co}}L_1}{\partial \xi_j}\right)(x,\xi). \tag{2.6}$$

The following definition will be also useful throughout the paper:

DEFINITION 2.1. Let $A(x,\xi) \in S^m(\Omega; M_N(\mathbb{C}))$ and ρ a point in $T^*\Omega$. We write:

$$A(x,\xi) \equiv_k 0 \quad \text{at } \rho,$$

iff the matrix symbol $A(x,\xi)$ vanishes of order k at ρ, i.e. its entries are symbols vanishing at least of order k at ρ.

We are now in a position to state our main result:

Theorem 2.1. *Assume that the differential operator L given by* (2.2) *satisfies* (H1)–(H4) *and assume that the Cauchy problem* (2.1) *for L is correctly posed. Let ρ be a triple characteristic point for $h = \det L_1$ and suppose that*

$$H_\Lambda(\rho) \in \Gamma^\sigma_{Q_2}(\rho) \cap \ker F_{Q_2}(\rho), \tag{2.7}$$

where both $\Lambda(x,\xi) = \xi_0 - \ell_1(x,\xi')$ and Q_2 are defined in Assumption (H3)(i); *then there exists a smooth matrix $\alpha(x) = [\alpha_{kr}(x)]_{k,r=1,\ldots,N}$, which can be explicitly computed* (*see Section* 3 *below and Theorem* 4.1), *defined in a suitable neighborhood of the origin in $\mathbb{R}^{n+1}$, such that if*

$$C(x,\xi) = \sum_{k,r=1}^{N} \alpha_{kr}(x) \sum_{s=1}^{N} ({}^{\mathrm{co}}L_1)_{ks}\mathcal{L}_{sr} = \mathrm{Tr}\ \left({}^t\alpha\,{}^{\mathrm{co}}L_1\mathcal{L}\right) \tag{2.8}$$

we have

$$\mathrm{Im}\ C \equiv_2 0, \quad \text{at } \rho; \tag{2.9}$$

$$\mathrm{Re}\ C \equiv_1 0, \quad \text{at } \rho; \tag{2.10}$$

and furthermore

$$\mathrm{Tr}^+ F_{Q_2}(\rho) H_\Lambda(\rho) \pm H_{\mathrm{Re}\ C}(\rho) \in \Gamma^\sigma_h(\rho). \tag{2.11}$$

Before stating the proof of Theorem 2.1 some comments are in order.

- The Levi conditions (2.9), (2.10) contain the condition

$$\mathcal{L}(\rho) = 0 \ \text{ in } \ \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N)/L_1(\rho)\mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N),$$

given by T. Nishitani [9] at a characteristic point ρ of multiplicity greater than 2.
- It is a known fact in the theory of first order hyperbolic systems with multiple characteristics that even if the terms of order 0 are missing and the determinant of the principal symbol is a bona fide hyperbolic polynomial with multiple characteristics, Levi conditions may not be satisfied. In other words the mere principal symbol gives rise to an ill–posed Cauchy problem. This is reflected by conditions (2.9)–(2.11), as can be seen from the following model operator

$$L_1(x,\xi) = \begin{bmatrix} -\xi_0 & \xi_n & 0 & 0 \\ \xi_2 & -\xi_0 & \xi_n - \xi_2 & a(x_2)\xi_2 \\ x_2\xi_2 & -\xi_2 + x_2^2\xi_n & -\xi_0 + \xi_2 + x_2\xi_n & -a(x_2)\xi_2 \\ \xi_2 & 0 & 0 & -\xi_0 + a(x_2)\xi_n \end{bmatrix}$$

near $\rho = (0, e_n)$, where we chose $a(x_2) = 1 + x_2$.
- Assumption (H3)(ii)(b) is not strictly required. An analogous statement holds and a parallel proof can be carried out if $\mathrm{Tr}^+ F_{Q_2}(\rho) = 0$.
- Assumption (H3)(ii)(a) means that the fundamental matrix of Q_2 at ρ has no Jordan blocks of size 4 in its canonical form. A statement analogous to Theorem 1.1 holds if (H3)(ii)(a) is replaced by

$$\ker F^2_{Q_2}(\rho) \cap \mathrm{ran} F^2_{Q_2}(\rho) \neq \{0\}.$$

- Since the Levi conditions (2.9)–(2.11) are scalar in nature it is not a priori evident that they are invariant under canonical transformations leaving the initial data hypersurface invariant. Actually there is a close link between conditions (2.9), (2.10) and conditions —evidently invariant with respect to canonical transformations— of Nishitani type [9].
Both to shed light onto this fact and to follow a more pedagogical path, we postpone the discussion until Section 3 below (see also the Appendix).
- The case of characteristics of constant multiplicity with maximal rank $N - 1$ over a multiple point has been studied by V. Petkov, [12], constructing a parametrix and proving also a propagation of singularities result.

3. Preparations

Let $A_j(x)$, $B(x)$, $j = 1, \dots, n$, be $N \times N$ (complex) matrices with entries belonging to $C^\infty(\Omega)$, Ω open subset of $\mathbb{R}^{n+1}_x = \mathbb{R}_{x_0} \times \mathbb{R}^n_{x'}$; consider the following differential operator:

$$L(x, D) = \sum_{j=0}^{n} A_j(x) D_j + B(x). \tag{3.1}$$

Due to our assumption (H1), (H2) we may suppose that $A_0(x)$ is non singular; hence without loss of generality we may assume that $A_0(x) = -I_N$ —the $N \times N$ identity matrix:

$$L(x, D) = -I_N D_0 + \sum_{j=1}^{n} A_j(x) D_j + B(x). \tag{3.2}$$

We denote by $L_1(x, \xi)$ the principal symbol of L, i.e. $-I_N \xi_0 + \sum_{j=1}^{n} A_j(x)\xi_j$, and by L_0 the zero order part of L, i.e. $L_0(x) = B(x)$. Put $h(x, \xi) = \det L_1(x, \xi)$. By Assumption (H3), $h(x, \xi)$ is a hyperbolic polynomial with respect to ξ_0 admitting roots of multiplicity at most 3.

Let ρ a triple characteristic point for $h(x, \xi)$; without loss of generality we may suppose (possibly using a symplectic transformation — actually a linear coordinate change) that $\rho = (0, e_n)$, where $e_n = (0, \dots, 0, 1)$. Hence

$$L_1(\rho) = A_n(0),$$

which, by our assumptions, turns out to be a degenerate matrix of rank $N - 1$, with the zero eigenvalue of multiplicity 3. As a consequence we may find a $N \times N$ non singular matrix $U(x)$ with smooth (i.e. C^∞ near the origin) entries such that

$$U(x)^{-1} A_n(x) U(x) = \operatorname{diag}(\tilde{A}_n(x), G(x)) + O(|x|^k), \quad \text{as} \quad x \to 0, \tag{3.3}$$

where k is a suitably large positive integer, $G(x)$ is a non singular $(N-3) \times (N-3)$ matrix with smooth entries and $\tilde{A}_n(x)$ is a 3×3 matrix with smooth entries. Since $A_n(0)$ has rank $N - 1$, from (3.3) we obtain that

$$\begin{cases} \tilde{A}_n(0) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = J_3, \\ \det G(0) \neq 0. \end{cases} \tag{3.4}$$

Applying Arnold's results (see e.g. [1]) we may find a 3×3 smooth matrix $\tilde{U}(x)$ such that

$$\tilde{U}(x)^{-1} \tilde{A}_n(x) \tilde{U}(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_3(x) & a_2(x) & a_1(x) \end{bmatrix} + O(|x|^k), \tag{3.5}$$

as $x \to 0$, k a suitably large positive integer; here the $a_j(x)$'s are smooth functions such that

$$a_j(x) = O(|x|^j) \quad \text{as } x \to 0,\ j = 1, 2, 3. \tag{3.6}$$

Defining $\tilde{\tilde{U}}(x) = \operatorname{diag}(\tilde{U}(x), I_{N-3})$, we have

$$\tilde{\tilde{U}}(x)^{-1} \operatorname{diag}(\tilde{A}_n(x), G(x)) \tilde{\tilde{U}}(x) = \operatorname{diag}(\bar{A}(x), G(x)) + O(|x|^k), \tag{3.7}$$

as $x \to 0$, where

$$\bar{A}(x) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_3(x) & a_2(x) & a_1(x) \end{bmatrix}, \tag{3.8}$$

and $G(x)$ has been defined in (3.3). From now on we shall assume that $A_n(x)$ is given by the right hand side of Equation (3.7). The following notation will be useful for our purposes:

$$A(x, \xi') = \sum_{j=1}^{n} A_j(x) \xi_j = \begin{bmatrix} A_{11}(x, \xi') & A_{12}(x, \xi') \\ A_{21}(x, \xi') & A_{22}(x, \xi') \end{bmatrix} \tag{3.9}$$

$$\begin{cases} A_{11}(x, \xi') = A''_{11}(x, \xi'') + \bar{A}(x)\xi_n + O(|x|^k)\xi_n, \\ A_{22}(x, \xi') = A''_{22}(x, \xi'') + G(x)\xi_n + O(|x|^k)\xi_n, \\ A_{ij}(x, \xi') = A''_{ij}(x, \xi'') + O(|x|^k)\xi_n, \quad i \neq j, i, j \in \{1, 2\}. \end{cases} \tag{3.10}$$

$$A_{11}(x, \xi') = [a_{kr}(x, \xi')]_{k,r=1,2,3} = \sum_{j=1}^{n} \left[a^j_{kr}(x)\right]_{k,r=1,2,3} \xi_j. \tag{3.11}$$

Thus from (3.7), (3.8) and (3.11) we obtain that

$$a^n_{sr}(x) = O(|x|^k) \quad \text{if } (s, r) \neq (1, 2),\ (2, 3),\ s < 3,$$

$$a^n_{sr}(x) = 1 + O(|x|^k) \quad \text{if } (s, r) = (1, 2),\ (2, 3),$$

$$a^n_{3r}(x) = O(|x|^{3-r+1}),\ r = 1, 2, 3.$$

Here we denote by ξ'' the vector $\xi'' = (\xi_1, \dots, \xi_{n-1})$. As a result we shall work near $\rho = (0, e_n)$ with the first order system

$$L(x, D) = -D_0 I_N + A(x, D') + B(x),$$

with $A(x, D')$ satisfying (3.9)–(3.11).

First of all we want to deduce some conditions on the matrices $A_j(x)$ implied by Assumption (H2). Let ρ be a triple characteristic point for $h(x, \xi)$; then

$$h \equiv_3 0, \quad \text{at } \rho. \tag{3.12}$$

Since h is a polynomial of order N in the ξ_0 variable, (3.12) is equivalent to

$$\xi_0^2 c_2(x, \xi') + \xi_0 c_1(x, \xi') + c_0(x, \xi') \equiv_3 0 \quad \text{at } \rho. \tag{3.13}$$

We have

$$\begin{aligned} c_0(x, \xi') &= \det A(x, \xi') \\ c_1(x, \xi') &= \sum_{j=1}^{N} \det A_{(j)}(x, \xi'), \\ c_2(x, \xi') &= \sum_{\substack{j,k=1 \\ j \neq k}}^{N} \det A_{(j,k)}(x, \xi'), \end{aligned}$$

where $A_{(j)}$ denotes the matrix A when its j–th row has been replaced by the j–th row of $-I_N$ (the identity matrix in $\mathbb{C}^N$) and $A_{(j,k)}$ denotes the matrix obtained when both the j–th and k–th row of A have been replaced by the j–th and k–th row of $-I_N$; (3.13) becomes then

$$c_0 \equiv_3 0, \quad c_1 \equiv_2 0, \quad c_2 \equiv_1 0 \quad \text{at } \rho. \tag{3.14}$$

Let us first consider c_2. We start assuming that both j, k are different from 3. Then $\det A_{(j,k)}$ vanishes at ρ since the third row of A vanishes at ρ. Assume now that $j = 3$. If $k \neq 1$ then the first column of $A_{(3,k)}$ vanishes at ρ, so that the only non trivial case is obtained for $j = 3$, $k = 1$. Then the second column of $A_{(3,1)}$ vanishes at ρ. Thus we always have

$$c_2(\rho) = 0.$$

Let us now consider c_1; since $c_1(x, \xi') = \mathrm{Tr}({}^{\mathrm{co}}L_1(x, \xi))|_{\xi_0 = 0}$, then the second condition in (3.14) is equivalent to

$$\mathrm{Tr}({}^{\mathrm{co}}A(x, \xi')) \equiv_2 0, \quad \text{at } \rho. \tag{3.15}$$

We shall explicit (3.15) later in this section. Finally consider c_0; by definition we have

$$c_0(x,\xi') = \det(A_n(x)\xi_n + A''(x,\xi'')).$$

thus c_0 can be regarded as an N-th order polynomial with respect to ξ_n. Hence the coefficient of ξ_n^{N-j} is the sum of the determinants of all the matrices obtained from $A_n(x)$ replacing j rows with the corresponding j rows of $A''(x,\xi'')$. $c_0 \equiv_3 0$ at ρ is then equivalent to

$$\begin{aligned}(3.16)\qquad \xi_n^N \det A_n(x) &+ \xi_n^{N-1}\sum_{j=1}^{N} \det A_{n\ (j)}(x,\xi'') \\ &+ \xi_n^{N-2}\sum_{\substack{j,k=1\\ j\neq k}}^{N} \det A_{n\ (j,k)}(x,\xi'') \equiv_3 0,\end{aligned}$$

at ρ. Here $A_{n(j)}$, $A_{n(j,k)}$ means that the j–th and/or the k–th row of $A_n(x)$ have been replaced by the j–th and/or the k–th row of $A''(x,\xi'')$. (3.16) then implies

$$(3.17)\qquad \det A_n(x) \equiv_3 0, \quad \text{at } \rho.$$

$$(3.18)\qquad \sum_{j=1}^{N} \det A_{n(j)}(x,\xi'') \equiv_3 0, \quad \text{at } \rho.$$

$$(3.19)\qquad \sum_{\substack{j,k=1\\ j\neq k}}^{N} \det A_{n(j,k)}(x,\xi'') \equiv_3 0, \quad \text{at } \rho.$$

