

A LIOUVILLE TYPE THEOREM FOR p -HARMONIC MAPS

Dedicated to Professor Fumiyuki Maeda on his sixtieth birthday

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1. Introduction

In this note we prove a Liouville type theorem for p -harmonic maps. Let (M, g) , (N, h) be Riemannian manifolds, and let $p \geq 2$. By Nash's isometric embedding, we may assume that N is a submanifold of a Euclidean space \mathbb{R}^d . The Sobolev space $W_{loc}^{1,p}(M, N)$ is defined to be

$$W_{loc}^{1,p}(M, N) = \left\{ u \in W_{loc}^{1,p}(M, \mathbb{R}^d); u(x) \in N \text{ a.e. } x \in M \right\},$$

where $W_{loc}^{1,p}(M, \mathbb{R}^d)$ denotes the Sobolev space of \mathbb{R}^d -valued L_{loc}^p -functions on M whose derivative belong to L_{loc}^p . A p -harmonic map $u : M \rightarrow N$ is a weak solution of the equation

$$(1.1) \quad \text{Trace}(\nabla(\|du\|^{p-2}du)) = 0,$$

i.e., $u \in W_{loc}^{1,p}(M, N)$ satisfies

$$(1.2) \quad - \int_M \|du\|^{p-2} \nabla u \cdot \nabla \varphi + \int_M \|du\|^{p-2} A(u)(\nabla u, \nabla u) \cdot \varphi = 0$$

for any $\varphi \in C_0^\infty(M, N)$, where A denotes the second fundamental form of N . A p -harmonic map u is characterized as a critical point of the p -energy functional

$$(1.3) \quad E_p(u) = \int_M \|du\|^p$$

in $W_{loc}^{1,p}(M, N)$, if the value of this functional is finite. When $p \neq 2$, the degenerate ellipticity of the equation (1.1) gives only (partial) $C^{1,\alpha}$ -regularity even for minimizers of the functional (1.3), while in case $p = 2$, $C^{1,\alpha}$ -regularity implies C^∞ -regularity. So we are concerned with p -harmonic maps which belong to the C_{loc}^1 -class, for general p .

Several studies are given for 2-harmonic maps or harmonic maps. (See Eells and Lemaire [3], [4].) For these harmonic maps, there are Liouville type theorems,

which states that a harmonic map u is constant under some conditions. A typical one of such conditions is the boundedness of u , where we say that u is bounded if its image is contained in a compact set. As a result with assumptions to images of maps, a Liouville type property is known when the image is enveloped by a convex function. (See Gordon [6] for $p = 2$, L.-F.Cheung and P.-F.Leung [1] for general $p \geq 2$.) The finiteness of the energy is another typical condition; precisely speaking, a harmonic map u is constant if $E_2(u) < \infty$, when M is complete and noncompact with $\text{Ric}_M \geq 0$ and $\text{Sect}_N \leq 0$, where Ric_M denotes the Ricci curvature of M , and Sect_N denotes the sectional curvature of N . (See Schoen and Yau [8], Hildebrandt [7].) In this note we extend this result for general $p \geq 2$.

Theorem 1. *Let M be complete and noncompact. Assume $\text{Ric}_M \geq 0$ and $\text{Sect}_N \leq 0$. Let $u : M \rightarrow N$ is a p -harmonic map of C^1_{loc} -class such that $E_p(u) < \infty$. Then u is a constant map.*

In [9], Takeuchi proved Theorem 1, using Hildebrandt’s argument [7], under the condition $E_{2p-2}(u) < \infty$ instead of $E_p(u) < \infty$. The exponent $2p - 2$, however, is not compatible in our case, since $2p - 2 \neq p$ when $p \neq 2$. Our proof of Theorem 1 has two steps; the first step (Section 3) for C^3_{loc} -maps and the second step (Section 4) in general case. The first step is based on a Bochner type formula (Section 2) and a standard argument of cutoff functions, and the second step depends on the approximation argument, which is used in [2] when M is an open set in a Euclidean space.

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2. Bochner type formula

In this section we give the following Bochner type formula.

