Taniguchi, T.
Osaka J. Math.
**35** (1998), 147–164

# ISOLATION PHENOMENA FOR QUATERNIONIC YANG-MILLS CONNECTIONS

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(Received December 12, 1994)

## 1. Introduction and statement of results

In this paper, we shall study a certain class of Yang-Mills connections on a quaternionic Kähler manifold, called *quaternionic Yang-Mills connections.*

Our basic setting is the following. Let $E$ be an associated Riemannian vector bundle of a principal bundle with a compact Lie group $G$ as the structure group over a compact oriented Riemannian manifold $(M, g)$. Let $\mathcal{A}$ be the space of connections on $E$. For a connection $\nabla \in \mathcal{A}$, we denote by $d^{\nabla}$ and $\delta^{\nabla}$ the covariant exterior derivative and its formal adjoint respectively acting on $\mathrm{End}(E)$-valued $p$-forms.

The Yang-Mills energy functional $YM : \mathcal{A} \longrightarrow \mathbb{R}$ is defined by

$$YM(\nabla) = \frac{1}{2}\int_M \|F^{\nabla}\|^2 dv_g,$$

where $F^{\nabla}$ is the curvature of a connection $\nabla \in \mathcal{A}$. A connection $\nabla$ is called a *Yang-Mills connection* , if $\nabla$ is a critical point of the Yang–Mills energy functional $YM(\nabla)$; namely, if it satisfies the Euler-Lagrange equation

$$\delta^{\nabla} F^{\nabla} = 0.$$

By the Bianchi identity $d^{\nabla} F^{\nabla} = 0$, the Euler-Lagrange equation is equivalent to

$$\Delta^{\nabla} F^{\nabla} = 0,$$

which says that $F^{\nabla}$ is harmonic, where $\Delta^{\nabla} = d^{\nabla}\delta^{\nabla} + \delta^{\nabla} d^{\nabla}$.

Nitta ([6]), Mamone Capria-Salamon ([2]) independently found higher dimensional analogues of the notion of self-dual and anti-self-dual connections on a quaternionic Kähler manifold. A quaternionic Kähler manifold is a Riemannian $4n$-manifold whose holonomy group lies in $Sp(n)\cdot Sp(1)$, $n > 1$. In the case of $n = 1$, we add the condition that $M$ is Einstein and half-conformally flat. The bundle of 2-forms on a quaternionic Kähler manifold $(M, g)$ has the following irreducible decomposition as a representation of $Sp(n) \cdot Sp(1)$:

$$\wedge^2 T^* M = S^2\mathbb{H} \oplus S^2\mathbb{E} \oplus (S^2\mathbb{H} \oplus S^2\mathbb{E})^{\perp}, \tag{1.1}$$

where $\mathbb{H}$ and $\mathbb{E}$ are the vector bundles associated with the standard representations of $Sp(1)$ and $Sp(n)$, respectively. Corresponding to the decomposition (1.1), we write the curvature $F^{\nabla}$ as

$$F^{\nabla} = F^1 + F^2 + F^3,$$

where $F^1 \in \Gamma(M; S^2\mathbb{H} \otimes \mathrm{End}(E))$, $F^2 \in \Gamma(M; S^2\mathbb{E} \otimes \mathrm{End}(E))$ and $F^3 \in \Gamma(M; (S^2\mathbb{H} \oplus S^2\mathbb{E})^{\perp} \otimes \mathrm{End}(E))$. A connection $\nabla$ is said to be *$c_i$-self-dual* ($i=1$, 2 or 3) if $F^j = 0$ for all $j \neq i$. In the case of $n = 1$, we have $F^1 = F^+$, $F^2 = F^-$ and $F^3 = 0$ where $F^+$ (resp. $F^-$) is the (resp. anti-) self-dual part of the curvature $F^{\nabla}$. We shall confine ourself to the case where $(M, g)$ is a compact quaternionic Kähler $4n$-manifold.

Recall that each $c_i$-self-dual connection is a Yang-Mills connection (cf. [6], [2], [3]). Moreover, if $M$ is compact, a $c_1$ or $c_2$-self-dual connection is minimizing the Yang-Mills energy functional $YM(\nabla)$ (cf. [3], [2]). As far as we know, there is no example of non-flat $c_3$-self-dual connections. If they exist, they are believed to be unstable. Indeed, it is known ([7]) that any non-flat $c_3$-self-dual connection over the quaternionic projective space $\mathbb{H}P^n$ is, if it exists, unstable. Nagatomo ([5]) proved that there is a unique non-flat $c_1$-self-dual connection over any simply-connected quaternionic Kähler $4n$-manifold with $n > 1$.

Let us recall some results on Yang-Mills connections. Bourguignon and Lawson ([1]) discussed gap-phenomena for Yang-Mills connections. They gave explicit $C^0$-neighborhoods of the minimal Yang-Mills fields which contain no other Yang-Mills fields up to gauge equivalent. They obtained the following.