By a direct calculation we have:

$$(3.20)\qquad \det A_n(x) = a_{31}^n(x)\det G(x) \equiv_3 0, \quad \text{at } x=0,$$

by (3.3), (3.6). Exploiting (3.20), (3.18) can be rewritten as

$$\begin{aligned}(3.21)\qquad &\sum_{j=1}^{N} A_{n(j)}(x,\xi'') \\ &= \sum_{j=1}^{3} A_{n(j)}(x,\xi'') \\ &= \left(\det \begin{bmatrix} a''_{11} & a''_{12} & a''_{13} \\ 0 & 0 & 1 \\ a^n_{31} & a^n_{32} & a^n_{33}\end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ a''_{21} & a''_{22} & a''_{23} \\ a^n_{31} & a^n_{32} & a^n_{33}\end{bmatrix} \right. \\ &\qquad \left. + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a''_{31} & a''_{32} & a''_{33}\end{bmatrix}\right) \det G(x).\end{aligned}$$

Neglecting the terms vanishing of order 3 at ρ in (3.21) we obtain

$$a''_{31} - a''_{21}a^{n}_{33} \equiv_3 0, \quad \text{at } \rho. \tag{3.22}$$

As a consequence of (3.22) we get the following, which we state for future reference:

$$a''_{31} \equiv_2 0, \quad \text{at } \rho; \tag{3.23}$$

$$a_{31} \equiv_2 0, \quad \text{at } \rho. \tag{3.24}$$

Let us now deal with (3.19). It is readily verified that if both k and j are different from 3 then the matrix $A_{n(j,k)}(x,\xi'')$ has 3 rows with entries vanishing at ρ. Thus (3.19) is equivalent to the following condition

$$\sum_{\substack{j=1\\ j\neq 3}}^{N} \det A_{n(3,j)}(x,\xi'')$$

$$= \det \begin{bmatrix} a''_{11} & a''_{12} & a''_{13} \\ 0 & 0 & 1 \\ a''_{31} & a''_{32} & a''_{33} \end{bmatrix} \det G(x) + \det \begin{bmatrix} 0 & 1 & 0 \\ a''_{21} & a''_{22} & a''_{23} \\ a''_{31} & a''_{32} & a''_{33} \end{bmatrix} \det G(x)$$

$$+\sum_{j=4}^{N} \det \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & \cdots & a_{3N} \\ \hline 0 & 0 & 0 & & & \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & & & \\ a_{j1} & a_{j2} & a_{j3} & & G_{(j)} & \\ 0 & 0 & 0 & & & \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & & & \end{array}\right] \equiv_3 0, \quad \text{at } \rho.$$

In the above formula $G_{(j)}$ stands for the matrix obtained from $G(x)$ by replacing the $(j-3)$–rd row with the corresponding row of $A''_{22}(x,\xi'')$. By means of cumbersome algebra and neglecting the terms vanishing of order at least 3 at ρ we find that (3.19) is equivalent to the following condition

$$-(a''_{11}a''_{32} + a''_{21}a''_{33}) \det G(x) - \sum_{j,\ell=4}^{N} a''_{3\ell}({}^{\text{co}}G(x))_{\ell-3\, j-3} a''_{j1} \equiv_3 0, \text{ at } \rho. \tag{3.25}$$

We summarize what has been done in the following

Proposition 3.1. *Assume* (H1), (H2) *and* (H4) *and let* ρ *be a triple characteristic point of* $h(x,\xi) = \det L_1(x,\xi)$. *Then the following conditions are equivalent*:
1.

$$h \equiv_3 0, \quad \textit{at } \rho.$$

2. (i)

$$\mathrm{Tr}({}^{\mathrm{co}}A(x,\xi')) \equiv_2 0, \quad \textit{at } \rho.$$

(ii)

$$a''_{31} - a''_{21}a^{n}_{33} \equiv_3 0, \quad \textit{at } \rho.$$

(iii)

$$(a''_{11}a''_{32} + a''_{21}a''_{33}) \det G(x) + \sum_{j,\ell=4}^{N} a''_{3\ell}({}^{\mathrm{co}}G(x))_{\ell-3\, j-3}a''_{j1} \equiv_3 0, \;\; \textit{at } \rho.$$

In particular 2(ii) *implies that* $a_{31} \equiv_2 0$, *at* ρ.

The remaining part of the present section is devoted to the study of the matrix–symbol $({}^{\mathrm{co}}L_1)(x,\xi)$, which plays a crucial rôle in the Levi conditions. Using the same block–form notation as in (3.9) we write

$$L_1(x,\xi) = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}(x,\xi). \tag{3.26}$$

We have:

Lemma 3.1. *Let* L_1 *be a* $N \times N$ *matrix–symbol as in* (3.26). *Then we have*:
(a) $({}^{\mathrm{co}}L_1)_{ij}(x,\xi) = ({}^{\mathrm{co}}L_{11})_{ij}(x,\xi) \det L_{22}(x,\xi) + O(|(\xi_0,\xi'')|^2)$, $1 \le i,j \le 3$.
(b) $({}^{\mathrm{co}}L_1)_{ij}(x,\xi) = O(|x| + |(\xi_0,\xi'')|)$, $4 \le i,j \le N$.

Proof. (a) By definition of cofactor matrix we have:

$$({}^{\mathrm{co}}L_1)_{ij} = (-1)^{i+j} \det(T_1 + T_2), \tag{3.27}$$

where

$$T_1 = \begin{bmatrix} (L_{11})^i_j & 0 \\ 0 & L_{22} \end{bmatrix},$$

$$T_2 = \begin{bmatrix} 0 & (L_{12})_j \\ \\ (L_{21})^i & 0 \end{bmatrix};$$

here M_j^i stands for the matrix obtained from M deleting the i–th column and the j–th row. Then the determinant in (3.27) is the sum of the determinants of all the matrices obtained replacing k rows of T_1 with the corresponding k rows of T_2, $k = 1, \dots, N$. Since all the entries of the matrix T_2 vanish of order at least 1 at ρ, (3.27) can be written as

$$(3.28) \qquad ({}^{\mathrm{co}}L_1)_{ij} = (-1)^{i+j} \left(\det T_1 + \sum_{k=1}^{N} \det(T_1)_k \right) + O\left(|(\xi_0, \xi'')|^2\right).$$

Now the matrix $(T_1)_k$ is block–triangular and it can be easily seen that $\det(T_1)_k = 0$, for every $k = 1, \dots, N$. The above equation then becomes:

$$({}^{\mathrm{co}}L_1)_{ij} = (-1)^{i+j} (\det T_1) + O\left(|(\xi_0, \xi'')|^2\right),$$

which is assertion (a).

(b) Using the same kind of argument as in the proof of the preceding assertion we may write:

$$\begin{aligned} ({}^{\mathrm{co}}L_1)_{ij} &= (-1)^{i+j} \det \left(\begin{bmatrix} L_{11} & 0 \\ \\ 0 & (L_{22})_{j-3}^{i-3} \end{bmatrix} + \begin{bmatrix} 0 & (L_{12})^{i-3} \\ \\ (L_{21})_{j-3} & 0 \end{bmatrix} \right) \\ &= (-1)^{i+j} \det(\tilde{T}_1 + \tilde{T}_2) \\ &= (-1)^{i+j} \left[\det L_{11} \det(L_{22})_{j-3}^{i-3} + \sum_{k=1}^{N} \det(\tilde{T}_1)_k \right] + O(|(\xi_0, \xi'')|^2) \\ &= (-1)^{i+j} \det L_{11} \det(L_{22})_{j-3}^{i-3} + O(|(\xi_0, \xi'')|^2). \end{aligned}$$

And assertion (b) follows keeping in mind that $\det L_{11}(x, \xi) = O((|x| + |\xi|)^2)$. □

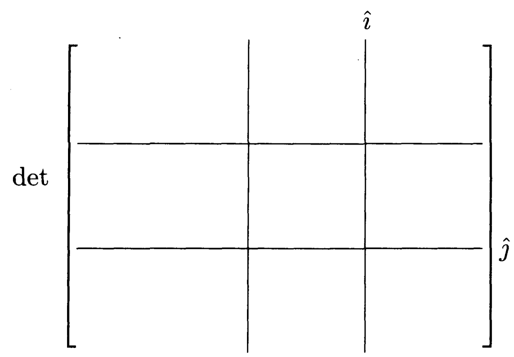
Corollary 3.1. *We have*

$$(3.29) \qquad ({}^{\mathrm{co}}L_1)_{31} - ({}^{\mathrm{co}}L_{11})_{31} \det L_{22} \equiv_3 0, \quad \text{at } \rho.$$

Proof. Going back to (3.28) we remark that $\det(T_1)_k = 0$. Hence the remainder term is generated replacing two rows of T_1 with the corresponding two rows of T_2, let's say the k–th and ℓ–th row, $k < \ell$. Then $k \in \{2, 3\}$ and $\ell \in \{4, \dots, N\}$, otherwise we get zero. In both cases we obtain a matrix with 3 rows vanishing at ρ and this proves the assertion. □

Lemma 3.2. *We have*
(a) $({}^{co}L_1)_{ij} \equiv_1 0$, *at* ρ, *if* $(i,j) \neq (1,3)$.
(b) $({}^{co}L_1)_{ij} \equiv_2 0$, *at* ρ, *if* $i \geq 2$, $j \in \{1,2,4,\ldots,N\}$.

Proof. (a) Easy by inspection.
(b) First assume $i \in \{2,3\}, j \in \{1,2\}$. Then the assertion follows from Lemma 3.1(a), computing the elements of ${}^{co}L_{11}$. Assume then $j \in \{1,2\}$, $i \geq 4$. Then $({}^{co}L_1)_{ij} = (-1)^{i+j}\det(L_1)^i_j$. The assertion follows noting that in $(L_1)^i_j$ both the 1–st column and either the 2–nd or the 3–rd column vanish at ρ. If $j \geq 4$ and $i \in \{2,3\}$ we argue in the same way exchanging rows with columns. We are left with the case $i,j \in \{4,\ldots,N\}$. We must compute the determinant of a matrix of the form



where the j–th row and i–th column have been suppressed. If we compute the above determinant according to the first column, we see that $({}^{co}((L_1)^i_j))_{1k}$ vanishes at ρ if $k \neq 3$; if $k = 3$ then $({}^{co}((L_1)^i_j))_{13} \neq 0$ at ρ, but by (3.24) $(L_1)_{31} \equiv_2 0$, at ρ, and this ends the proof of the lemma. □

Corollary 3.2. *Condition* 2.(i) *of Proposition* 3.1 *is equivalent to*

$$a_{32} + a_{21} \equiv_2 0.$$

Proof. It suffices to use Lemma 3.1 and Lemma 3.2 noting that the vanishing pattern of the entries of the matrix ${}^{co}A$ is the same as that of the matrix ${}^{co}L_1$. □

In the remaining part of the section we list some results which will be useful in the next section.

Lemma 3.3.

$$({}^{\mathrm{co}}L_1)_{j1} \equiv_3 (-1)^{2+j} a_{23} a_{32} \det \begin{bmatrix} a_{41} & \\ \vdots & (L_{22})^j \\ a_{4N} & \end{bmatrix}, \quad j \geq 4. \tag{3.30}$$

Proof. By a calculation:

$$\begin{aligned} ({}^{\mathrm{co}}L_{11})_{j1} &\equiv_3 (-1)^{1+j} a_{23} \det((L_1)_1^j)_1^3 \\ &\equiv_3 (-1)^{1+j} a_{23} \det \begin{bmatrix} 0 & a_{32} & 0 & \cdots & 0 \\ a_{41} & a_{42} & a_{44}-\xi_0 & \cdots & a_{4N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & a_{N4} & \cdots & a_{NN}-\xi_0 \end{bmatrix}, \end{aligned}$$

which yields the conclusion. □

Lemma 3.4.

$$({}^{\mathrm{co}}L_1)_{31} \equiv_3 ({}^{\mathrm{co}}L_{11})_{31} \det L_{22}. \tag{3.31}$$

Proof. We have $({}^{\mathrm{co}}L_1)_{31} = \det(L_1)_1^3$. Arguing as in the proof of Lemma 3.1(a) we see that $\det(L_1)_1^3$ is obtained computing the determinant of a block–diagonal matrix in which one line at least has been replaced by the corresponding line of a block–anti–diagonal matrix. It is easy to see that if only one line is replaced we get zero; thus we must replace at least two lines, one in the upper part of the matrix and the other in the lower part. So $({}^{\mathrm{co}}L_1)_{31}$ is given, modulo terms vanishing of the fourth order at ρ, by a sum whose typical representative is

$$\det\left[\begin{array}{cc|cccc} a_{21} & a_{22} & 0 & \cdots\cdots & & 0 \\ 0 & 0 & a_{34} & \cdots\cdots & & a_{3N} \\ \hline 0 & 0 & a_{44}-\xi_0 & \cdots\cdots & & a_{4N} \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & a_{k-1\,4} & \cdots\cdots & & a_{k-1\,N} \\ a_{k1} & a_{k2} & 0 & \cdots\cdots & & 0 \\ 0 & 0 & a_{k+1\,4} & \cdots\cdots & & a_{k+1\,N} \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & a_{4N} & \cdots\cdots & & a_{NN}-\xi_0 \end{array}\right]$$

which is $\equiv_3 0$ at ρ. This ends the proof of the Lemma. □

Lemma 3.5.

$$(3.32)\qquad ({}^{\mathrm{co}}L_1)_{32} \equiv_3 ({}^{\mathrm{co}}L_{11})_{32} \det L_{22} - a_{12} \sum_{\ell,h=4}^{N} a_{3\ell} a_{h1} ({}^{\mathrm{co}}L_{22})_{\ell-3\,h-3}.$$