Lemma 1. *Let $u : M \rightarrow N$ be a map of C^3_{loc} -class. Then the following equality holds:*

$$\begin{aligned}
 \text{(a)} \quad & \|du\|^{p-1} \Delta \|du\|^{p-1} + \langle \|du\|^{p-2} du, (d^\nabla \delta^\nabla + \delta^\nabla d^\nabla)(\|du\|^{p-2} du) \rangle \\
 & = 2(\|\nabla(\|du\|^{p-2} du)\|^2 - \|\nabla\|du\|^{p-1}\|^2) \\
 & \quad + \|du\|^{2p-4} \sum_{j=1}^m \langle \text{Ric}_M(du(e_j)), du(e_j) \rangle \\
 & \quad - \|du\|^{2p-4} \sum_{i,j=1}^m \langle \text{Riem}_N(du(e_i), du(e_j)) du(e_j), du(e_i) \rangle,
 \end{aligned}$$

hence

$$\begin{aligned}
 \text{(b)} \quad & \|du\| \triangle \|du\|^{p-1} + \langle du, (d^\nabla \delta^\nabla + \delta^\nabla d^\nabla)(\|du\|^{p-2} du) \rangle \\
 & \geq \|du\|^{p-2} \sum_{j=1}^m \langle \text{Ric}_M(du(e_j)), du(e_j) \rangle \\
 & \quad - \|du\|^{p-2} \sum_{i,j=1}^m \langle \text{Riem}_N(du(e_i), du(e_j)) du(e_j), du(e_i) \rangle,
 \end{aligned}$$

where $\{e_j\}_{j=1}^k$ is an orthonormal base of the tangent space of M , and d^∇ [resp. δ^∇] denotes the derivation with respect to ∇ [resp. the L^2 -adjoint operator of d^∇].

Therefore, when $\text{Ric}_M \geq 0$ and $\text{Sect}_N \leq 0$, we have

$$\text{(c)} \quad \|du\| \triangle \|du\|^{p-1} + \langle du, (d^\nabla \delta^\nabla + \delta^\nabla d^\nabla)(\|du\|^{p-2} du) \rangle \geq 0.$$

Proof of Lemma 1. Using the relation between the rough Laplacian \triangle and the Hodge-de Rham Laplacian $d^\nabla \delta^\nabla + \delta^\nabla d^\nabla$, we have

$$\begin{aligned}
 \text{(2.4)} \quad & \frac{1}{2} \triangle \|du\|^{2p-2} = \frac{1}{2} \triangle \| \|du\|^{p-2} du \|^2 \\
 & = \langle \|du\|^{p-2} du, \triangle(\|du\|^{p-2} du) \rangle + \|\nabla(\|du\|^{p-2} du)\|^2 \\
 & = -\langle \|du\|^{p-2} du, (d^\nabla \delta^\nabla + \delta^\nabla d^\nabla)(\|du\|^{p-2} du) \rangle \\
 & \quad + \|\nabla(\|du\|^{p-2} du)\|^2 + Q(u),
 \end{aligned}$$

where

$$\begin{aligned}
 Q(u) = & \|du\|^{2p-4} \sum_{i=1}^n \langle \text{Ric}_M(du(e_i)), du(e_i) \rangle \\
 & - \|du\|^{2p-4} \sum_{i,j=1}^n \langle \text{Riem}_N(du(e_i), du(e_j)) du(e_j), du(e_i) \rangle.
 \end{aligned}$$

(cf. Eells and Lemaire [3, p.8, (2.20)] for $p = 2$; Note that for any harmonic map ($p = 2$), $d^\nabla(du) = \delta^\nabla(du) = 0$.) On the other hand, we see

$$\begin{aligned}
 \text{(2.5)} \quad & \frac{1}{2} \triangle \|du\|^{2p-2} = \frac{1}{2} \triangle (\|du\|^{p-1})^2 \\
 & = \|du\|^{p-1} \triangle \|du\|^{p-1} + \|\nabla \|du\|^{p-1}\|^2.
 \end{aligned}$$

Then from (2.4) and (2.5), we have (a). From (a), we have

$$\begin{aligned}
 (2.6) \quad & \|du\|^{p-1} \triangle \|du\|^{p-1} + \langle \|du\|^{p-2} du, (d^\nabla \delta^\nabla + \delta^\nabla d^\nabla)(\|du\|^{p-2} du) \rangle \\
 & \geq \|du\|^{2p-4} \sum_{j=1}^m \langle \text{Ric}_M(du(e_j)), du(e_j) \rangle \\
 & \quad - \|du\|^{2p-4} \sum_{i,j=1}^m \langle \text{Riem}_N(du(e_i), du(e_j)) du(e_j), du(e_i) \rangle,
 \end{aligned}$$

since

$$\|\nabla \|du\|^{p-1}\| = \|\nabla \|du\|^{p-2} du\| \leq \|\nabla(\|du\|^{p-2} du)\|.$$