**Theorem A.** ([1]) *Let $\nabla$ be a Yang-Mills connection on $(S^4, g_0)$. If the self-dual part $F^+$ of the curvature of $\nabla$ satisfies the pointwise inequality $\|F^+\|^2 < 3$, then $F^+ = 0$. The same is true for the anti-self-dual part $F^-$ of the curvature of $\nabla$.*

They next examined the case where the inequality $\|F^{\nabla}\|^2 < 3$ is relaxed on $(S^4, g_0)$.

**Theorem B.** ([1]) *Let $\nabla$ be a Yang-Mills connection on a Riemannian vector bundle $E$ over $(S^4, g_0)$. If $F^{\nabla}$ satisfies the pointwise inequality $\|F^{\nabla}\|^2 \leq 3$, then either $E$ is flat or $E = E_0 \oplus S$ where $E_0$ is flat and where $S$ is one of the 4-dimensional bundles of tangent spinors with the canonical Riemannian connections.*

The purpose of this paper is to generalize these results to quaternionic Kähler manifolds. We introduce the following notion for connections:

Definition 1.1. A connection $\nabla$ on a Riemannian vector bundle over a compact quaternionic Kähler manifold is called a *quaternionic Yang-Mills connection* if $\Delta^\nabla(F^\nabla \wedge \Omega^{n-1}) = 0$ where $\Omega$ is the fundamental 4-form on $(M, g)$ (See §2).

Note that in the case of $n = 1$, the quaternionic Yang-Mills connections are the Yang-Mills connections, and vice versa. It is easy to see that the $c_1$-, $c_2$- and $c_3$-self-dual connections introduced above are quaternionic Yang-Mills connections.

**Proposition 1.1.** *If a connection $\nabla$ is a quaternionic Yang-Mills connection, then $\nabla$ is a Yang-Mills connection.*

We shall give a proof of Proposition 1.1 in § 3.

Wolf ([9]) classified the compact simply-connected quaternionic Kähler symmetric spaces, called *Wolf spaces*. The only examples of the Wolf spaces are the following.

$$\mathbb{H}P^n, \quad Gr_2(\mathbb{C}^{n+2}), \quad Gr_4(\mathbb{R}^{n+4}), \quad \frac{G_2}{SO(4)},$$
$$\frac{F_4}{Sp(3)\cdot Sp(1)}, \quad \frac{E_6}{SU(6)\cdot Sp(1)}, \quad \frac{E_7}{Spin(12)\cdot Sp(1)}, \quad \frac{E_8}{E_7\cdot Sp(1)}.$$

From now on, we suppose that $(M, g)$ is a Wolf space. Note that the Riemannian curvature operator $R$ acting on $\wedge^2 TM$ has also a splitting $R = R_1 + R_2 + R_3$ with respect to the decomposition (1.1). By ([4]) we can write the curvature operator $R_i$ as $R_i = \mu_i I_{\wedge^2 TM}$ where $\mu_i$ ($i = 1$ or $2$) is a positive constant. Since $R_3$ is negative semi-definite, we put $\mu_3 = 0$. We set $\lambda_i = s/(2n) - 2\mu_i$ ($i = 1$, $2$ or $3$) where $s$ is the scalar curvature of $(M, g)$. Then we shall state the following.

**Theorem 1.1.** *Let $\nabla$ be a quaternionic Yang-Mills connection on a Wolf space $(M,g)$, $(n \geq 1)$, and assume $F^3 = 0$, i.e., the $c_3$-self-dual part $F^3$ of the curvature of $\nabla$ vanishes.*

(1) *If the $c_1$-self-dual part $F^1$ of the curvature of $\nabla$ satisfies the pointwise inequality*

$$\|F^1\|^2 < \frac{n(4n-1)\lambda_1^2}{16(2n-1)^2},$$

*then $F^1 = 0$, that is, $\nabla$ is a $c_2$-self-dual connection.*

(2) *If the $c_2$-self-dual part $F^2$ of the curvature of $\nabla$ satisfies the pointwise inequality*

$$\|F^2\|^2 < \frac{n(4n-1)\lambda_2^2}{16(2n-1)^2},$$

*then $F^2 = 0$, that is, $\nabla$ is a $c_1$-self-dual connection.*

Theorem 1.1 for $M = \mathbb{H}P^1$ coincides with Theorem A. It seems that the assumption $F^3 = 0$ is necessary to get the generalization of Theorem A. We next show that the $c_3$-self-dual connections can be characterized as follows if they exist.