Proof. We have

$$\begin{aligned} &({}^{\mathrm{co}}L_1)_{32} \\ &\equiv_3 -\left[(a_{11}-\xi_0)\det((L_1)_2^3)_1^1 - a_{12}\det((L_1)_2^3)_1^2\right] \\ &\equiv_3 -(a_{11}-\xi_0)a_{32}\det L_{22} + a_{12}a_{31}\det L_{22} \\ &\qquad -(a_{11}-\xi_0)\sum_{\ell=4}^{N}(-1)^{\ell-1}a_{3\ell}\det\begin{bmatrix} a_{42} & \\ \vdots & (L_{22})^{\ell-3} \\ a_{N2} & \end{bmatrix} \\ &\qquad +a_{12}\sum_{\ell=4}^{N}(-1)^{\ell-1}a_{3\ell}\det\begin{bmatrix} a_{41} & \\ \vdots & (L_{22})^{\ell-3} \\ a_{N1} & \end{bmatrix} \\ &= ({}^{\mathrm{co}}L_{11})_{32}\det L_{22} + a_{12}\sum_{\ell,s=4}^{N}(-1)^{\ell+s-3}a_{3\ell}a_{s1}\det(L_{22})_{s-3}^{\ell-3}, \end{aligned}$$

which proves the assertion. □

Lemma 3.6. *Let $j \geq 4$, then*

$$({}^{\mathrm{co}}L_1)_{j2} \equiv_3 a_{12}(a_{33} - \xi_0) \sum_{\ell=4}^{N} a_{\ell\,1}({}^{\mathrm{co}}L_{22})_{j-3\,\ell-3}.$$

Proof. Analogous to the proof of Lemma 3.3. □

Lemma 3.7. *Let $j \geq 4$, then*

$$({}^{\mathrm{co}}L_1)_{1j} \equiv_2 -a_{12}a_{23} \sum_{s=4}^{N} a_{3s}({}^{\mathrm{co}}L_{22})_{s-3\,j-3}.$$

Proof. It's again a calculation analogous to the proof of Lemma 3.3. □

Lemma 3.8. *Let $j \geq 4$, then*

$$({}^{\mathrm{co}}L_1)_{3j} \equiv_3 a_{12}a_{21} \sum_{h=1}^{N} a_{3h}({}^{\mathrm{co}}L_{22})_{h-3\,j-3}.$$

Lemma 3.9. *Let $j \geq 4$, then*

$$({}^{\mathrm{co}}L_1)_{j3} \equiv_2 -a_{12}a_{23} \sum_{\ell=4}^{N} a_{\ell 1}\,({}^{\mathrm{co}}G)_{j-3\,\ell-3}\,\xi_n^{N-4}.$$

We need also to compute the first terms in the asymptotic development of a cofactor matrix:

Lemma 3.10. *Let $A(x,\xi) \in S^1(\Omega, M_N(\mathbb{C}))$, $B(x,\xi) \in S^0(\Omega, M_N(\mathbb{C}))$. Then*

$$ {}^{\mathrm{co}}(A+B) \sim {}^{\mathrm{co}}A + T_B + C, \tag{3.33}$$

where $T_B \in S^{N-2}(\Omega, M_N(\mathbb{C}))$, $C \in S^{N-3}(\Omega, M_N(\mathbb{C}))$ and furthermore

$$AT_B = \mathrm{Tr}\;(B\,{}^{\mathrm{co}}A)\,I_n - B\,{}^{\mathrm{co}}A. \tag{3.34}$$

Proof. Suppose that A is an non-singular matrix symbol; then (3.34) is a consequence of the formula

$$T_B = \mathrm{Tr}\;(B\,{}^{\mathrm{co}}A)\,A^{-1} - {}^{\mathrm{co}}ABA^{-1}.$$

The general case can be recovered by a density argument. □

4. The Levi Condition

In this Section we want to study conditions (2.9), (2.10), establishing a link with those introduced by T. Nishitani [9]. As a by–product we show that the scalar quantity C defined in (2.8) is the subprincipal symbol of a certain scalar operator, which will allow us to complete the proof of Theorem 2.1 in the next section.

First of all we remark that due to the result of Appendix A and performing the change of u–variables in $\mathbb{C}^N$ of Section 3, we want actually to prove the following theorem:

Theorem 4.1. *Assume that the differential operator L, given by* (2.2) *satisfies* (H1)–(H4) *and assume that the Cauchy problem* (2.1) *for L is correctly posed. Let* ρ *be a triple characteristic point for* $h = \det L_1$ *and suppose that* (2.7) *is satisfied.*

Then defining

$$C(x,\xi) = \sum_{s=1}^{N} ({}^{\mathrm{co}}L_1)_{1s}\mathcal{L}_{s3}(x,\xi), \tag{4.1}$$

we have

$$\mathrm{Im}\, C \equiv_2 0, \quad \text{at } \rho; \tag{4.2}$$

$$\mathrm{Re}\, C \equiv_1 0, \quad \text{at } \rho, \tag{4.3}$$

and furthermore

$$\mathrm{Tr}^+ F_{Q_2}(\rho) H_\Lambda(\rho) \pm H_{\mathrm{Re}\ C}(\rho) \in \Gamma_h^\sigma(\rho). \tag{4.4}$$

Actually Theorem 4.1 implies Theorem 2.1 in the coordinates of Section 3. Moreover it becomes evident what the choice of the coefficients $\alpha_{kr}(x)$, $k, r = 1, \ldots, N$, means. From now on we shall argue in the coordinate framework of Theorem 4.1.

We start with the following

Proposition 4.1. *Using the same notation as in the preceding section, the following assertions are equivalent*

(a) *There exists a* $N \times N$ (*complex*) *symbol–matrix* $T(x,\xi)$ *such that*

$$\mathcal{L} - L_1 T = 0, \tag{4.5}$$

modulo terms vanishing of order 2 *at* ρ.

(b) *The symbol–matrices* $\mathcal{L}$ *and* L_1 *satisfy the "compatibility" conditions*

$$\sum_{s=1}^{N} ({}^{\mathrm{co}}L_1)_{1s}\mathcal{L}_{sk} = 0, \quad \forall k = 1, \ldots, N. \tag{4.6}$$

modulo terms vanishing of order 2 *at* ρ.

Proof. We shall argue on each column of T. Let us consider the ℓ–th column of equation (4.5); we have

$$\mathcal{L}_{j\ell} - \sum_{k=1}^{N} L_{jk}T_{k\ell} \equiv_2 0, \quad \text{at } \rho, \tag{4.7}$$

where for sake of brevity, we wrote L for the principal symbol L_1; this will cause no misunderstanding and will be used only in this argument.

Due to the particular structure of L (see e.g. Section 3), we may isolate the following system of $N-1$ equations:

$$\begin{cases} \sum_{k=2}^{N} L_{jk}T_{k\ell} \equiv_2 \mathcal{L}_{j\ell} - L_{j1}T_{1\ell}, \; j = 1,2 \\ \sum_{k=2}^{N} L_{jk}T_{k\ell} \equiv_2 \mathcal{L}_{j\ell} - L_{j1}T_{1\ell}, \; j = 4,\dots,N, \end{cases} \tag{4.8}$$

at ρ. When ℓ is fixed between 1 and N and $T_{1\ell}$ is for the moment regarded as a "parameter", we may think of (4.8) as though it were a linear system with a coefficient matrix of the form

$$L_3^1 = \left[\begin{array}{cc|cccc} * & & & & & \\ & * & & & & \\ \hline & & * & & & \\ & & & \ddots & & \\ & & & & & \\ & & & & & * \end{array}\right];$$

where the notation L_3^1 has been defined in the proof of assertion (a) of Lemma 3.1, $*$ means a symbol elliptic at ρ and all the off–diagonal terms vanish at ρ. Hence $\det L_3^1$ is an elliptic symbol and we may write

$$T_{k\ell} = (\det L_3^1)^{-1} \sum_{\substack{s=1 \\ s\neq 3}}^{N} ({}^{\mathrm{co}}(L_3^1))_{k-1\, s^*} (\mathcal{L}_{s\ell} - L_{s1}T_{1\ell}), \tag{4.9}$$

where $k = 2,\dots,N$ and for $s \in \{1,2,4,\dots,N\}$, s^* is defined by

$$s^* = \begin{cases} s, & \text{if } s = 1,2 \\ s-1, & \text{if } s = 4,\dots,N. \end{cases}$$

We would like to point out that even though formula (4.9) is not a polynomial with respect to the ξ_0 variable, the term $(\det L_3^1)^{-1}(x,\xi)$ differs from $(\det L_3^1)^{-1}(x,0,\xi')$ for terms vanishing at ρ. Furthermore since both L_{s1}, $s = 1,2,4,\dots,N$, and $\mathcal{L}_{s\ell}$ vanish at ρ (see T. Nishitani [9]), we get error terms vanishing of order at least 2 at ρ. We shall keep writing $(\det L_3^1)^{-1}$ and, since there is no danger of misundertanding, we shall not bother in specifying the dependence on the variables $(x,0,\xi')$.

Let us now turn to the third equation in (4.7); it reads, modulo terms vanishing of order 2 at ρ:

$$\sum_{k=2}^{N} L_{3k}T_{k\ell} = \mathcal{L}_{3\ell}, \tag{4.10}$$

and this becomes

$$(\det L_3^1)^{-1}\sum_{k=2}^{N}\sum_{\substack{s=1\\ s\neq 3}}^{N} L_{3k}(\,{}^{\mathrm{co}}(L_3^1))_{k-1\,s^*}\mathcal{L}_{s\ell} \equiv_2 \mathcal{L}_{3\ell}, \text{ at } \rho. \tag{4.11}$$

Now we have, if $s \geq 4$,

$$\begin{aligned} &\sum_{k=2}^{N} L_{3k}(\,{}^{\mathrm{co}}(L_3^1))_{k-1\,s^*} \\ &= \sum_{k=2}^{N} L_{3k}(-1)^{s^*+k-1}\det((L_3^1)_{s^*}^{k-1}) \\ &= \sum_{k=2}^{N} (-1)^{k+3-1}L_{3k}(-1)^{s^*+1}\det((L_3^1)_{s^*}^{k-1}) \\ &= (-1)^{s^*+1}\det L_s^1 \\ &= -(\,{}^{\mathrm{co}}L)_{1s}. \end{aligned} \tag{4.12}$$

The same result is easily seen to hold if $s = 1,2$. Hence from (4.11), (4.12) we may rewrite the third equation of (4.7) as

$$-(\det L_3^1)^{-1}\sum_{\substack{s=1\\ s\neq 3}}^{N}(\,{}^{\mathrm{co}}L)_{1s}\mathcal{L}_{s\ell} = \mathcal{L}_{3\ell}, \quad \text{at } \rho,$$

i.e.

$$\sum_{s=1}^{N}(\,{}^{\mathrm{co}}L)_{1s}\mathcal{L}_{s\ell} \equiv_2 0, \quad \text{at } \rho,\ \ell = 1,\dots,N. \tag{4.13}$$

and this proves the Proposition. □

Our next goal is to show that if our assumptions are satisfied then most of the conditions (4.13) — actually all of them but the one corresponding to $\ell = 3$ — are also verified.

Lemma 4.1. *Condition* (4.6) *for* $k = 1$ *holds true if Assumptions* (H1)–(H4) *are satisfied.*

Proof. Due to Proposition 4.1 we shall actually show that we can find the first column of the matrix T in such a way that the first column of the matrix $\mathcal{L} - L_1 T$ in (4.5) vanishes of order 2 at ρ. Now this implies that

$$\frac{a_{32}}{a_{12}}\mathcal{L}_{11} + \frac{a_{33} - \xi_0}{a_{23}}\mathcal{L}_{21} \equiv_2 \mathcal{L}_{31} - (L_{12}G(x)^{-1}\xi_n^{-1}\mathcal{L}_{21})_{31}. \tag{4.14}$$

Here the inner couple of indices denotes the block, as in (3.26), while the outer pair labels the place in the resulting matrix. Let us first compute $\mathcal{L}_{11}$, the 1,1–entry of $\mathcal{L}$, defined in (2.4), (2.5). We have

$$\mathcal{L}_{11} = \sum_{j=1}^{N} \Big(b_{1j}({}^{\text{co}}L_1)_{j1} + \frac{i}{2}(\langle \partial_x, \partial_\xi \rangle L_1)_{1j}({}^{\text{co}}L_1)_{j1} - \frac{i}{2}\{(L_1)_{1j}, ({}^{\text{co}}L_1)_{j1}\} \Big). \tag{4.15}$$