On any point such that $du = 0$, the inequality (b) holds trivially. On the other points, we divide the both sides of (2.6) by $\|du\|^{p-2}$, and then we have the inequality (b). □

3. Proof of Theorem 1 for C_{loc}^3 -maps

Throughout this paper, all positive constants C_1, C_2, \dots depend only on p, M, N if there is no special mention. Since u is a p -harmonic map,

$$\delta^\nabla(\|du\|^{p-2} du) = 0.$$

Then by Lemma 1 (c), we have

$$(3.7) \quad \|du\| \triangle \|du\|^{p-1} + \langle du, \delta^\nabla d^\nabla(\|du\|^{p-2} du) \rangle \geq 0.$$

since $\text{Ric}_M \geq 0, \text{Sect}_N \leq 0$. Take any point $x \in M$. Let η be a cutoff function satisfying that

$$(3.8) \quad \eta : \begin{cases} = 1 & \text{on } B_\rho(x) \\ \in [0, 1] & \text{on } B_{2\rho}(x) - B_\rho(x) \\ = 0 & \text{on } M - B_{2\rho}(x) \end{cases}$$

and that

$$(3.9) \quad \|\nabla \eta\|^2 \leq \frac{C_1}{\rho^2}.$$

Then from (3.7), we get

$$(3.10) \quad \int_M \|du\| \eta^2 \triangle \|du\|^{p-1} + \int_M \langle \eta^2 du, \delta^\nabla d^\nabla(\|du\|^{p-2} du) \rangle \geq 0.$$

We have

$$\begin{aligned}
 (3.11) \quad & \int_M \|du\| \eta^2 \Delta \|du\|^{p-1} \\
 &= - \int_M \nabla(\|du\| \eta^2) \cdot \nabla \|du\|^{p-1} \\
 &= -(p-1) \int_M \|du\|^{p-2} \eta^2 \|\nabla \|du\|\|^2 - 2(p-1) \int_M \|du\|^{p-2} \eta \nabla \|du\| \cdot \nabla \eta \\
 &= -\frac{4(p-1)}{p^2} \int_M \|\nabla \|du\|^{p/2}\|^2 \eta^2 - \frac{4(p-1)}{p} \int_M \|du\|^{p/2} \eta \nabla \|du\|^{p/2} \cdot \nabla \eta \\
 &\leq -\frac{4(p-1)}{p^2} \int_M \|\nabla \|du\|^{p/2}\|^2 \eta^2 + \varepsilon \int_M \|\nabla \|du\|^{p/2}\|^2 \eta^2 + \frac{C_2}{\varepsilon} \int_M \|du\|^p \|\nabla \eta\|^2
 \end{aligned}$$

for any $\varepsilon > 0$. We see

$$(3.12) \quad \int_M \langle \eta^2 du, \delta^\nabla d^\nabla (\|du\|^{p-2} du) \rangle = \int_M \langle d^\nabla (\eta^2 du), d^\nabla (\|du\|^{p-2} du) \rangle.$$

Since

$$\|d^\nabla (\varphi du)\| \leq C_3 \|\nabla \varphi\| \|du\|,$$

we have

$$\begin{aligned}
 (3.13) \quad & \left| \langle d^\nabla (\eta^2 du), d^\nabla (\|du\|^{p-2} du) \rangle \right| \\
 &\leq \|d^\nabla (\eta^2 du)\| \|d^\nabla (\|du\|^{p-2} du)\| \\
 &\leq C_4 \|\nabla \eta^2\| \|du\| \|\nabla \|du\|^{p-2}\| \|du\| \\
 &= C_5 \eta \|\nabla \eta\| \|du\|^{p-1} \|\nabla \|du\|\| \\
 &= C_6 \eta \|\nabla \eta\| \|du\|^{p/2} \|\nabla \|du\|^{p/2}\| \\
 &\leq \varepsilon \|\nabla \|du\|^{p/2}\|^2 \eta^2 + \frac{C_7}{\varepsilon} \|du\|^p \|\nabla \eta\|^2.
 \end{aligned}$$

From (3.12) and (3.13), we get

$$\begin{aligned}
 (3.14) \quad & \left| \int_M \langle \eta^2 du, \delta^\nabla d^\nabla (\|du\|^{p-2} du) \rangle \right| \\
 &\leq \varepsilon \int_M \|\nabla \|du\|^{p/2}\|^2 \eta^2 + \frac{C_7}{\varepsilon} \int_M \|du\|^p \|\nabla \eta\|^2.
 \end{aligned}$$