**Theorem 1.2.** *Let $\nabla$ be a quaternionic Yang-Mills connection on a Wolf space $(M, g)$, $(n \geq 1)$. If the $c_1$-self-dual part $F^1$and the $c_2$-self-dual part $F^2$of the curvature of $\nabla$ respectively satisfy the pointwise inequalities*

$$\|F^1\|^2 < \frac{n(4n-1)\lambda_1^2}{16(2n-1)^2}, \qquad \|F^2\|^2 < \frac{n(4n-1)\lambda_2^2}{16(2n-1)^2},$$

*then $F^1 = F^2 = 0$, that is, $\nabla$ is a $c_3$-self-dual connection.*

To generalize Theorem B, we suppose that the base manifold $M$ is a quaternionic projective space $(\mathbb{H}P^n, g_0)$. Let $g_0$ be the Riemannian metric on $\mathbb{H}P^n$ with the scalar curvature $s = 4n(2n-1)(n+2)$. With respect to this metric $g_0$, we calculate $\lambda_1$ and $\lambda_2$ of Theorem 1.1. Then we can read Theorem 1.1 as follows.

**Corollary 1.1.** *Let $\nabla$ be a quaternionic Yang-Mills connection on $(\mathbb{H}P^n, g_0)$, $(n \geq 1)$, and assume that $F^3 = 0$.*

(1) *If $F^1$ satisfies the pointwise inequality*

$$\|F^1\|^2 < n(4n-1),$$

*then $F^1 = 0$, that is, $\nabla$ is a $c_2$-self-dual connection.*

(2) *If $F^2$ satisfies the pointwise inequality*

$$\|F^2\|^2 < \frac{n(4n-1)(n+1)^2}{4},$$

*then $F^2 = 0$, that is, $\nabla$ is a $c_1$-self-dual connection.*

Using Corollary 1.1, we examine what happens when the inequality $\|F^\nabla\|^2 < n(4n-1)$ is relaxed on $(\mathbb{H}P^n, g_0)$.

**Theorem 1.3.** *Let $\nabla$ be a quaternionic Yang-Mills connection on a Riemannian vector bundle $E$ with any structure group $G$ over $(\mathbb{H}P^n, g_0)$, $(n \geq 1)$, and assume that $F^3 = 0$. If $F^\nabla$ satisfies the pointwise inequality $\|F^\nabla\|^2 \leq n(4n-1)$, then either $E$ is a flat vector bundle or $E = E_0 \oplus \mathbb{H}$, where $E_0$ is a flat vector bundle and where $\mathbb{H}$ is the tautological quaternion line bundle.*

In the case of $n = 1$, Theorem 1.3 coincides with Theorem B. We next obtain the following theorem in which the assumption of $F^3 = 0$ is not necessary.

**Theorem 1.4.** *Let $\nabla$ be a quaternionic Yang-Mills connection on a Wolf space $(M, g)$, $(n \geq 2)$.*
(1) *If $F^1$, $F^2$ and $F^3$ satisfy the pointwise inequalities*

$$\|F^1\| < \frac{\lambda_1}{\sqrt{2}},$$
$$\|F^3\| < \frac{\lambda_3}{\sqrt{2}} - (n+3)\|F^1\| - \|F^2\|,$$

*then $F^1 = F^3 = 0$, that is, $\nabla$ is a $c_2$-self-dual connection. Moreover if $\nabla$ is non-flat, then the $c_2$-self-dual part $F^2$ satisfies*

$$\frac{\lambda_2}{\sqrt{2}} \leq \|F^2\| < \frac{\lambda_3}{\sqrt{2}}.$$

(2) *If $F^1$, $F^2$ and $F^3$ satisfy the pointwise inequalities*

$$\|F^2\| < \frac{\lambda_2}{\sqrt{2}},$$
$$\|F^3\| < \frac{\lambda_3}{\sqrt{2}} - \frac{3n+4}{n}\|F^2\| - \|F^1\|,$$

*then $F^2 = F^3 = 0$, that is, $\nabla$ is a $c_1$-self-dual connection. Moreover if $\nabla$ is non-flat, then the $c_1$-self-dual part $F^1$ satisfies*

$$\frac{\lambda_1}{\sqrt{2}} \leq \|F^1\| < \frac{\lambda_3}{\sqrt{2}}.$$

## 2. Preliminaries

In this section, we fix notation. Let $(M, g)$ be a compact quaternionic Kähler $4n$-manifold, and $P$ a principal $G$-bundle over $(M, g)$ with a compact Lie group $G$ as structure group. We denote by $\mathfrak{g}$ the Lie algebra of $G$. For a faithful orthogonal representation $\rho : G \longrightarrow O(N)$, we consider a Riemannian vector bundle $E = P \times_\rho \mathbb{R}^N$ associated with $P$ by $\rho$. Each connection on $P$ corresponds to a connection $\nabla$ on $E$. We denote by $\mathcal{A}$ the set of the connections on $E$. To each connection $\nabla$ on $E$, the curvature $F^\nabla$, given by the formula $F^\nabla_{X,Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ for tangent vectors $X$, $Y$, is a 2-form on $M$ with values in the bundle $\mathfrak{so}_E$ whose fibre $\mathfrak{so}_{E,x}$, $x \in M$, consists of skew-symmetric endomorphisms of the fibre $E_x$ of $E$. The pointwise norm of $F^\nabla$ at each point $x$ is given by