Since $a_{32} \equiv_1 0$ at ρ, we may neglect in (4.15) every term vanishing at ρ; thus, by Lemma 3.2, (4.15) becomes

$$\mathcal{L}_{11} = -\frac{i}{2}\{(L_1)_{11}, ({}^{\text{co}}L_1)_{11}\}, \tag{4.16}$$

modulo terms vanishing at ρ. By (3.10) the r.h.s. of (4.16) is equal to

$$-\frac{i}{2}\langle \partial_\xi (L_1)_{11}, \partial_x ({}^{\text{co}}L_1)_{11} \rangle$$

modulo terms vanishing at ρ. By Lemma 3.1(a) we get that

$$\partial_x({}^{\text{co}}L_{11})_{11} \equiv_1 (\partial_x({}^{\text{co}}L_1)_{11}) \det L_{22} \equiv_1 \partial_x(-a_{23}a_{32}) \det L_{22} \equiv_1 0,$$

by (3.10). Hence

$$\mathcal{L}_{11} \equiv_1 0. \tag{4.17}$$

The same argument yields

$$\mathcal{L}_{21} \equiv_1 0. \tag{4.18}$$

Consider then the second term in the r.h.s. of (4.14); since, by (3.10), $L_{12} \equiv_1 0$ at ρ, again we may neglect the terms in $\mathcal{L}_{21}$ vanishing at ρ. We have that

$$\mathcal{L}_{j1} = \sum_{k=1}^{N} \left[b_{jk}({}^{\mathrm{co}}L_1)_{k1} + \frac{i}{2}(\langle \partial_x, \partial_\xi \rangle L_1)_{jk}({}^{\mathrm{co}}L_1)_{k1} - \frac{i}{2}\{(L_1)_{jk}, ({}^{\mathrm{co}}L_1)_{k1}\} \right] \equiv_1 0, \tag{4.19}$$

if $j \geq 4$, by Lemma 3.4. We are left with

$$\begin{aligned} \mathcal{L}_{31} &= \sum_{j=1}^{N} \left[b_{3j}({}^{\mathrm{co}}L_1)_{j1} + \frac{i}{2}(\langle \partial_x, \partial_\xi \rangle L_1)_{3j}({}^{\mathrm{co}}L_1)_{j1} - \frac{i}{2}\{(L_1)_{3j}, ({}^{\mathrm{co}}L_1)_{j1}\} \right] \\ &\equiv_2 \frac{i}{2} \left[\langle \partial_x, \partial_\xi \rangle a_{31}({}^{\mathrm{co}}L_1)_{11} - \sum_{j=1}^{N} \{(L_1)_{3j}, ({}^{\mathrm{co}}L_1)_{j1}\} \right], \end{aligned} \tag{4.20}$$

where, to obtain the last equality we used Lemma 3.2 and the result of Lemma 1.1 in [9]. Furthermore if $j \geq 4$ $\{(L_1)_{3j}, ({}^{\mathrm{co}}L_1)_{j1}\} \equiv_2 0$, by Lemma 3.4 and Lemma 3.5 ($\langle \partial_\xi (L_1)_{3j}, \partial_x ({}^{\mathrm{co}}L_1)_{j1} \rangle \equiv_2 0$ if $j \geq 4$, due to Lemma 3.3 and (3.6)). Hence we obtain that

$$\begin{aligned} \mathcal{L}_{31} &= \frac{i}{2} \det L_{22} \Bigg[\langle \partial_x, \partial_\xi \rangle a_{31} \, ({}^{\mathrm{co}}L_{11})_{11} \\ &\quad + \sum_{j=1}^{3} (\langle \partial_x a_{3j}, \partial_\xi ({}^{\mathrm{co}}L_{11})_{j1} \rangle - \langle \partial_\xi (a_{3j} - \xi_0 \delta_{3j}), \partial_x ({}^{\mathrm{co}}L_{11})_{j1} \rangle) \Bigg] \\ &= \frac{i}{2} \det L_{22} \left[\langle \partial_x, \partial_\xi \rangle a_{31}(-a_{23}a_{32}) + \langle \partial_x a_{31}, \partial_\xi (-a_{23}a_{32}) \rangle \right. \\ &\quad \left. + \langle \partial_x a_{33}, \partial_\xi (a_{21}a_{32}) \rangle - \langle \partial_\xi a_{32}, \partial_x (a_{31}a_{23} - a_{21}a_{33}) \rangle \right], \end{aligned} \tag{4.21}$$

due to (3.6), (3.10). Finally (4.21) can be rewritten as

$$\begin{aligned} \mathcal{L}_{31} &\equiv_2 \frac{i}{2} \det L_{22} \left[-\langle \partial_x, \partial_\xi \rangle a_{31} a_{32} \xi_n + \langle \partial_x a_{33}^n, \partial_\xi a_{21}'' \rangle a_{32} \xi_n \right. \\ &\quad - \langle \partial_x a_{31}'', \partial_\xi a_{32} \rangle \xi_n + \langle \partial_x a_{33}^n, \partial_\xi a_{32} \rangle a_{21}'' \xi_n \\ &\quad \left. - \langle \partial_\xi a_{32}, \partial_x a_{31}'' - \partial_x a_{33}^n a_{21}'' \rangle \xi_n \right] \\ &\equiv_2 0, \end{aligned} \tag{4.22}$$

because of Proposition 3.1. This ends the proof of the Lemma. □

The next result is concerned with the second column of the matrix T in (4.5):

Lemma 4.2. *Condition* (4.6) *for* $k = 2$ *holds true if Assumptions* (H1)–(H4) *are satisfied.*

Proof. We shall argue along the same lines of the proof of the preceding Lemma. The analog of Equation (4.14) is now

$$\frac{a_{32}}{a_{12}}\mathcal{L}_{12} + \frac{a_{33}-\xi_0}{a_{23}}\mathcal{L}_{22} \equiv_2 \mathcal{L}_{32} - (L_{12}G(x)^{-1}\xi_n^{-1}\mathcal{L}_{21})_{32}, \tag{4.23}$$

with the same convention on the inner and outer pairs of indices.

Since

$$\mathcal{L}_{12} = \sum_{j=1}^{N}\left[b_{1j}({}^{\mathrm{co}}L_1)_{j2} + \frac{i}{2}(\langle\partial_x,\partial_\xi\rangle L_1)_{1j}({}^{\mathrm{co}}L_1)_{j2} - \frac{i}{2}\{(L_1)_{1j},({}^{\mathrm{co}}L_1)_{j2}\}\right],$$

by Lemma 3.4 we obtain

$$\frac{a_{32}}{a_{12}}\mathcal{L}_{12} \equiv_2 -\frac{i}{2}\frac{a_{32}}{a_{12}}\{(L_1)_{11},({}^{\mathrm{co}}L_1)_{12}\}, \tag{4.24}$$

and analogously

$$\frac{a_{33}-\xi_0}{a_{23}}\mathcal{L}_{22} \equiv_2 -\frac{i}{2}\frac{a_{33}-\xi_0}{a_{23}}\{(L_1)_{21},({}^{\mathrm{co}}L_1)_{12}\}, \tag{4.25}$$

$$\mathcal{L}_{32} \equiv_2 \frac{i}{2}(\langle\partial_x,\partial_\xi\rangle L_1)_{31}({}^{\mathrm{co}}L_1)_{12} - \frac{i}{2}\sum_{k=1}^{N}\{(L_1)_{3k},({}^{\mathrm{co}}L_1)_{k2}\}, \tag{4.26}$$

$$\begin{aligned}&(L_{12}G(x)^{-1}\xi_n^{-1}\mathcal{L}_{21})_{32}\\ &\equiv_2 -\frac{i}{2}\frac{1}{\det L_{22}}\sum_{s,h=1}^{N-3}(L_1)_{3\,h+3}({}^{\mathrm{co}}L_{22})_{hs}\{(L_1)_{s+3\,1},({}^{\mathrm{co}}L_1)_{12}\},\end{aligned} \tag{4.27}$$

where in (4.27) $(\det L_{22})^{-1} = (\det L_{22}(x,0,\xi'))^{-1} \equiv_1 (\det G(x))^{-1}\xi_n^{-(N-3)}$ and where we used the fact that

$$\mathcal{L}_{j+3\,2} \equiv_1 -\frac{i}{2}\{(L_1)_{j+3\,1},({}^{\mathrm{co}}L_1)_{12}\}.$$

In what follows the quantity $(\det L_{22})^{-1}$ will always mean $(\det L_{22}(x,0,\xi'))^{-1}$.

Assembling (4.24)–(4.27) and using Lemma 3.5, 3.6, we find that condition (4.23) becomes:

$$-\frac{i}{2}\frac{a_{32}}{a_{12}}\{(L_1)_{11},({}^{\mathrm{co}}L_{11})_{12}\}\det L_{22} - \frac{i}{2}\frac{a_{33}-\xi_0}{a_{23}}\{(L_1)_{21},({}^{\mathrm{co}}L_{11})_{12}\}\det L_{22}$$

$$
\begin{aligned}
&\equiv_2 \frac{i}{2}(\langle\partial_x,\partial_\xi\rangle L_1)_{31}({}^{\mathrm{co}}L_{11})_{12}\det L_{22} - \frac{i}{2}\{(L_1)_{31},({}^{\mathrm{co}}L_{11})_{12}\}\det L_{22}\\
&-\frac{i}{2}\{(L_1)_{32},({}^{\mathrm{co}}L_{11})_{22}\}\det L_{22}\\
&-\frac{i}{2}\left\{(L_1)_{33},({}^{\mathrm{co}}L_{11})_{32}\det L_{22} - a_{12}\sum_{\ell,h=4}^{N} a_{3\ell}a_{h1}({}^{\mathrm{co}}L_{22})_{\ell-3\,h-3}\right\}\\
&-\frac{i}{2}\sum_{k=4}^{N}\left\{(L_1)_{3k},a_{12}(a_{33}-\xi_0)\sum_{\ell=4}^{N}a_{\ell 1}({}^{\mathrm{co}}L_{22})_{k-3\,\ell-3}\right\}\\
&+\frac{i}{2}\sum_{h,s=4}^{N}(L_1)_{3h}({}^{\mathrm{co}}L_{22})_{h-3\,s-3}\{(L_1)_{s1},({}^{\mathrm{co}}L_{11})_{12}\}.
\end{aligned}
\tag{4.28}
$$

Using (3.10), from (4.28) we obtain:

$$
\begin{aligned}
&-2a_{32}\{a_{11}-\xi_0,a_{33}\}\det L_{22} - (a_{33}-\xi_0)\{a_{21},a_{33}\}\det L_{22}\\
&-\sum_{k=1}^{n}\partial_{x_k}a_{31}^k(a_{33}-\xi_0)\xi_n\det L_{22} + 2\xi_n\{a_{31},a_{33}-\xi_0\}\det L_{22}\\
&\qquad -2\{a_{32},a_{33}-\xi_0\}(a_{11}-\xi_0)\det L_{22}\\
&+2\xi_n\sum_{\ell,h=4}^{N}({}^{\mathrm{co}}L_{22})_{\ell-3\,h-3}\left[\{a_{33}-\xi_0,a_{3\ell}\}a_{h1}+\{a_{33}-\xi_0,a_{h1}\}a_{3\ell}\right]\equiv_2 0.
\end{aligned}
\tag{4.29}
$$

Here we used the notation

$$
a_{ij}(x,\xi') = \sum_{\ell=1}^{N} a_{ij}^{\ell}(x)\xi_\ell, \tag{4.30}
$$

(see e.g. (3.11)). The expression in the l.h.s of (4.29) is actually a first order polynomial w.r.t. the variable ξ_0 (recall that here $\det L_{22} = \det L_{22}(x,0,\xi')$). In order to show that Equation (4.29) holds we first show that coefficient of $\xi_0\xi_n\det L_{22}$ is zero modulo terms vanishing of order 1 at ρ. This coefficient is, modulo terms vanishing of order 1 at ρ,

$$
\begin{aligned}
&\sum_{k=1}^{N}\partial_{x_k}a_{31}^k + \xi_n^{-1}\{a_{21},a_{33}\} + 2\xi_n^{-1}\{a_{32},a_{33}-\xi_0\}\\
&\qquad\equiv_1 \sum_{k=1}^{N} a_{21}^k\partial_{x_k}a_{33}^n - \xi_n^{-1}\{a_{21},a_{33}\}\equiv_1 0,
\end{aligned}
\tag{4.31}
$$

because of relations 2(ii) in Proposition 3.1.

Let us now consider the terms that do not contain ξ_0 in (4.29); we have the quantity:

$$
\begin{aligned}
(4.32)\qquad & -2a_{32}(-\partial_{x_0}a_{33}^n\xi_n+\sum_{j=1}^{n-1}a_{11}^j\partial_{x_j}a_{33}^n\xi_n)\\
& -a_{33}\sum_{k=1}^{n-1}a_{21}^k\partial_{x_k}a_{33}^n\xi_n\\
& -\sum_{j=1}^{n}\sum_{k=1}^{n-1}\partial_{x_k}a_{31}^k a_{33}^j\xi_j\xi_n+2\xi_n\partial_{x_0}a_{31}\\
& +2\xi_n\left(\sum_{k=1}^{n}a_{31}^k\partial_{x_k}a_{33}^n\xi_n-\sum_{k,j=1}^{n-1}a_{33}^k\partial_{x_k}a_{31}^j\xi_j\right)\\
& -2a_{11}\sum_{k=1}^{n-1}a_{32}^k\partial_{x_k}a_{33}^n\xi_n\\
& +2\xi_n\sum_{\ell,h=4}^{N}\frac{({}^{\mathrm{co}}L_{22})_{\ell-3\,h-3}}{\det L_{22}}a_{h1}\left(\sum_{k=1}^{n-1}a_{33}^k\partial_{x_k}a_{3\ell}^n\xi_n-\sum_{k=1}^{n-1}a_{3\ell}^k\partial_{x_k}a_{33}^n\xi_n\right)\\
& +2\xi_n\sum_{\ell,h=4}^{N}\frac{({}^{\mathrm{co}}L_{22})_{\ell-3\,h-3}}{\det L_{22}}a_{h1}\left(\sum_{k=1}^{n-1}a_{33}^k\partial_{x_k}a_{h1}^n\xi_n-\sum_{k=1}^{n-1}a_{h1}^k\partial_{x_k}a_{33}^n\xi_n\right);
\end{aligned}
$$

using Proposition 3.1, (4.32) can be rewritten as (modulo terms vanishing of order 2 at ρ)

$$
\begin{aligned}
(4.33)\qquad & -2\sum_{s=1}^{n-1}a_{32}^s\xi_s\sum_{j=1}^{n-1}a_{11}^j\partial_{x_j}a_{33}^n\xi_n-\sum_{s=1}^{n}a_{33}^s\xi_s\sum_{j=1}^{n-1}a_{21}^k\partial_{x_k}a_{33}^n\xi_n\\
& -\sum_{k=1}^{n-1}\partial_{x_k}a_{33}^n a_{21}^k\sum_{j=1}^{n}a_{33}^j\xi_j\xi_n+2\xi_n^2\sum_{k=1}^{n-1}a_{31}^k\partial_{x_k}a_{33}^n\\
& -2\xi_n\sum_{k=1}^{n-1}a_{33}^k\partial_{x_k}a_{33}\sum_{j=1}^{n-1}a_{21}^j\xi_j-2\sum_{s=1}^{n-1}a_{11}^s\xi_s\sum_{k=1}^{n-1}a_{32}^k\partial_{x_k}a_{33}^n\xi_n\\
& -2\xi_n^2\sum_{\ell,h=4}^{N}\frac{({}^{\mathrm{co}}L_{22})_{\ell-3\,h-3}}{\det L_{22}}\sum_{s=1}^{n-1}a_{h1}^s\xi_s\sum_{k=1}^{n-1}a_{3\ell}^k\partial_{x_k}a_{33}^n\\
& -2\xi_n^2\sum_{\ell,h=4}^{N}\frac{({}^{\mathrm{co}}L_{22})_{\ell-3\,h-3}}{\det L_{22}}\sum_{s=1}^{n-1}a_{3\ell}^s\xi_s\sum_{k=1}^{n-1}a_{h1}^k\partial_{x_k}a_{33}^n.
\end{aligned}
$$