Then from (3.10), (3.11) and (3.14),

$$\left(\frac{4(p-1)}{p^2} - 2\varepsilon \right) \int_M \|\nabla \|du\|^{p/2}\|^2 \eta^2 \leq \frac{C_8}{\varepsilon} \int_M \|du\|^p \|\nabla \eta\|^2.$$

Let $\varepsilon = \frac{p-1}{p^2}$, and then we obtain

$$(3.15) \quad \int_{B_\rho(x)} \|\nabla \|du\|^{p/2}\|^2 \leq C_9 \int_M \|du\|^p \|\nabla \eta\|^2 \leq \frac{C_{10}}{\rho^2} \int_M \|du\|^p.$$

Let ρ go to infinity, and then we see that $\nabla \|du\|^{p/2} \equiv 0$, i.e., $\|du\|$ is constant on M . Note that the volume of M is infinite, since $\text{Ric}_M \geq 0$. Then by the condition $E_p(u) < \infty$, we conclude that the constant $\|du\|$ is zero. Therefore u is a constant map. \square

4. Proof of Theorem 1

In this section we complete our proof of Theorem 1 using an approximation. We use the arguments in Duzaar and Fuchs [2]. We may assume $p > 2$, since Theorem 1 holds for $p = 2$. As mentioned in the introduction, we may assume that the target manifold N is a submanifold of a Euclidean space \mathbb{R}^d , and that u is a map into \mathbb{R}^d . Then we know

Proposition 1. *For any p -harmonic map u of C^1_{loc} -class, $\|du\|^{p/2-1} du$ belong to $W^{1,2}_{loc}(TM, \mathbb{R}^d)$.*

When M is a domain of a Euclidean space, Proposition 1 is Lemma 2.2 in Duzaar and Fuchs [2]. In the above general situation, Proposition 1 can be proved with slight modifications.

Using Proposition 1, we will complete our proof of Theorem 1. Let $M_+ := \{x \in M; \|du\|(x) \neq 0\}$. By Proposition 1, we see that du is of $W^{1,2}_{loc}$ -class on M_+ , since

$$\nabla du = \nabla(\|du\|^{p/2-1} du) \|du\|^{1-p/2} - \frac{p-2}{p} \nabla(\|du\|^{p/2}) \|du\|^{-p/2} du,$$

hence

$$\begin{aligned} \|\nabla du\| &\leq \|du\|^{1-p/2} \|\nabla(\|du\|^{p/2-1} du)\| + \frac{p-2}{p} \|du\|^{1-p/2} \|\nabla(\|du\|^{p/2})\| \\ &\leq \left(1 + \frac{p-2}{p}\right) \|du\|^{1-p/2} \|\nabla(\|du\|^{p/2-1} du)\|. \end{aligned}$$

Then we can find an approximating sequence $\{u_k\}_{j=1}^\infty \subset C^\infty_{loc}(M, \mathbb{R}^d)$ such that as k goes to infinity,

- (a) u_k converges to u in $C^1_{loc}(M)$,
- (b) u_k converges to u weakly in $W^{1,p}_{loc}(M)$, and

(c) u_k converges to u weakly in $W_{loc}^{2,2}(M_+)$.

By Lemma 1, we have

$$\begin{aligned} & \|du_k\| \triangleq \|du_k\|^{p-1} \\ & + \frac{p}{2(p-1)} \langle du_k, (d^\nabla \delta^\nabla + \delta^\nabla d^\nabla)(\|du_k\|^{p-2} du_k) \rangle \geq 0, \end{aligned}$$

since $\text{Ric}_M \geq 0$, $\text{Sect}_N \leq 0$. Let η be a cutoff function on M satisfying (3.8) and (3.9), and let

$$\varphi_\varepsilon(x) := \begin{cases} \frac{\|du\|(x)}{\max\{\|du\|(x), \varepsilon\}} & \text{on } M_+ \\ 0 & \text{on } M - M_+. \end{cases}$$