$$\|F^\nabla\|^2 = \sum_{i<j} \|F^\nabla_{e_i,e_j}\|^2,$$

where $\{e_1, \cdots, e_{4n}\}$ is an orthonormal basis of the tangent space $T_xM$, $x \in M$, and the inner product of the fibre $\mathfrak{so}_{E,x}$ is given by

$$\langle A, B\rangle = -\frac{1}{2}\mathrm{tr}(A \circ B) \tag{2.1}$$

for $A$, $B \in \mathfrak{so}_{E,x}$. There exists a subbundle $\mathfrak{g}_E$ of $\mathfrak{so}_E$ corresponding to a bundle $\mathfrak{g}_P = P \times_{Ad} \mathfrak{g}$ through $\rho$. Let $A^p(\mathfrak{g}_E)$, $0 \le p \le 4n$, be the space of $\mathfrak{g}_E$-valued $p$-forms on $M$. We get the exterior differential $d^\nabla : A^p(\mathfrak{g}_E) \longrightarrow A^{p+1}(\mathfrak{g}_E)$ and the adjoint operator $\delta^\nabla : A^p(\mathfrak{g}_E) \longrightarrow A^{p-1}(\mathfrak{g}_E)$ corresponding to $\nabla \in \mathcal{A}$. $\Delta^\nabla = d^\nabla\delta^\nabla + \delta^\nabla d^\nabla$ is the *Laplacian* for $\mathfrak{g}_E$-valued $p$-forms. There is another second order operator $\nabla^*\nabla$, called the *rough Laplacian* , acting on $\mathfrak{g}_E$-valued differential forms. It is given by the formula $\nabla^*\nabla\varphi = -\sum_{j=1}^{4n}(\nabla^2_{e_j,e_j}\varphi)$, $\varphi \in A^p(\mathfrak{g}_E)$, where $\nabla^2_{X,Y} = \nabla_X\nabla_Y - \nabla_{D_XY}$.

The bundle of 2-forms on a quaternionic Kähler manifold $M$ has the following irreducible decomposition as a representation of $Sp(n) \cdot Sp(1)$:

$$\wedge^2 T^*M = S^2\mathbb{H} \oplus S^2\mathbb{E} \oplus (S^2\mathbb{H} \oplus S^2\mathbb{E})^\perp, \tag{2.2}$$

where $\mathbb{H}$ and $\mathbb{E}$ are the vector bundles associated to the standard representations of $Sp(1)$ and $Sp(n)$, respectively. A connection whose $\mathfrak{g}_E$-valued curvature 2-form lies in $S^2\mathbb{H}$, $S^2\mathbb{E}$ or $(S^2\mathbb{H} \oplus S^2\mathbb{E})^\perp$ is called a $c_1$, $c_2$ or $c_3$-self-dual connection respectively. Corresponding to the decomposition (2.2), we write the curvature $F^\nabla$ as

$$F^\nabla = F^1 + F^2 + F^3.$$

In the case of $n = 1$, corresponding to the fact that $SO(4) = Sp(1) \cdot Sp(1)$, $\wedge^2T^*M$ is decomposed as

$$\wedge^2T^*M = \wedge^2_+ \oplus \wedge^2_-. \tag{2.3}$$

A connection whose $\mathfrak{g}_E$-valued curvature 2-form lies in $\wedge^2_+$ or $\wedge^2_-$ is called a self-dual or anti-self-dual connection respectively. Corresponding to the decomposition (2.3), we write the curvature $F^\nabla$ as

$$F^\nabla = F^+ + F^-.$$

The associated bundles $\mathbb{H}$, $\mathbb{E}$ for this case are precisely the half-spinor bundles of $M$. The vector bundle $S^2\mathbb{H}$ is a subbundle of $\mathrm{End}(TM)$ of real rank 3. Locally $S^2\mathbb{H}$ has a basis $\{I, J, K\}$ satisfying

$$I^2 = J^2 = -1, \qquad IJ = -JI = K.$$

The metric $g$ on $M$ satisfies $g(IX, IY) = g(JX, JY) = g(KX, KY) = g(X, Y)$ for all $X$, $Y \in T_xM$. Local 2-forms $\{\omega_I, \omega_J, \omega_K\}$ are defined by

$$\omega_I(X, Y) = g(IX, Y), \quad \omega_J(X, Y) = g(JX, Y), \quad \omega_K(X, Y) = g(KX, Y).$$