Now this quantity becomes, again by Proposition 3.1,

$$(4.34)\quad 2\xi_n \sum_{k=1}^{n-1} \partial_{x_k} a^n_{33} \Bigg[a^k_{11} \sum_{s=1}^{n-1} a^s_{21}\xi_s - a^k_{21} \sum_{s=1}^{n} a^s_{33}\xi_s + \xi_n a^n_{33} a^k_{21} - a^k_{33} \sum_{s=1}^{n-1} a^s_{21}\xi_s$$
$$- a^k_{32} \sum_{s=1}^{n-1} a^s_{11}\xi_s - \xi_n \sum_{\ell,h=4}^{N} \frac{({}^{\mathrm{co}}L_{22})_{\ell-3\,h-3}}{\det L_{22}} \sum_{s=1}^{n-1} (a^s_{h1} a^k_{3\ell} + a^s_{3\ell} a^k_{h1})\xi_s \Bigg].$$

The symbol in square brackets in (4.34) is

$$\Bigg[-\sum_{s=1}^{n-1} \left(a^k_{11} a^s_{32} + a^k_{32} a^s_{11} + a^k_{21} a^s_{33} + a^k_{33} a^s_{21} \right)$$
$$+ \sum_{\ell,h=4}^{N} \frac{({}^{\mathrm{co}}L_{22})_{\ell-3\,h-3}}{\det L_{22}} (a^k_{3\ell} a^s_{h1} + a^s_{3\ell} a^k_{h1}) \Bigg] \xi_s \equiv_2 0,$$

by condition (iii) of item 2 in Proposition 3.1. This ends the proof of Lemma 4.2. □

The next lemma takes care of all the remaining conditions but one:

Lemma 4.3. *Condition* (4.6) *for* $k = 4, \ldots, N$ *holds true if Assumptions* (H1)–(H4) *are satisfied.*

Proof. Let $j \in \{1, \ldots, N-3\}$, then the above mentioned conditions are equivalent to the following:

$$(4.35)\qquad \frac{a_{32}}{a_{12}} (\mathcal{L}_{12})_{1j} + \frac{a_{33} - \xi_0}{a_{23}} (\mathcal{L}_{12})_{2j} \equiv_2 (\mathcal{L}_{12} - L_{12} L_{22}^{-1} \mathcal{L}_{22})_{3j},$$

$j = 1, \ldots, N-3$. Here we are using the same convention as above on the matrix indices and on the expression L_{22}^{-1}. We have

$$\begin{aligned} (\mathcal{L}_{12})_{1j} &= \mathcal{L}_{1\,j+3} \\ &= \sum_{\ell=1}^{N} \Bigg[b_{1\ell} ({}^{\mathrm{co}}L_1)_{\ell\,j+3} + \frac{i}{2} \langle \partial_x, \partial_\xi \rangle (L_1)_{1\ell} ({}^{\mathrm{co}}L_1)_{\ell\,j+3} \\ &\qquad - \frac{i}{2} \{ (L_1)_{1\ell}, ({}^{\mathrm{co}}L_1)_{\ell\,j+3} \} \Bigg], \end{aligned}$$

$j \in \{1, \ldots, N-3\}$. Since $a_{32} \equiv_1 0$, at ρ, it is enough to neglect the terms in the above expression vanishing of order 1 at ρ. By Lemma 3.4 we see that in this case everything is negligible except the term $-(i/2)\{(L_1)_{11}, ({}^{\mathrm{co}}L_1)_{1\,j+3}\}$, corresponding to $\ell = 1$ in the above summation. Now applying Lemma 3.7 and using (3.10) it is easy to see that also the latter term vanishes of order 1 at ρ. Thus

$$\frac{a_{32}}{a_{12}}(\mathcal{L}_{12})_{1j} \equiv_2 0, \tag{4.36}$$

at ρ, for every $j \in \{1, \dots, N-3\}$.

The same argument applies to the subsequent term, giving

$$\frac{a_{33}-\xi_0}{a_{23}}(\mathcal{L}_{12})_{2j} \equiv_2 0, \tag{4.37}$$

at ρ, for every $j \in \{1, \dots, N-3\}$. The last term is then

$$(\mathcal{L}_{12} - L_{12}L_{22}^{-1}\mathcal{L}_{22})_{3j} = \mathcal{L}_{3\,j+3} - \sum_{\ell,s=1}^{N-3} (L_1)_{3\,\ell+3}(L_{22}^{-1})_{\ell s}\mathcal{L}_{s+3\,j+3}, \tag{4.38}$$

$j \in \{1, \dots, N-3\}$. Furthermore since

$$\mathcal{L}_{s+3\,j+3} = \sum_{h=1}^{N}\Big[b_{s+3\,h}({}^{\text{co}}L_1)_{h\,j+3} + \frac{i}{2}\langle\partial_x, \partial_\xi\rangle(L_1)_{s+3\,h}({}^{\text{co}}L_1)_{h\,j+3} - \frac{i}{2}\{(L_1)_{s+3\,h}, ({}^{\text{co}}L_1)_{h\,j+3}\Big],$$

by Lemmas 3.4 and 3.9 we obtain that $\mathcal{L}_{s+3\,j+3} \equiv_1 0$, $\forall s, j \in \{1, \dots, N-3\}$. Since, by (3.10), every term $(L_1)_{3\,\ell+3}$, $\ell \in \{1, \dots, N-3\}$, vanishes of the first order, we see that the sum in the r.h.s. of (4.38) is negligible. Thus the only surviving term in (4.38) is $\mathcal{L}_{3\,j+3}$, $j \in \{1, \dots, N-3\}$. We have:

$$\begin{aligned}(4.39)\ \mathcal{L}_{3\,j+3} &= \sum_{h=1}^{N}\Big[b_{3h}({}^{\text{co}}L_1)_{h\,j+3} + \frac{i}{2}\langle\partial_x, \partial_\xi\rangle(L_1)_{3h}({}^{\text{co}}L_1)_{h\,j+3} \\ &\qquad - \frac{i}{2}\{(L_1)_{3h}, ({}^{\text{co}}L_1)_{h\,j+3}\}\Big] \\ &\equiv_2 \frac{i}{2}\left[\langle\partial_x, \partial_\xi\rangle a_{31}({}^{\text{co}}L_1)_{1\,j+3} + \langle\partial_x a_{31}, \partial_\xi({}^{\text{co}}L_1)_{1\,j+3}\rangle\right. \\ &\qquad \left. - \langle\partial_\xi(a_{33}-\xi_0), \partial_x({}^{\text{co}}L_1)_{3\,j+3}\rangle + \langle\partial_x a_{33}, \partial_\xi({}^{\text{co}}L_1)_{3\,j+3}\rangle\right],\end{aligned}$$

by Lemma 3.2 and 3.6. Applying Lemma 3.7 and 3.8 we may write the above quantity as

$$\begin{aligned}&\mathcal{L}_{3\,j+3} \\ &\equiv_2 \frac{i}{2}\Big[-(\langle\partial_x, \partial_\xi\rangle a_{31})\xi_n^2 \sum_{s=4}^{N} a_{3s}({}^{\text{co}}L_{22})_{s-3\,j}\end{aligned}$$

$$
\begin{aligned}
&-\langle \partial_x a_{31}, \xi_n^2 \sum_{s=4}^{N} \partial_\xi a_{3s} ({}^{\mathrm{co}}L_{22})_{s-3\,j}\rangle \\
&+\langle \partial_x a_{33}^n, \xi_n^2 \sum_{s=4}^{N} (a_{3s}\partial_\xi a_{21} + a_{21}\partial_\xi a_{3s})({}^{\mathrm{co}}L_{22})_{s-3\,j}\rangle \Big] \\
\equiv_2\ & 0,
\end{aligned}
$$

at ρ, by Proposition 3.1. This ends the proof of the Lemma. □

The remaining part of this section is devoted to the "true" condition (4.6) for $k = 3$. More precisely we want to rewrite the quantity in (4.6) when $k = 3$ as the subprincipal symbol of a certain scalar operator.

Define

$$
M(x,\xi) = \begin{bmatrix} a_{12} & a_{13} \\ a_{22} - \xi_0 & a_{23} \end{bmatrix}, \tag{4.40}
$$

which is an elliptic symbol near ρ, and let

$$
\begin{cases}
\tau_j(x,\xi) = \dfrac{1}{i}\{({}^{\mathrm{co}}L_{11})_{j3}, \det L_{22}\}, & \text{if } 1 \le j \le 3, \\
\tau_j(x,\xi) & \\
= -\dfrac{1}{i}(\det G(x))^{-1} \left\{ \xi_n \displaystyle\sum_{s=4}^{N} a_{s1}({}^{\mathrm{co}}G)_{j-3\,s-3}, \det L_{22} \right\}, & \text{if } 4 \le j \le N.
\end{cases} \tag{4.41}
$$

We need also the following $N \times N$ symbol of order $N-1$, elliptic near ρ:

$$
R(x,\xi) = [r_{ij}(x,\xi)]_{\substack{1\le i\le N \\ 1\le j\le N}} \tag{4.42}
$$

$$
= \left[\begin{array}{c|c|c}
{}^{\mathrm{co}}(L_1+B)_{13} + \tau_1 & 0 \quad\quad 0 & \\
\vdots & \det G\, {}^{\mathrm{co}}M \xi_n^{N-2} & 0 \\
\vdots & [(-\, {}^{\mathrm{co}}G(L_{21}+B_{21}))_{jk}]_{\substack{k=2,3 \\ 4\le j\le N}} \xi_n^{N-2} & \xi_n^{N-1}\, {}^{\mathrm{co}}G \\
{}^{\mathrm{co}}(L_1+B)_{N3} + \tau_N & &
\end{array}\right]
$$

and put

$$
K(x,D) = L(x,D)R(x,D) = [k_{ij}(x,D)]_{\substack{1\le i\le N \\ 1\le j\le N}}. \tag{4.43}
$$

Proposition 4.2. *We have*

$$k_{31}^s(x,\xi) \equiv_2 \sum_{s=1}^{N} ({}^{\mathrm{co}}L_1)_{1s}\mathcal{L}_{s3}, \tag{4.44}$$

where k_{31}^s denotes the subprincipal symbol of the $(3,1)$–entry of K.

Proof. We compute first k_{31}^s. We have

$$\begin{aligned} k_{31}^s & \\ \equiv_2 & \sum_{j=1}^{3} b_{3j}({}^{\mathrm{co}}L_1)_{j3} + \sum_{j=4}^{N} b_{3j}(-{}^{\mathrm{co}}GL_{21}\xi_n^{N-2})_{j-3\,1} \\ & -b_{11}a_{23}a_{32}\det L_{22} - (a_{33}-\xi_0)a_{12}b_{21}\det L_{22} \\ & +\sum_{j=4}^{N} a_{3j}(-{}^{\mathrm{co}}GB_{21})_{j-3\,1}\xi_n^{N-2} + \frac{1}{i}\sum_{j=1}^{3}\partial_\xi(L_1)_{3j}\partial_x({}^{\mathrm{co}}L_1)_{j3} \\ & +\frac{1}{i}\sum_{j=4}^{N}\partial_\xi(L_1)_{3j}\partial_x(-{}^{\mathrm{co}}GL_{21})_{j-3\,1}\xi_n^{N-2} \\ & +\frac{i}{2}\langle\partial_x,\partial_\xi\rangle\left[\sum_{j=1}^{N}(L_1)_{3j}({}^{\mathrm{co}}L_1)_{j3}\right] + \sum_{j=1}^{N}(L_1)_{3j}\tau_j, \end{aligned} \tag{4.45}$$

where we used Lemma 3.9 and Lemma 3.10.

Let us now turn to condition (4.6) with $k = 3$. The latter is equivalent to evaluating modulo terms vanishing of order 2 at ρ the quantity:

$$-\frac{a_{32}}{a_{12}}\mathcal{L}_{13} - \frac{a_{33}-\xi_0}{a_{23}}\mathcal{L}_{23} + \mathcal{L}_{33} - (L_{12}L_{22}^{-1}\mathcal{L}_{21})_{33} \overset{def}{\equiv} \Lambda \tag{4.46}$$

Now, due to Lemma 3.9, we have

$$\begin{aligned} \mathcal{L}_{13} \equiv_1 & \ b_{11}({}^{\mathrm{co}}L_{11})_{13}\det L_{22} + \frac{i}{2}\langle\partial_x,\partial_\xi\rangle a_{11}({}^{\mathrm{co}}L_{11})_{13}\det L_{22} \\ & -\frac{i}{2}\{a_{11}-\xi_0, ({}^{\mathrm{co}}L_1)_{13}\}, \end{aligned} \tag{4.47}$$

$$\mathcal{L}_{23} \equiv_1 b_{21}({}^{\mathrm{co}}L_{11})_{13}\det L_{22} + \frac{i}{2}\langle\partial_x,\partial_\xi\rangle a_{21}({}^{\mathrm{co}}L_{11})_{13}\det L_{22} \tag{4.48}$$

$$-\frac{i}{2}\{a_{21}, ({}^{\mathrm{co}}L_1)_{13}\}. \tag{4.49}$$

Plugging these results into (4.46) we obtain:

$$\Lambda \equiv_2 -a_{32}b_{11}\xi_n\det L_{22} - (a_{33}-\xi_0)b_{21}\xi_n\det L_{22} \tag{4.50}$$

$$+\sum_{j=1}^{3} b_{3j}(\,^{\mathrm{co}}L_{11})_{j3}\det L_{22}+\sum_{j=4}^{N} b_{3j}(-\,^{\mathrm{co}}GL_{21}\xi_n^{N-2})_{j-3\,1}$$
$$+\sum_{j=4}^{N}(-\,^{\mathrm{co}}GB_{21})_{j-3\,1}\xi_n^{N-2}$$
$$-\frac{i}{2}(\langle\partial_x,\partial_\xi\rangle a_{11})a_{32}\xi_n\det L_{22}-\frac{i}{2}(\langle\partial_x,\partial_\xi\rangle a_{21})(a_{33}-\xi_0)\xi_n\det L_{22}$$
$$+\frac{i}{2}a_{32}\xi_n^{-1}\{a_{11}-\xi_0,(\,^{\mathrm{co}}L_{11})_{13}\det L_{22}\}$$
$$+\frac{i}{2}(a_{33}-\xi_0)\xi_n^{-1}\{a_{21},(\,^{\mathrm{co}}L_{11})_{13}\det L_{22}\}$$
$$+\sum_{j=1}^{3}\frac{i}{2}\left[\langle\partial_x,\partial_\xi\rangle(L_1)_{3j}(\,^{\mathrm{co}}L_1)_{j3}-\{(L_1)_{3j},(\,^{\mathrm{co}}L_1)_{j3}\}\right]$$
$$+\sum_{j=4}^{N}\frac{i}{2}\left[\langle\partial_x,\partial_\xi\rangle(L_1)_{3j}(\,^{\mathrm{co}}L_1)_{j3}-\{(L_1)_{3j},(\,^{\mathrm{co}}L_1)_{j3}\}\right]$$
$$-\frac{i}{2}\sum_{h,k=4}^{N}a_{3h}\frac{(\,^{\mathrm{co}}L_{22})_{h-3\,k-3}}{\det L_{22}}\Bigg[(\langle\partial_x,\partial_\xi\rangle a_{k1})\xi_n^2\det L_{22}$$
$$-\sum_{j=4}^{N}\{(L_1)_{kj},(\,^{\mathrm{co}}L_1)_{j3}\}-\langle\partial_\xi a_{k1},\partial_x\det L_{22}\rangle\xi_n^2\Bigg].$$