Then $\varphi_\varepsilon \in L_0^{1,2}(M_+)$, where $L_0^{1,2}(M_+)$ is the completion of $C_0^\infty(M_+)$, and $\varphi_\varepsilon \rightarrow 1$, $\nabla \varphi_\varepsilon \rightarrow 0$ in $L_0^{1,2}(M_+)$. Using the function $\varphi_\varepsilon \eta^2$ instead of η^2 , we apply the estimates (3.11), (3.14) for smooth maps u_k . Then we have, instead of (3.15),

$$\begin{aligned} (4.16) \quad & \int_{M_+} \|\nabla \|du_k\|^{p/2}\|^2 \varphi_\varepsilon \eta^2 + \int_{M_+} \|du_k\|^{p/2} \eta^2 \nabla \|du_k\|^{p/2} \cdot \nabla \varphi_\varepsilon \\ & \leq \frac{C_{11}}{\rho^2} \int_{M_+} \|du_k\|^p \varphi_\varepsilon + C_{12} \int_{M_+} \langle \varphi_\varepsilon \eta^2 du_k, d^\nabla \delta^\nabla (\|du_k\|^{p-2} du_k) \rangle \\ & \leq \frac{C_{11}}{\rho^2} \int_{M_+} \|du_k\|^p \varphi_\varepsilon + C_{12} \int_{M_+} \delta^\nabla (\varphi_\varepsilon \eta^2 du_k) \delta^\nabla (\|du_k\|^{p-2} du_k). \end{aligned}$$

Since u_k converge to u in $C_{loc}^1(M_+) \cap W_{loc}^{2,2}(M_+)$,

$$\int_{M_+} \delta^\nabla (\varphi_\varepsilon \eta^2 du_k) \delta^\nabla (\|du_k\|^{p-2} du_k) \rightarrow \int_{M_+} \delta^\nabla (\varphi_\varepsilon \eta^2 du) \delta^\nabla (\|du\|^{p-2} du) = 0.$$

as k goes to infinity, where the last equality follows from the fact that u satisfies the p -harmonic map equation $\delta^\nabla (\|du\|^{p-2} du) = 0$. Therefore, let k goes to infinity in (4.16), and then we have

$$\begin{aligned} & \int_{M_+} \|\nabla \|du\|^{p/2}\|^2 \varphi_\varepsilon \eta^2 + \int_{M_+} \|du\|^{p/2} \eta^2 \nabla \|du\|^{p/2} \cdot \nabla \varphi_\varepsilon \\ & \leq \frac{C_{11}}{\rho^2} \int_{M_+} \|du\|^p \varphi_\varepsilon. \end{aligned}$$

Let ε go to zero, and then we get

$$(4.17) \quad \int_{M_+} \|\nabla \|du\|^{p/2}\|^2 \eta^2 \leq \frac{C_{11}}{\rho^2} \int_{M_+} \|du\|^p$$

for any $\eta \in C_0^1(M)$ satisfying (3.8) and (3.9). Here we have used the facts that $\nabla\varphi_\varepsilon$ goes to a measure-valued tensor μ whose support is in ∂M_+ , and that $\|du\|$ vanishes there. Let ρ go to infinity in (4.17), and then we have $du = 0$ on M_+ , i.e., u is constant on M_+ . Hence u is a constant map on M . \square

5. A remark.

When M is compact, we have the following result, which is an extension of facts in harmonic map case ($p = 2$). (See Eells and Sampson [5].)

Theorem 2. *Let M be compact ($\partial M = \emptyset$). Let $u : M \rightarrow N$ be a p -harmonic map of C^1 -class.*

- (a) *Assume $\text{Ric}_M \geq 0$ and $\text{Sect}_N \leq 0$. Then u is totally geodesic.*
- (b) *In addition to (a), if $\text{Ric}_M > 0$ somewhere, then u is a constant map.*
- (c) *In addition to (a), if $\text{Sect}_N < 0$, then u is a constant map, or u maps onto a closed geodesic in N .*

Proof. We take an approximating sequence $\{u_k\}_{k=1}^\infty$ such as that in Section 4. Take the function φ_ε in Section 4. From Lemma 1 (b),

$$\int_{M_+} \|du_k\| \varphi_\varepsilon \Delta \|du_k\|^{p-1} + \int_{M_+} \langle \varphi_\varepsilon du_k, (d^\nabla \delta^\nabla + \delta^\nabla d^\nabla)(\|du_k\|^{p-2} du_k) \rangle \geq 0.$$

Using integration by parts, we have

$$\begin{aligned} & \int_{M_+} \|du_k\| \varphi_\varepsilon \Delta \|du_k\|^{p-1} \\ &= - \int_{M_+} \varphi_\varepsilon \nabla \|du_k\| \cdot \nabla \|du_k\|^{p-1} - \int_{M_+} \|du_k\| \nabla \|du_k\|^{p-1} \cdot \nabla \varphi_\varepsilon \\ &= - \frac{4(p-1)}{p^2} \int_{M_+} \|\nabla \|du_k\|^{p/2}\|^2 \varphi_\varepsilon - \int_{M_+} \|du_k\| \nabla \|du_k\|^{p-1} \cdot \nabla \varphi_\varepsilon. \end{aligned}$$