$\{\omega_I, \omega_J, \omega_K\}$ is a local orthogonal frame of $S^2\mathbb{H}$. We define a global 4-form $\Omega$ by

$$\Omega = \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K.$$

$\Omega$ is a nondegenerate and parallel form on $M$, called the *fundamental* $4$-*form* on $M$. A connection $\nabla$ on the quaternionic Kähler $4n$-manifold $(M, g)$ is a $c_i$-self-dual connection ($i = 1$, 2 or 3) if and only if its curvature $F^\nabla$ satisfies

$$*F^\nabla = c_i F^\nabla \wedge \Omega^{n-1}, \tag{2.4}$$

where $*$ is the Hodge star operator and $c_1 = (6n)/((2n+1)!)$, $c_2 = -1/((2n-1)!)$ and $c_3 = 3/((2n-1)!)$ ([3]). Note that the equation (2.4) can be viewed as the self-dual or anti-self-dual equation on a oriented Riemannian 4-manifold.

Let $(M, g)$ be a compact quaternionic Kähler $4n$-manifold. At each point, we consider $F^\nabla$ as a linear map

$$F^\nabla : \wedge^2 TM \longrightarrow \mathfrak{g}_E.$$

In $\wedge^2 TM$ we have the identities

$$[e_i \wedge e_j, e_k \wedge e_l] = \delta_{il} e_k \wedge e_j + \delta_{jl} e_i \wedge e_k + \delta_{ik} e_j \wedge e_l + \delta_{jk} e_l \wedge e_i \tag{2.5}$$

for all $i$, $j$, $k$, $l$, where $\{e_1, \ldots, e_{4n}\}$ is an orthonormal basis of the tangent space $T_x M$. For any $\varphi$ in $A^2(\mathfrak{g}_E)$, the Bochner-Weitzenböck formula is

$$\langle \Delta^\nabla \varphi, \varphi\rangle - \langle \nabla^* \nabla \varphi, \varphi\rangle = \langle \varphi \circ \left(\frac{s}{2n} I - 2R\right), \varphi\rangle - \rho(\varphi),$$

where

$$\rho(\varphi) = \langle \kappa(\varphi), \varphi\rangle = \langle [F^\nabla, \varphi], \varphi\rangle, \quad \kappa(\varphi)_{X,Y} = \sum_{i=1}^{4n} \{[F^\nabla_{e_i,X}, \varphi_{e_i,Y}] - [F^\nabla_{e_i,Y}, \varphi_{e_i,X}]\}$$

and $R$ is the Riemannian curvature operator acting on $\wedge^2 TM$.
For $\varphi = F^\nabla$, this formula implies that

$$\langle \Delta F^\nabla, F^\nabla\rangle - \langle \nabla^* \nabla F^\nabla, F^\nabla\rangle = \langle F^\nabla \circ \left(\frac{s}{2n} I - 2R\right), F^\nabla\rangle - \rho(F^\nabla), \tag{2.6}$$

where

$$\rho(F^\nabla) = \sum_{i,j,k=1}^{4n} \langle [F^\nabla_{e_i,e_j}, F^\nabla_{e_j,e_k}], F^\nabla_{e_k,e_i}\rangle. \tag{2.7}$$

We now examine the term $\rho$ given by (2.7). We now introduce an inner product on the bundle $\mathfrak{g}_E$ as follows. Recall that we have $\mathfrak{g}_E \subseteq \mathfrak{so}_E$, the bundle of skew-symmetric endomorphisms of $E$. Given two endomorphisms $A$ and $B$ of $E_x$, we define $\langle A,B\rangle := 1/2tr({}^tA\circ B)$. There is a natural bundle isomorphism $\wedge^2E\simeq\mathfrak{so}_E$ determined by the requirement that

$$(u\wedge v)(w)=\langle u,w\rangle v-\langle v,w\rangle u$$

for $u$, $v$, $w\in E_x$. The elements $\{\xi_i\wedge\xi_j\}_{i<j}$ form an orthonormal basis of $(\mathfrak{so}_E)_x$ whenever $(\xi_1,\dots,\xi_N)$ is an orthonormal basis of $E_x$. In particular, there is a canonical isometry $\wedge^2TM\simeq\mathfrak{so}_M$. We have also $\mathfrak{g}\subseteq\wedge^2T_xM\simeq\mathfrak{so}(N)$. For any Lie algebra $\mathfrak{g}$ with a fixed $Ad$-invariant inner product $\langle\cdot,\cdot\rangle$, we have the associated fundamental 3-form $\Phi_{\mathfrak{g}}$ given by $\Phi_{\mathfrak{g}}(X,Y,Z)=\langle[X,Y],Z\rangle$ for $X,Y,Z\in\mathfrak{g}$ and $\Phi_{\wedge^2TM}(\alpha,\beta,\gamma)=\langle[\alpha,\beta],\gamma\rangle$ for $\alpha,\beta,\gamma\in\wedge^2TM$. We may rewrite (2.7) as