Using (3.10) Λ becomes (modulo terms vanishing of order two at ρ):

$$\Lambda\equiv_2 k_{31}^s-\sum_{j=1}^{N}(L_1)_{3j}\tau_j \tag{4.51}$$
$$+ia_{32}\xi_n\langle\partial_\xi(a_{11}-\xi_0),\partial_x\det L_{22}\rangle+i(a_{33}-\xi_0)\xi_n\langle\partial_\xi a_{21},\partial_x\det L_{22}\rangle$$
$$-\frac{i}{2}\sum_{j=4}^{N}(L_1)_{3j}\left[\xi_n^{N-2}(-\,^{\mathrm{co}}G\langle\partial_x,\partial_\xi\rangle L_{21})_{j-3\,1}+\xi_n^{N-2}(\langle\partial_x(-\,^{\mathrm{co}}G),\partial_\xi L_{21}\rangle)_{j-3\,1}\right]$$
$$+\frac{i}{2}\sum_{h=4}^{N}a_{3h}(-\,^{\mathrm{co}}G\langle\partial_x,\partial_\xi\rangle L_{21})_{h-3\,1}\xi_n^{N-2}-\frac{i}{2}(L_{12}\langle\partial_x\,^{\mathrm{co}}L_{22},\partial_\xi L_{21}\rangle)_{31}\xi_n^2$$
$$-\frac{i}{2}(L_{12}\,^{\mathrm{co}}L_{22}\det L_{22}\langle\partial_x(\det L_{22})^{-1},\partial_\xi L_{21}\rangle)_{31}\xi_n^2$$
$$+\frac{i}{2}\langle(L_{12}(\det L_{22})^{-1}\,^{\mathrm{co}}L_{22}\partial_\xi L_{21})_{31},\partial_x\det L_{22}\rangle\xi_n^2,$$

where the scalar product means a summation over the gradient components label (i.e. $(\langle\partial_x A,\partial_\xi B\rangle)_{hk}=\sum_{j=0}^{n}\sum_{s=1}^{N}\partial_{x_j}A_{hs}\partial_{\xi_j}B_{sk}$). Furthermore to get (4.51) we used the following identities:

$$\begin{aligned}
&\frac{i}{2}\sum_{h,k=4}^{N} a_{3h}\frac{({}^{\mathrm{co}}L_{22})_{h-3\,k-3}}{\det L_{22}}\sum_{j=4}^{N}\{(L_1)_{kj},({}^{\mathrm{co}}L_1)_{j3}\}\\
&=-\frac{i}{2}\sum_{j=4}^{N}\langle (L_{12}(\det L_{22})^{-1}\,{}^{\mathrm{co}}L_{22}\partial_x L_{22})_{3\,j-3},\partial_\xi(-\xi_n^2\,{}^{\mathrm{co}}L_{22}L_{21})_{j-3\,1}\rangle\\
&=-\frac{i}{2}(\langle L_{12}\partial_x\,{}^{\mathrm{co}}L_{22},\partial_\xi L_{21}\rangle)_{31}\xi_n^2\\
&\quad-\frac{i}{2}(\langle L_{12}\partial_x(\det L_{22})^{-1}\det L_{22}\,{}^{\mathrm{co}}L_{22},\partial_\xi L_{21}\rangle)_{31}\xi_n^2
\end{aligned}$$

and

$$\begin{aligned}
&\frac{i}{2}\sum_{h,k=4}^{N} a_{3h}\frac{({}^{\mathrm{co}}L_{22})_{h-3\,k-3}}{\det L_{22}}\langle\partial_\xi(L_1)_{k1},\partial_x\det L_{22}\rangle\xi_n^2\\
&=\frac{i}{2}\langle (L_{12}(\det L_{22})^{-1}\,{}^{\mathrm{co}}L_{22}\partial_\xi L_{21})_{31},\partial_x\det L_{22}\rangle\xi_n^2.
\end{aligned}\tag{4.52}$$

Thus

$$\begin{aligned}
\Lambda &\equiv_2 k_{31}^s-\sum_{j=1}^{N}(L_1)_{3j}\tau_j\\
&\quad -ia_{32}\{({}^{\mathrm{co}}L_{11})_{23},\det L_{22}\}-i(a_{33}-\xi_0)\{({}^{\mathrm{co}}L_{11})_{33},\det L_{22}\}\\
&\quad -i\sum_{j=4}^{N}a_{3j}(\det G)^{-1}\{-({}^{\mathrm{co}}GL_{21})_{j-3\,1},\det L_{22}\}\xi_n\\
&\equiv_2 k_{31}^s,
\end{aligned}\tag{4.53}$$

due to (4.41). This completes the proof of the Proposition. □

5. Proof of the Theorem

In order to prove Theorem 4.1 we need information on the growth rate of the elements of the matrix $K(x,D_x)$ after the symplectic dilation

$$\begin{cases} y_j=\rho^{s/2}x_j, & j=0,\dots,n-1,\\ y_n=\rho^s x_n. \end{cases}\tag{5.1}$$

and the corresponding contragredient transform for the dual variables; here s, ρ denote suitable positive parameters to be chosen later.

In what follows we denote by $\sigma_k(A)$ the symbol (positively) homogeneous of order k of the (pseudo)differential operator $A(x, D_x)$. Let us consider first the two blocks K_{12} and K_{22}. From the definition of the matrix R we have

$$\begin{aligned}\sigma_N(K_{12})(x,\xi) &= L_{11}(x,\xi)\sigma_{N-1}(R_{12})(x,\xi) + L_{12}(x,\xi)\sigma_{N-1}(R_{22})(x,\xi) \\ &= L_{12}(x,\xi)\,{}^{\mathrm{co}}G(x)\xi_n^{N-1}.\end{aligned}$$

Hence, performing the symplectic dilation (5.1), we have

$$K_{12}(y, D_y) = O\left(\rho^{(N-(1/2))s}\right).$$

For the block K_{22} we obtain

$$\begin{aligned}\sigma_N(K_{22})(x,\xi) &= L_{21}(x,\xi)\sigma_{N-1}(R_{12})(x,\xi) + L_{22}(x,\xi)\sigma_{N-1}(R_{22})(x,\xi) \\ &= (-\xi_0 I_{N-3} + A''_{22}(x,\xi'') + G(x)\xi_n + O(|x|^k)\xi_n)\,{}^{\mathrm{co}}G(x)\xi_n^{N-1} \\ &= \det G(x) I_{N-3}\xi_n^N + C(x,\xi) + O(|x|^k)D(x,\xi),\end{aligned}$$

where $C(x,\xi)$ is a $(N-3)\times(N-3)$ matrix whose entries are polynomials with respect to ξ, homogeneous of degree at most $N-1$ in ξ_n and at least 1 in (ξ_0, ξ''). Noting that $\det G(x) = \det\left(G\left(\rho^{-s/2}y_0, \rho^{-s/2}y'', \rho^{-s}y_n\right)\right) = \det G(0) + O\left(\rho^{-s/2}\right)$, we may write

$$K_{22}(y, D_y) = \det G(0) I_{N-3} D_n^N \rho^{Ns} + O\left(\rho^{(N-(1/2))s}\right).$$

Consider now the block K_{21}. The principal symbol of a generic element of the second column of K_{21} is given by

$$\begin{aligned}\sigma_N(k_{i2})(x,\xi) &= \sum_{k=1}^{N}(L_1)_{ik}(x,\xi)\sigma_{N-1}(r_{k2})(x,\xi) \\ &= a_{i2}(x,\xi')\sigma_{N-1}(r_{22})(x,\xi) + a_{i3}(x,\xi')\sigma_{N-1}(r_{32})(x,\xi) \\ &\quad + \sum_{k=4}^{N}(-L_{22}\,{}^{\mathrm{co}}G)_{i-3\,k-3}(x,\xi)a_{k2}(x,\xi')\xi_n^{N-2},\end{aligned}$$

where $i = 4, \ldots, N$.

Recalling (3.10), we easily see that

$$k_{i2}(y, D_y) = O\left(\rho^{(N-1)s}\right), \quad i = 4\ldots, N.$$

An analogous calculation, repeated for the third column, yields:

$$k_{i3}(y, D_y) = O\left(\rho^{(N-1)s}\right), \quad i = 4\ldots, N.$$

As far as the first column of the block K_{21} is concerned, we get that

$$\begin{aligned}\sigma_N(k_{j1})(x,\xi) &\equiv 0,\\ \sigma_{N-1}(k_{j1})(x,\xi) = \sum_{k=1}^{3}\frac{1}{i}\langle\partial_\xi(L_1)_{jk},(\partial_x({}^{\mathrm{co}}L_1)_{k3})\rangle &+ \sum_{k=4}^{N}\frac{1}{i}\langle\partial_\xi(L_1)_{jk},\partial_x({}^{\mathrm{co}}L_1)_{k3}\rangle\\ &+\sum_{k=1}^{3}(L_1)_{jk}\tau_k+\sum_{k=4}^{N}(L_1)_{jk}\tau_k.\end{aligned}$$

The terms in the second and third sum vanish of the first order in the (ξ_0,ξ'')–variables by Lemma 3.9, hence in the coordinates defined by (5.1) they are $O\left(\rho^{(N-(3/2))s}\right)$; the same holds, by Lemma 3.1(a), for the last two terms of the first sum. Thus we are left with

$$\frac{1}{i}\left[\langle\partial_\xi a_{j1},\partial_x({}^{\mathrm{co}}L_1)_{13}\rangle+\sum_{s,k=4}^{N}a^n_{jk}(\det G(x))^{-1}\xi_n^2\right.$$
$$\left.\times\langle(\partial_\xi a_{s1})({}^{\mathrm{co}}G)_{k-3\,s-3},\partial_x\det L_{22}\rangle\right](y,\eta)+O\left(\rho^{(N-(3/2))s}\right).$$

Computing the term in square brackets we obtain

$$\begin{aligned}&\langle\partial_\xi a_{j1},\partial_x\det L_{22}\rangle a_{12}a_{23}\\ &\quad\left.-\xi_n^2\sum_{s=4}^{N}\langle\partial_\xi a_{s1},\partial_x\det L_{22}\rangle\sum_{k=4}^{N}(G)_{j-3\,k-3}(G^{-1})_{k-3\,s-3}\right|_{(y,\eta)}\\ &= O\left(\rho^{(N-3/2)s}\right).\end{aligned}$$

Summing up our results, we have

$$K(y,D_y) \tag{5.2}$$

$$=\left[\begin{array}{ccc|c} & K_{11}(y,D_y) & & O(\rho^{(N-(1/2))s})\\ \hline O(\rho^{(N-(3/2))s}) & O(\rho^{(N-1)s}) & O(\rho^{(N-1)s}) & \rho^{Ns}\det G(0)I_{N-3}D_n^N\\ \vdots & \vdots & \vdots & +O(\rho^{(N-(1/2))s})\end{array}\right].$$

Eventually we study the block $K_{11}(y,D_y)$. Easily we have

$$K_{11}(y,D_y) \tag{5.3}$$

$$= \begin{bmatrix} O\left(\rho^{(N-(3/2))s}\right) & \rho^{Ns} \det G(0) I_2 D_n^N + O\left(\rho^{(N-(1/2))s}\right) \\ k_{31}(y, D_y) & O\left(\rho^{(N-(1/2))s}\right) \end{bmatrix}.$$

In fact for the upper right block we have

$$\begin{aligned} M(x,\xi) \det G(x)\, {}^{\text{co}}M(x,\xi)\xi_n^{N-2} &= \det G(x) \det M(x,\xi) I_2 \xi_n^{N-2} \\ &= \det G(x) I_2 \xi_n^N + O(|x|^{2k})\xi_n^N + C(x,\xi), \end{aligned}$$

where $C(x,\xi)$ is homogeneous of order at most $N-1$ w.r.t. ξ_n. Let us now turn to $(K_{11})_{11}$ and $(K_{11})_{21}$. The principal symbol of $(K_{11})_{11}$ is trivially zero, whereas the terms of order $N-1$ are given by

$$\sum_{j=1}^{N} \frac{1}{i} \langle \partial_\xi (L_1)_{1j}, \partial_x ({}^{\text{co}}L_1)_{j3} \rangle + \sum_{j=1}^{N} (L_1)_{1j} \tau_j.$$

Now, by Lemma 3.2, $({}^{\text{co}}L_1)_{j3}$ vanishes of the first order w.r.t. the variables (ξ_0, ξ'') when $j > 1$ and $(L_1)_{1j}$ vanishes of the first order in the variables (ξ_0, ξ'') when $j \neq 2$. Hence, performing the symplectic dilation in (5.1), we obtain

$$\begin{aligned} \sigma_{N-1}((K_{11})_{11}) = \frac{1}{i} & [\langle \partial_\xi (a_{11} - \xi_0), \partial_x ({}^{\text{co}}L_1)_{13} \rangle \\ & + a_{12} (\det G(x))^{-1} \langle \partial_\xi ({}^{\text{co}}L_1)_{23}, \partial_x \det L_{22} \rangle] \, (y, \eta) + O\left(\rho^{(N-(3/2))s}\right). \end{aligned}$$

Using Lemma 3.1(a) and replacing $({}^{\text{co}}L_{11})_{12}$, $({}^{\text{co}}L_{11})_{23}$ with their expressions we conclude:

$$\begin{aligned} \sigma_{N-1}((K_{11})_{11}) &= \frac{1}{i} [\langle \partial_\xi (a_{11} - \xi_0), \partial_x \det L_{22} \rangle a_{12} a_{23} \\ & \qquad - a_{12} a_{23} \langle \partial_\xi (a_{11} - \xi_0), \partial_x \det L_{22} \rangle] + O\left(\rho^{(N-(3/2))s}\right) \\ &= O\left(\rho^{(N-(3/2))s}\right). \end{aligned}$$

A quite analogous argument yields that

$$K_{21}(y, D_y) = O\left(\rho^{(N-(3/2))s}\right).$$

Next, in order to prove our statement we want to construct an asymptotic solution for the operator $L(x, D)$ depending on the large positive parameter ρ. Actually we construct an asymptotic solution for $K(x, D_x)$, which amounts to the same thing, since $R(x, D_x)$ is an elliptic operator.