Then from the above inequality, we get

$$\begin{aligned} & \frac{4(p-1)}{p^2} \int_{M_+} \|\nabla \|du_k\|^{p/2}\|^2 \varphi_\varepsilon \\ & \leq \int_{M_+} \langle \delta^\nabla(\varphi_\varepsilon du_k), \delta^\nabla(\|du_k\|^{p-2} du_k) \rangle \\ & \quad + \int_{M_+} \langle d^\nabla(\varphi_\varepsilon du_k), d^\nabla(\|du_k\|^{p-2} du_k) \rangle \\ & = \int_{M_+} \langle \delta^\nabla(\varphi_\varepsilon du_k), \delta^\nabla(\|du_k\|^{p-2} du_k) \rangle \end{aligned}$$

$$+ \int_{M_+} \|\nabla\varphi_\varepsilon\| \|du_k\| \|d^\nabla(\|du_k\|^{p-2} du_k)\|.$$

We have used the fact $\|d^\nabla(\varphi_\varepsilon du_k)\| \leq C_3 \|\nabla\varphi_\varepsilon\| \|du_k\|$. Let $k \rightarrow \infty$ and let $\varepsilon \rightarrow 0$, then we have

$$\frac{4(p-1)}{p^2} \int_{M_+} \|\nabla\|du\|^{p/2}\|^2 \leq 0.$$

Therefore $\|du\|$ is constant on M_+ , hence on M , since $\|du\|$ is continuous. From Lemma 1 (a), we have

$$\begin{aligned} & - \int_{M_+} \|\nabla(\|du_k\|^{p-2} du_k)\|^2 \varphi_\varepsilon + \int_{M_+} \|du_k\|^{p/2} \nabla\|du_k\|^{p/2} \cdot \nabla\varphi_\varepsilon \\ & + \int_{M_+} \langle \|du_k\|^{p-2} \varphi_\varepsilon du_k, (d^\nabla \delta^\nabla + \delta^\nabla d^\nabla)(\|du_k\|^{p-2} du_k) \rangle \\ & \geq 0. \end{aligned}$$

We have applied the integration by parts and used the assumption $\text{Ric}_M \geq 0$ and $\text{Sect}_N \leq 0$ and the fact that

$$\|\nabla(\|du_k\|^{p-2} du_k)\|^2 \geq \|\nabla\|du_k\|^{p-2} du_k\|^2 \geq \|\nabla\|du_k\|^{p-1}\|^2.$$

Let $k \rightarrow \infty$, and let $\varepsilon \rightarrow 0$, then we get

$$0 \geq 2 \int_{M_+} \|\nabla(\|du\|^{p-2} du)\|^2$$

on M_+ , since $\|du\|$ is constant. Hence $\nabla du = 0$, i.e., u is totally geodesic on M_+ . Since $\|du\|$ is constant, u is a harmonic map; u is totally geodesic on M . We have (a).

We know, by the proof of (a),

$$\begin{aligned} \|du\| & \equiv \text{Const.} =: C_0 \text{ on } M \\ \nabla du & = 0 \quad \text{on } M_+ \end{aligned}$$

Then from Lemma 1 (a) again, we have

$$\begin{aligned} (5.18) \quad 0 & \leq C_0^{2p-4} \sum_{j=1}^m \langle \text{Ric}_M(du(e_j)), du(e_j) \rangle \\ & = C_0^{2p-4} \sum_{i,j=1}^m \langle \text{Riem}_N(du(e_j), du(e_j)) du(e_j), du(e_i) \rangle \\ & \leq 0 \end{aligned}$$

on M_+ . If $\text{Ric}_M > 0$ somewhere, the inequality (5.18) implies $C_0 = 0$, or $du = 0$ at this point. Then $C_0 = 0$, and we have (b). If $\text{Sect}_N < 0$ at a point $x^* \in M$, then $C_0 = 0$ or $\dim(\text{Image}(du(x^*))) \leq 1$. If $\dim(\text{Image}(du(x^*))) = 0$, then $\|du\|(x) = C_0 = \|du\|(x^*) = 0$ for any $x \in M$, i.e. u is a constant map. If $\dim(\text{Image}(du(x))) = 1$, we have (c) since u is totally geodesic. \square

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