$$\begin{aligned}\rho(F^\nabla)&=\sum_{i,j,k=1}^{4n}\Phi_{\mathfrak{g}_E}(F^\nabla_{e_i,e_j},F^\nabla_{e_j,e_k},F^\nabla_{e_k,e_i})\\&=\sum_{i,j,k=1}^{4n}(F^{\nabla*}\Phi_{\mathfrak{g}_E})(e_i\wedge e_j,e_j\wedge e_k,e_k\wedge e_i)\\&=(F^{\nabla*}\Phi_{\mathfrak{g}_E},\Phi_{\wedge^2TM}),\end{aligned}$$

where, for notational convenience, we define the inner product in $\wedge^3(\wedge^2T^*M)$ by $(\Phi,\Psi)=\sum_{U,V,W}\Phi(U,V,W)\Psi(U,V,W)$, where $U$, $V$ and $W$ are an orthonormal basis of $\wedge^2TM$. Therefore, we have the following basic result. Let $F^\nabla$ be a curvature 2-form on $E$ and let $\lambda$ be the minimal eigenvalue of the operator $(s/2n)I-2R$ on 2-forms over a compact quaternionic Kähler manifold $M$. Then

$$\text{(2.8)}\quad \langle\nabla^*\nabla F^\nabla,F^\nabla\rangle-\langle\Delta^\nabla F^\nabla,F^\nabla\rangle\le-\{\lambda\|F^\nabla\|^2-(F^{\nabla*}\Phi_{\mathfrak{g}_E},\Phi_{\wedge^2TM})\}.$$

At each point $x\in M$, we want to estimate $(F_x^{\nabla*}\Phi_{\mathfrak{g}_E},\Phi_{\wedge^2T_xM})$ in terms of $\|F^\nabla\|^2$ where $F_x^\nabla:\mathfrak{so}(4n)\longrightarrow\mathfrak{g}$ is a linear map and where $\mathfrak{g}$ is any Lie subalgebra of $\mathfrak{so}(N)$. Recall that an inner product on $\mathfrak{g}$ is induced from the canonical one on $\mathfrak{so}(N)$ defined by $\langle A,B\rangle=-(1/2)\mathrm{tr}(\rho(A)\cdot\rho(B))$. Consequently $F_x^{\nabla*}\Phi_{\mathfrak{g}}=F_x^{\nabla*}\Phi_{\mathfrak{so}(N)}$. Therefore, in the argument of this paper, we can ignore $\mathfrak{g}$.

The norm $\|\cdot\|$ induced by the inner product (2.1) has the property that

$$\text{(2.9)}\qquad \|[A,B]\|\le\sqrt{2}\|A\|\cdot\|B\|$$

for all $A$, $B$ in which the equality holds if and only if the pair $A$, $B$ is orthogonally equivalent to the following matrices:

$$\text{(2.10)}\qquad \left(\begin{array}{c|c}\mathrm{i}&0\\\hline 0&0\end{array}\right),\quad\left(\begin{array}{c|c}\mathrm{j}&0\\\hline 0&0\end{array}\right),$$

where

$$\mathrm{i} = \left(\begin{array}{cc|cc} 0 & -t & 0 & 0 \\ t & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -t \\ 0 & 0 & t & 0 \end{array}\right), \quad \mathrm{j} = \left(\begin{array}{cc|cc} 0 & 0 & -t & 0 \\ 0 & 0 & 0 & t \\ \hline t & 0 & 0 & 0 \\ 0 & -t & 0 & 0 \end{array}\right).$$

We shall also state the following result, which is used in proving our theorems.

**Lemma 2.1.** *Let $S = ((s_{ij}))$ be a symmetric $4n \times 4n$ matrix with $s_{ij} \geq 0$ and $s_{ii} = 0$. If $\mathrm{tr}S^2 = (4n(4n-1)\lambda^2)/((4n-2)^2 2^2)$ for any positive real number $\lambda$, then*

$$\mathrm{tr}S^3 \leq \frac{4n(4n-1)\lambda^3}{(4n-2)^2 2^3}$$

*with equality holding if and only if $s_{ij} = (\lambda)(4(2n-1))$, $i \neq j$.*

The proof of Lemma 2.1 is entirely similar to the argument for Lemma (5.14) in [1].

Denoting $F^{\alpha}_{e_i,e_j}$ by $F^{\alpha}_{ij}$, we have the following.