Consider first the differential operator $k_{31}(x, D_x)$. Using symplectic dilations of the form

$$(5.4) \qquad \begin{cases} y_j = \rho^{s/2+\mu_j} x_j, & j = 0, 1, \ldots, n-1, \\ y_n = \rho^s x_n, \end{cases}$$

and the canonical conjugate in the dual variables, by Lemma 2.1 in [3] we may find a linear symplectic change of variables leaving the vector $(0, e_0) \in T_\rho T^*\Omega$ fixed and leaving invariant the Lagrangean plane $x = 0$, such that

$$
\begin{aligned}
(5.5) \qquad k_{31}(y, D_y) = \rho^{(N-(3/2))s}\Big\{ &(D_0 - \langle \lambda^{(1)}, y^{(1)}\rangle - \langle \lambda^{(2)}, D^{(2)}\rangle)\times \\
\Big[-D_0^2 + 2D_0L_1(y^{(1)}D_n, D^{(2)}) &+ 2D_0L_2(y^{(3)}D_n, D^{(3)}) + Q^{(1)}(y^{(1)}D_n, D^{(2)}) \\
+Q^{(2)}(y^{(3)}D_n, D^{(3)}) &+ Q^{(3)}(y^{(1)}D_n, D^{(2)}; y^{(3)}D_n, D^{(3)})\Big] \\
+ \Big(c_0D_0 + \langle c^{(1)}, y^{(1)}\rangle D_n + \langle c^{(2)}, D^{(2)}\rangle &+ \langle c_1^{(3)}, y^{(3)}\rangle D_n + \langle c_2^{(3)}, D^{(3)}\rangle\Big) D_n\Big\} D_n^{N-3} \\
&+O\left(\rho^{(N-2)s}\right),
\end{aligned}
$$

where we used the notation

$$
\begin{aligned}
y^{(1)} &= (y_1, \ldots, y_d), \qquad & y^{(2)} &= (y_{d+1}, \ldots, y_\ell), \\
y^{(3)} &= (y_{\ell+1}, \ldots, y_{n'}), \qquad & y^{(4)} &= (y_{n'+1}, \ldots, y_{n+1}),
\end{aligned}
$$

d, ℓ, n' being suitable positive integers determined in terms of the geometric situation in $T_\rho T^*\Omega$ (see e.g. [3]); $\lambda^{(1)}$, $\lambda^{(2)}$ are suitable vectors, L_1, L_2 are linear forms, $Q^{(1)}$, $Q^{(2)}$ are positive definite quadratic forms and $Q^{(3)}$, is a real bilinear form on $\mathbb{R}^\ell \times \mathbb{R}^{2(n'-\ell)}$. We point out that the numbers μ_j, $j = 0, 1, \ldots, n-1$, and s in (5.4) can be chosen in such a way that the remainder term in (5.5) be negligible w.r.t. the principal part. The next step is just a hack in order to straighten things, i.e. put on the diagonal the important contributions.
Define

$$
\begin{aligned}
\Lambda &= \operatorname{diag}\left(\begin{bmatrix} 0 & 1 \\ I_2 & 0 \end{bmatrix}, I_{N-3} \right); \\
W_\rho &= \operatorname{diag}\left(\rho^{s/4}, I_{N-1}\right); \\
V_\rho &= \operatorname{diag}\left(\rho^{-(N-3/2)s}, I_2\rho^{-Ns}, I_{N-3}\rho^{-Ns}\right); \\
K'(y, D_y) &= W_\rho \Lambda K(y, D_y) W_\rho^{-1} V_\rho.
\end{aligned}
$$

Because of (5.2) and (5.3) we obtain

$$
(5.6) \qquad K'(y, D_y)
$$

$$= \begin{bmatrix} P'(y,D_y) & O(\rho^{a(1,2)-s/4}) & O(\rho^{a(1,3)-s/4}) \\ O(\rho^{a(2,1)-7s/4}) & \det G(0)D_n^N I_2 + O(\rho^{a(2,2)-s/2}) & O(\rho^{a(2,3)-s/2}) \\ O(\rho^{a(3,1)-7s/4}) & O(\rho^{a(3,2)-s}) & \det G(0)D_n^N I_{N-3} + O(\rho^{a(3,3)-s/2}) \end{bmatrix},$$

where $P'(y,D_y) = \rho^{-(N-3/2)s}k_{31}(y,D_y)$ and the $a(i,j)$'s are integers not depending on s but only on $\mu = (\mu_0,\dots,\mu_{n-1})$. In order to prove our conditions we need another localization of $K'(x,D)$. Following [3] we define the symplectic dilation

$$S_\rho\left(y_0,y^{(1)},y^{(2)},y^{(3)},y_n\right) = \left(y_0/\rho^2, y^{(1)}/\rho^3, y^{(2)}, y^{(3)}/\rho^2, y_n/\rho^4\right)$$

and then put $K''(y,D_y) = (K'(y,D_y)\circ S_\rho)U_\rho$, where

$$U_\rho = \mathrm{diag}(\rho^{-(4N-6)}, \rho^{-4N}I_{N-1}).$$

Then we have

$$K''(y,D_y)$$

$$= \begin{bmatrix} P''(y,D_y) & O(\rho^{a(1,2)-s/4}) & O(\rho^{a(1,3)-s/4}) \\ O(\rho^{a(2,1)-s/4}) & \det G(0)D_n^N I_2 + O(\rho^{a(2,2)-s/2}) & O(\rho^{a(2,3)-s/2}) \\ O(\rho^{a(3,1)-7s/4}) & O(\rho^{a(3,2)-s}) & \det G(0)D_n^N I_{N-3} + O(\rho^{a(3,3)-s/2}) \end{bmatrix}$$

where

$$\begin{aligned} P''(y,D) = \Big\{&(D_0 - \langle\lambda^{(1)},y^{(1)}\rangle\rho^{-1}D_n - \rho^{-2}\langle\lambda^{(2)},D^{(2)}\rangle)\times \\ &[-D_0^2 + 2D_0L_1(\rho^{-1}y^{(1)}D_n,\rho^{-2}D^{(2)}) + 2D_0L_2(y^{(3)}D_n,D^{(3)}) \\ &+Q^{(1)}(\rho^{-1}y^{(1)}D_n,\rho^{-2}D^{(2)}) + Q^{(2)}(y^{(3)}D_n,D^{(3)}) \\ &+Q^{(3)}(\rho^{-1}y^{(1)}D_n,\rho^{-2}D^{(2)};y^{(3)}D_n,D^{(3)})] \\ &+(c_0D_0 + \langle c^{(1)},\rho^{-1}y^{(1)}\rangle D_n + \rho^{-2}\langle c^{(2)},D^{(2)}\rangle \\ &+\langle c_1^{(3)},y^{(3)}\rangle D_n + \langle c_2^{(3)},D^{(3)}\rangle)D_n\Big\}D_n^{N-3} + O(\rho^{-k}), \end{aligned}$$

where k is a suitably large positive integer. In Proposition 4.2 we proved that

$$C(x,\xi) \equiv_2 k_{31}^s(x,\xi) \quad \text{at } \rho.$$

This implies that $H_C(\rho) = H_{k^s_{31}}(\rho)$, hence condition (4.2) is equivalent to

$$\text{Im}\,(c_0, \mathcal{C}, c_1^{(3)}, c_2^{(3)}) = 0, \tag{5.7}$$

where $\mathcal{C} = (c^{(1)}, c^{(2)})$.

We want to show that if condition (5.7) (i.e. condition (4.2)) is violated then there exists a null asymptotic solution u_ρ of K violating the a priori estimate implied by the well–posedness of the Cauchy problem.

Suppose first that Im $\left(c_1^{(3)}, c_2^{(3)}\right) \neq 0$. Let us consider the operator $K'(y, D)$; using symplectic dilations we may throw away the "involutive variables" $\left(y^{(1)}, \eta^{(2)}\right)$. Put

$$y^{(3)} = \left(y_{\ell+1}, y^{(5)}\right), \quad \left(c_1^{(3)}\right)_{\ell+1} = \alpha + i\beta, \quad \left(c_2^{(3)}\right)_{\ell+1} = \alpha' + i\beta';$$

here we assume that $(\beta, \beta') \neq (0,0)$ and shall actually argue when $\beta' > 0$. Perform the symplectic dialation

$$\left(y_0, y_{\ell+1}, y^{(5)}, y_n\right) \longmapsto \left(y_0, \rho^{-\kappa} y_{\ell+1}, y^{(5)}, \rho^{-\kappa} y_n\right),$$

where κ is a suitable positive integer. Denoting by $\tilde{K}'(y, D_y) = (K' \circ \tilde{S}_\rho)(y, D_y) U_\rho^{(2)}$, where $U_\rho^{(2)} = \text{diag}(\rho^{-(N-1)\kappa}, I_{N-1}\rho^{-N\kappa})$, we obtain

$$\tilde{K}'(y, D_y)$$
$$= \begin{bmatrix} \tilde{P}'(y, D_y) & O(\rho^{a'(1,2)-s/4}) & O(\rho^{a'(1,3)-s/4}) \\ O(\rho^{a'(2,1)-7s/4}) & \begin{matrix}\det G(0) D_n^N I_2 \\ +O(\rho^{a'(2,2)-s/2})\end{matrix} & O(\rho^{a'(2,3)-s/2}) \\ O(\rho^{a'(3,1)-7s/4}) & O(\rho^{a'(3,2)-s}) & \begin{matrix}\det G(0) D_n^N I_{N-3} \\ +O(\rho^{a'(3,3)-s/2})\end{matrix} \end{bmatrix}.$$

Denote by E_ρ the multiplication by

$$I_N \exp\,(i\rho^2 y_{\ell+1} + i\rho^3 y_n + i\rho\varphi(y)),$$

and consider

$$H' = E_\rho^{-1} \tilde{K}' E_\rho U_\rho^{(3)} = [h_{ik}]_{\substack{1 \le i \le N \\ 1 \le k \le N}},$$

where

$$U_\rho^{(3)} = \text{diag}\left(\rho^{-(3N-4)}, \rho^{-3N} I_{N-1}\right).$$

The elements of $H'(y, D_y)$ can be written as

$$h_{ki}(y, D_y) \sim \sum_{j=0}^{\infty} \rho^{-j} h_{ki}^{j}(y, D_y), \quad i, k = 1, \dots, N,$$

where, by the definition of $\tilde{K}'$, we have

$$(5.8) \qquad \begin{cases} h_{ii} = 1 + O(\rho^{-1}), & i = 2, \dots, N; \\ h_{ij} = O(\rho^{a''(i,j)-s/4}), & a''(i,j) \in \mathbb{Z},\ i \neq j,\ i, j = 1 \dots, N \\ h_{11} = \rho E_\rho^{-1} \tilde{P}' E_\rho + O(\rho^{-\mu}). \end{cases}$$

Proceeding as in [3] we have

$$(5.9) \qquad \begin{aligned} h_{11}^{(0)} &= q\varphi_{y_0} + \alpha' + i\beta', \\ h_{11}^{(1)} &= q(D_0 + \varphi_{y_0}\varphi_{y_{\ell+1}}), \end{aligned}$$

where $q = Q^{(2)}(1, 0; 0, 0)$ is a positive real number. Moreover the integers $a''(i, j)$ do not depend on s and we can choose s in such a way that

$$(5.10) \qquad \mu \geq 1, \quad h_{ij}^{(0)} = h_{ij}^{(1)} = 0, \quad \text{if } i \neq j.$$

We now want to construct an asymptotic solution for H' in the form

$$u \sim \sum_{d=0}^{\infty} \rho^{-d} u_d(y),$$

where

$$u_d(y) = \sum_{j=1}^{N} \sigma_j^d(y) e_j,$$

e_j being basis vectors in $\mathbb{C}^N$ and the σ_j^d being smooth scalar functions. We introduce the following notation:

$$\tilde{\sigma}^d = (\sigma_2^d, \dots, \sigma_N^d),$$

$$\tilde{h}^{(d)} = (h_{12}^{(d)}, \dots, h_{1N}^{(d)}),$$

$$\hat{h}^{(d)} = (h_{21}^{(d)}, \dots, h_{N1}^{(d)}),$$

$$H^{(d)} = [h_{ij}^{(d)}]_{\substack{2 \leq i \leq N \\ 2 \leq j \leq N}}, \quad d \geq 0.$$

From (5.8) and (5.10) we have

$$\tilde{h}^{(0)} = \tilde{h}^{(1)} = 0, \; \hat{h}^{(0)} = \hat{h}^{(1)} = 0, \; H^{(0)} = I_{N-1}. \tag{5.11}$$

In order to solve the equation

$$H'(y, D) \sum_{d=0}^{\infty} \rho^{-d} u_d(y) \sim 0, \tag{5.12}$$

we annihilate the coefficient of ρ^{-m}, $m = 0, 1, \ldots$ in the left hand side of (5.12). Thus (5.12) is equivalent to the set of equations

$$\sum_{p=0}^{m} h_{11}^{(m-p)} \sigma_1^p + \sum_{p=0}^{m} \tilde{h}^{(m-p)} \tilde{\sigma}^p = 0, \tag{5.13}$$

$$\sum_{p=0}^{m} \hat{h}^{(m-p)} \sigma_1^p + \sum_{p=0}^{m} H^{(m-p)} \tilde{\sigma}^p = 0, \tag{5.14}$$

for $m = 0, 1, \ldots$ Equations (5.13), (5.14) can be reduced to a family of equations in the unknown functions σ_1^d. Let us consider the first m equations in (5.14); they can be written in the form

$$\mathcal{H} \begin{bmatrix} \tilde{\sigma}^0 \\ \tilde{\sigma}^1 \\ \vdots \\ \tilde{\sigma}^m \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_m \end{bmatrix}, \tag{5.15}$$

where $b_j = -\sum_{p=0}^{j} \hat{h}^{(j-p)} \sigma_1^p$ and

$$\mathcal{H} = \begin{bmatrix} H^{(0)} & 0 & 0 & \cdots & 0 \\ H^{(1)} & H^{(0)} & 0 & \cdots & 0 \\ H^{(2)} & H^{(1)} & H^{(0)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H^{(m)} & H^{(m-1)} & H^{(m-2)} & \ldots & H^{(0)} \end{bmatrix},$$

i.e. $\mathcal{H}$ is a $(m+1)(N-1) \times (m+1)(N-1)$ matrix which is non singular since $\det \mathcal{H} = 1$. As a consequence

$$\begin{bmatrix} \tilde{\sigma}^0 \\ \tilde{\sigma}^1 \\ \vdots \\ \tilde{\sigma}^m \end{bmatrix} = {}^{\mathrm{co}}\mathcal{H} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_m \end{bmatrix}.$$

Let ${}^{co}\mathcal{H} = [C_{ij}]_{\substack{0\le i\le m\\ 0\le j\le m}}$, where C_{ij} is a $(N-1)\times(N-1)$ block $\forall\, i,j = 0,\dots,m$. We want to compute ${}^{co}\mathcal{H}$; we have ${}^{co}\mathcal{H}\mathcal{H} = I_{(m+1)(N-1)}$, which can be rewritten as

$$\sum_{k=0}^{m} H^{(i-k)} C_{kj} = \delta_{ij} I_{N-1},\ \ i,j = 0,\dots,m. \tag{5.16}$$

We make the convention that $H^{(m)} = 0$ if $m < 0$.

Let $i = 1$ in (5.16); we obtain

$$H^{(0)} C_{00} = I_{N-1}, \quad H^{(0)} C_{0k} = 0, \quad k > 0,$$

hence

$$C_{00} = I_{N-1}, \qquad C_{0k} = 0 \quad \forall k > 0. \tag{5.17}$$

Let now $j > i$; then (5.16) becomes

$$0 = \sum_{k=0}^{m} H^{(i-k)} C_{kj} = \sum_{k=0}^{i} H^{(i-k)} C_{kj}.$$

If $C_{kj} = 0$ for $k < i$, then $C_{ij} = 0$; thus, by an inductive argument, (5.17) implies

$$C_{ij} = 0 \qquad \forall\, i,j = 0,\dots,m,\ i > j. \tag{5.18}$$

Let now $j = i$; then

$$I_{N-1} = \sum_{k=0}^{m} H^{(i-k)} C_{ki} = C_{ii},\ \ i = 0,\dots,m. \tag{5.19}$$

Finally if $j < i$ (5.16) yields

$$\begin{aligned} 0 &= \sum_{k=0}^{m} H^{(i-k)} C_{kj} = \sum_{k=j}^{m} H^{(i-k)} C_{kj} \\ &= \sum_{k=j}^{i} H^{(i-k)} C_{kj} = H^{(0)} C_{ij} + \sum_{k=j}^{i-1} H^{(i-k)} C_{kj}, \end{aligned}$$

so that

$$C_{ij} = -\sum_{k=j}^{i-1} H^{(i-k)} C_{kj}, \quad j < i. \tag{5.20}$$

Lemma 5.1. *The following relations holds*:

$$C_{ij} = C_{i+1\,j+1}, \quad i,j = 0,\dots,m-1. \tag{5.21}$$

Proof. By (5.18), (5.19) we have that (5.21) is true if $j \geq i$. It then suffices to prove (5.21) when $j < i$. From (5.19) we have

$$C_{i+1\,j+1} = -\sum_{k=j}^{i-1} H^{(i-k)} C_{k+1\,j+1},$$

and this allows us to prove the assertion via an inductive argument. □

Solving Equation (5.15), we obtain

$$\begin{aligned} \tilde{\sigma}^m &= \sum_{k=0}^{m} C_{mk}\Big(-\sum_{p=0}^{k} \hat{h}^{(k-p)} \sigma_1^p\Big) \\ &= \sum_{p=0}^{m}\sum_{k=p}^{m} C_{mk}\big(-\hat{h}^{(k-p)} \sigma_1^p\big), \end{aligned} \tag{5.22}$$

whence

$$\begin{aligned} \sum_{p=0}^{m} \tilde{h}^{(m-p)} \tilde{\sigma}^p &= \sum_{j=0}^{m}\sum_{p=j}^{m}\sum_{k=j}^{p} \tilde{h}^{(m-p)} C_{mk}\big(-\hat{h}^{(k-j)}\big)\sigma_1^j \\ &= \sum_{j=0}^{m} P_j^{(m)} \sigma_1^j. \end{aligned}$$

The last line of the above equation can be rewritten as

$$\sum_{\ell=0}^{m} P_{m-\ell}^{(m)} \sigma_1^{m-\ell}.$$

Actually the operators $P_{m-\ell}^{(m)}$ do not depend on m: we have

$$\begin{aligned} P_{m-\ell}^{(m)} &= \sum_{p=m-\ell}^{m}\ \sum_{k=m-\ell}^{p} \tilde{h}^{(m-p)} C_{mk}\big(-\hat{h}^{(k-m+\ell)}\big) \\ &= \sum_{p'=0}^{\ell}\sum_{k'=0}^{p'} \tilde{h}^{(\ell-p')} C_{m\,k'+m-\ell}\big(-\hat{h}^{(k')}\big) \\ &= \sum_{p'=0}^{\ell}\sum_{k'=0}^{p'} \tilde{h}^{(\ell-p')} C_{\ell k'}\big(-\hat{h}^{(k')}\big) \\ &\overset{def}{\equiv} P_\ell \end{aligned}$$

by (5.21).

Lemma 5.2. *One can find a family of differential operators* $\{\hat{P}_k\}_{k\geq 0}$, *such that the equations* (5.13) *and* (5.14) *are equivalent to the equations*

$$\sum_{j=0}^{m} \hat{P}_j \sigma_1^{m-j} = 0, \quad m = 0, 1, \ldots, \tag{5.23}$$

where $\hat{P}_0 = h_{11}^{(0)}$, $\hat{P}_1 = h_{11}^{(1)}$, $\hat{P}_2 = h_{11}^{(2)}$. *More precisely if the functions* σ_1^d, $d = 0, 1, \ldots$, *satisfy* (5.23), *then* σ_1^d *and* $\tilde{\sigma}^d$ *given by* (5.22) *are solution of* (5.13) *and* (5.14).

Proof. Equation (5.13) can be written as

$$\sum_{p=0}^{m} h_{11}^{(m-p)} \sigma_1^p + \sum_{\ell=0}^{m} P_\ell \sigma_1^{m-\ell} = 0, \quad m = 0, 1, \ldots$$

Since from (5.11) $P_\ell = 0$ when $\ell = m-2$, $m-1$, m, define

$$\begin{aligned} \hat{P}_\ell &= h_{11}^{(\ell)}, \text{ if } \ell = 0, 1, 2, \\ \hat{P}_\ell &= h_{11}^{(\ell)} + P_\ell, \text{ if } \ell = 3, 4, \ldots \end{aligned}$$

Equation (5.13) then becomes

$$\sum_{j=0}^{m} \hat{P}_j \sigma_1^{m-j} = 0, \quad m = 0, 1, \ldots,$$

and this proves the Lemma. □

Recalling (5.9), in order to solve (5.23), we may proceed as in [3] choosing $\varphi(x) = -((\alpha' + i\beta')/q)y_0 + i\sum_{j=1}^{n} y_j^2$ and then arguing as in Hörmander [4].

The cases $\beta' < 0$ and $\beta \lesseqgtr 0$ can be handled in essentially the same way and we refer the reader to [3]. Hence we get Im $\left(c_1^{(3)}, c_2^{(3)}\right) = 0$, at ρ.

The remaining cases, i.e. Im $(c_0, \mathcal{C}) \neq 0$ and Im $(c_0, \mathcal{C}) = 0$, $H_{k_{31}^s} + \mathrm{Tr}^+ F_{Q_2} H_\ell$ does not belong to the propagation cone of h are dealt with along the same guidelines, starting from the operator $K''(y, D_y)$. This completes the proof of Theorem 4.1.

A Appendix

In this Appendix we study the invariance properties of the symbol $\mathcal{L}$ defined in (2.5) with respect to changes of coordinates in $\mathbb{C}^N$ depending only on x. More precisely denote by $U(x)$, x in a suitable neighborhood of the origin in $\mathbb{R}^{n+1}$, the matrix of the coordinate change in $\mathbb{C}^N$. Let $L'(x,D)$ be the transformed operator of $L(x,D)$:

$$L'(x,D) = -D_0 + \sum_{j=1}^{n} U^{-1}(x)A_j(x)U(x)D_j + U^{-1}(x)B(x)U(x) - U^{-1}(x)(D_0U)(x) \sum_{j=1}^{n} U^{-1}(x)A_j(x)(D_jU)(x) = L_1'(x,D) + B'(x). \tag{A.1}$$

Denote by $\mathcal{L}'(x,\xi)$ the result of definition (2.5) applied to the operator L' defined in (A.1). We have:

$$^{\mathrm{co}}(U^{-1}L_1U) = U^{-1}\,{}^{\mathrm{co}}L_1U, \tag{A.2}$$

so that ${}^{\mathrm{co}}L_1' = U^{-1}\,{}^{\mathrm{co}}L_1U$. From now on we drop the variables x and ξ to simplify the notation when this will cause no misunderstanding. Hence

$$\begin{aligned}\mathcal{L}' &= B'\,{}^{\mathrm{co}}L' + \frac{i}{2}\langle\partial_x,\partial_\xi\rangle L_1'\,{}^{\mathrm{co}}L_1' - \frac{i}{2}\{L_1',\,{}^{\mathrm{co}}L_1'\}\\ &= U^{-1}\mathcal{L}U + \frac{1}{i}\sum_{j=0}^{n} U^{-1}(\partial_{\xi_j}L_1)\partial_{x_j}UU^{-1}\,{}^{\mathrm{co}}L_1U\\ &\quad+\frac{i}{2}\sum_{j=0}^{n}\left[\partial_{x_j}U^{-1}\partial_{\xi_j}L_1U + U^{-1}\partial_{\xi_j}L_1\partial_{x_j}U\right]U^{-1}\,{}^{\mathrm{co}}L_1U\\ &\quad-\frac{i}{2}\sum_{j=0}^{n}\left[U^{-1}\partial_{\xi_j}L_1U\left(\partial_{x_j}U^{-1}\,{}^{\mathrm{co}}L_1U + U^{-1}\,{}^{\mathrm{co}}L_1\partial_{x_j}U\right)\right.\\ &\quad\left.-\left(\partial_{x_j}U^{-1}L_1U + U^{-1}L_1\partial_{x_j}U\right)U^{-1}\partial_{\xi_j}\,{}^{\mathrm{co}}L_1U\right].\end{aligned} \tag{A.3}$$

Now

$$\partial_{x_j}U^{-1}U = -U^{-1}\partial_{x_j}U, \tag{A.4}$$

$$\begin{cases}\partial_{x_j}L_1\,{}^{\mathrm{co}}L_1 \equiv_2 -L_1\partial_{x_j}\,{}^{\mathrm{co}}L_1, & \text{at } \rho;\\ \partial_{\xi_j}L_1\,{}^{\mathrm{co}}L_1 \equiv_2 -L_1\partial_{\xi_j}\,{}^{\mathrm{co}}L_1, & \text{at } \rho;\end{cases} \tag{A.5}$$

since $\det L_1 \equiv_3 0$, at ρ. We remark that the second, fourth and fifth term in the second equality of (A.3) cancel due to (A.4). By (A.5) the third and seventh terms cancel modulo a symbol vanishing of order 2 at ρ. Thus (A.3) becomes

$$\mathcal{L}' \equiv_2 U^{-1}\mathcal{L}U \tag{A.6}$$

$$+\frac{i}{2}\sum_{j=0}^{n} U^{-1}L_1UU^{-1}\partial_{\xi_j}{}^{\mathrm{co}}L_1\partial_{x_j}U$$
$$-\frac{i}{2}\sum_{j=0}^{n} U^{-1}L_1U\partial_{x_j}U^{-1}\partial_{\xi_j}{}^{\mathrm{co}}L_1U.$$

Summing up we proved the following

Proposition A.1. *Denote by $U(x)$ a smooth $N\times N$ non singular matrix defined in a neighborhood of 0 in $\mathbb{R}^{n+1}$. Then*

$$\mathcal{L}' \equiv_2 U^{-1}\mathcal{L}U + L_1'T,$$

for a certain matrix $T\in S^{N-2}(\Omega; M_N(\mathbb{C}))$.

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S. Benvenuti
Dipartimento di Matematica
Università di Bologna
Piazza di Porta S. Donato 5
40127 Bologna, Italia

A. Bove
Dipartimento di Matematica
Università di Bologna
Piazza di Porta S. Donato 5
40127 Bologna, Italia