**Proposition 2.1** ([3]). *Let $F^1$, $F^2$ and $F^3$ be respectively the $c_1$-self-dual, $c_2$-self-dual and $c_3$-self-dual parts.*

(1) *The $c_1$-self-dual part $F^1$ satisfies*

$$\begin{aligned} &F^1_{4k+1,4k+2} = F^1_{4k+3,4k+4} = F^1_{4l+1,4l+2} = F^1_{4l+3,4l+4}, \\ &F^1_{4k+1,4k+3} = F^1_{4k+4,4k+2} = F^1_{4l+1,4l+3} = F^1_{4l+4,4l+2}, \\ &F^1_{4k+1,4k+4} = F^1_{4k+2,4k+3} = F^1_{4l+1,4l+4} = F^1_{4l+2,4l+3}, \\ &F^1_{4p+1,4q+1} = F^1_{4p+2,4q+2} = F^1_{4p+3,4q+3} = F^1_{4p+4,4q+4} = 0, \\ &\qquad\qquad (\forall k, l, p, q), \\ &F^1_{4p+\alpha,4q+\beta} = 0, \quad (\forall p \neq q, \forall \alpha, \beta). \end{aligned}$$

(2) *The $c_2$-self-dual part $F^2$ satisfies*

$$\begin{aligned} &F^2_{4k+1,4k+2} = -F^2_{4k+3,4k+4}, \\ &F^2_{4k+1,4k+3} = F^2_{4k+2,4k+4}, \\ &F^2_{4k+1,4k+4} = -F^2_{4k+2,4k+3}, \\ &F^2_{4p+1,4q+1} = F^2_{4p+2,4q+2} = F^2_{4p+3,4q+3} = F^2_{4p+4,4q+4}, \\ &F^2_{4p+1,4q+2} = -F^2_{4p+2,4q+1} = -F^2_{4p+3,4q+4} = F^2_{4p+4,4q+3}, \\ &F^2_{4p+1,4q+3} = F^2_{4p+2,4q+4} = -F^2_{4p+3,4q+1} = -F^2_{4p+4,4q+2}, \\ &F^2_{4p+1,4q+4} = -F^2_{4p+2,4q+3} = F^2_{4p+3,4q+2} = -F^2_{4p+4,4q+1}, \\ &\qquad\qquad (\forall k), \quad (0 \leq p < q \leq n-1). \end{aligned}$$

(3) *The $c_3$-self-dual part $F^3$ satisfies*

$$\sum_{k=0}^{n-1} F^3_{4k+1,4k+2} = \sum_{k=0}^{n-1} F^3_{4k+1,4k+3} = \sum_{k=0}^{n-1} F^3_{4k+1,4k+4} = 0,$$
$$F^3_{4p+1,4q+2} + F^3_{4q+1,4p+2} = F^3_{4p+3,4q+4} + F^3_{4q+3,4p+4},$$
$$F^3_{4p+1,4q+3} + F^3_{4q+1,4p+3} = -(F^3_{4p+2,4q+4} + F^3_{4q+2,4p+4}),$$
$$F^3_{4p+1,4q+4} + F^3_{4q+1,4p+4} = F^3_{4p+2,4q+3} + F^3_{4q+2,4p+3},$$
$$\sum_{\alpha=1}^{4} F^3_{4p+\alpha,4q+\alpha} = 0,$$
$$(\forall p, q).$$

Proposition 2.1 follows from the argument for Theorem 2.2 in [3].

## 3. Some properties of quaternionic Kähler manifolds

In this section, we prepare a few propositions. First, we shall give a proof of Proposition 1.1.

Proof of Proposition 1.1. We see that $d^\nabla(F^\nabla \wedge \Omega^{n-1}) = 0$ by $d^\nabla F^\nabla = 0$ and $d\Omega = 0$. Hence if $M$ is compact, then the connection $\nabla$ satisfies $\Delta^\nabla(F^\nabla \wedge \Omega^{n-1}) = 0$ if and only if $\delta^\nabla(F^\nabla \wedge \Omega^{n-1}) = 0$. We shall prove that $\nabla$ satisfies $\delta^\nabla F^\nabla = 0$ if $\delta^\nabla(F^\nabla \wedge \Omega^{n-1}) = 0$. We take an orthonormal frame field $\{e_i; i = 1, 2, \dots, 4n\}$ such that $Ie_{4k+1} = e_{4k+2}$, $Je_{4k+1} = e_{4k+3}$, $Ke_{4k+1} = e_{4k+4}$, $(k = 0, 1, \dots, n-1)$, and denote the dual frame by $\{\theta^i; i = 1, 2, \dots, 4n\}$. The vector bundle $S^2\mathbb{H}$ has the following frame field, $\{\omega_I, \omega_J, \omega_K\}$:

$$\omega_I = \sum_{k=0}^{n-1} (\theta^{4k+1} \wedge \theta^{4k+2} + \theta^{4k+3} \wedge \theta^{4k+4}),$$
$$\omega_J = \sum_{k=0}^{n-1} (\theta^{4k+1} \wedge \theta^{4k+3} + \theta^{4k+4} \wedge \theta^{4k+2}),$$
$$\omega_K = \sum_{k=0}^{n-1} (\theta^{4k+1} \wedge \theta^{4k+4} + \theta^{4k+2} \wedge \theta^{4k+3}).$$

The fundamental 4-form is $\Omega = \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K$. Using the orthonormal frame $\{\theta^i; i = 1, 2, \dots, 4n\}$, we can write the curvature 2-form $F^\nabla$ as $F^\nabla = \sum_{i<j} F_{ij}\theta^i \wedge \theta^j$. From $\Omega^{n-1} = ((2n-1)!/6) * \Omega$ ([3]), $\delta^\nabla(F^\nabla \wedge \Omega^{n-1}) = 0$ is equivalent to $\delta^\nabla(F^\nabla \wedge *\Omega) = 0$. It is easy to see that the quaternionic Yang-Mills equation $\delta^\nabla(F^\nabla \wedge *\Omega) = 0$ is equivalent to

$$\nabla_i F_{ij} = 0, \quad (i, j = 1, \dots, 4n).$$

On the other hand, the Yang-Mills equation $\delta^\nabla F^\nabla = 0$ is equivalent to

$$\sum_i \nabla_i F_{ij} = 0, \quad (j = 1, \dots, 4n).$$

Therefore, if $\nabla$ satisfies $\delta^\nabla(F^\nabla \wedge \Omega^{n-1}) = 0$, then $\delta^\nabla F^\nabla = 0$. □

**Proposition 3.1.** *Let $F^1$, $F^2$ and $F^3$ be respectively the $c_1$-self-dual, $c_2$-self-dual and $c_3$-self-dual parts of the curvature $F^\nabla$ on a compact quaternionic Kähler manifold. Then the following are equivalent:*

$$\Delta^\nabla(F^\nabla \wedge \Omega^{n-1}) = 0; \tag{3.1}$$

$$\Delta^\nabla\left(\frac{c_\alpha - c_\gamma}{c_\alpha} F^\alpha + \frac{c_\beta - c_\gamma}{c_\beta} F^\beta\right) = 0 \tag{3.2}$$

*for any permutation* $(\alpha, \beta, \gamma)$ *of* $\{1, 2, 3\}$.

Proof. Let

$$F^\nabla = F^\alpha + F^\beta + F^\gamma \tag{3.3}$$

denote the curvature, for any $(\alpha, \beta, \gamma)$ as above. From (3.3), we have

$$c_\gamma F^\nabla \wedge \Omega^{n-1} = c_\gamma F^\alpha \wedge \Omega^{n-1} + c_\gamma F^\beta \wedge \Omega^{n-1} + c_\gamma F^\gamma \wedge \Omega^{n-1}. \tag{3.4}$$

Hence, we get

$$c_\gamma * (F^\nabla \wedge \Omega^{n-1}) = \frac{c_\gamma}{c_\alpha} F^\alpha + \frac{c_\gamma}{c_\beta} F^\beta + F^\gamma. \tag{3.5}$$

It follows from (3.3) and (3.5) that

$$\left(1 - \frac{c_\gamma}{c_\alpha}\right) F^\alpha + \left(1 - \frac{c_\gamma}{c_\beta}\right) F^\beta = F^\nabla - c_\gamma * (F^\nabla \wedge \Omega^{n-1}). \tag{3.6}$$

Applying $d^\nabla$ and $\delta^\nabla$ to (3.6), respectively, and using Bianchi identity $d^\nabla F^\nabla = 0$ and $d\Omega^{n-1} = 0$, we obtain

$$d^\nabla\left[\left(1 - \frac{c_\gamma}{c_\alpha}\right) F^\alpha + \left(1 - \frac{c_\gamma}{c_\beta}\right) F^\beta\right] = -c_\gamma * \delta^\nabla(F^\nabla \wedge \Omega^{n-1}),$$
$$\delta^\nabla\left[\left(1 - \frac{c_\gamma}{c_\alpha}\right) F^\alpha + \left(1 - \frac{c_\gamma}{c_\beta}\right) F^\beta\right] = \delta^\nabla F^\nabla.$$

From Proposition 1.1, $\nabla$ fullfills $\delta^\nabla F^\nabla = 0$ if it satisfies $\delta^\nabla(F^\nabla \wedge \Omega^{n-1}) = 0$. Hence, (3.1) and (3.2) are equivalent. This completes the proof of Proposition 2.1. □

In the case $n=1$, we conclude that the following three conditions are equivalent ([1]):
(1) $\delta^\nabla F^\nabla = 0$, (2) $\Delta^\nabla F^+ = 0$, (3) $\Delta^\nabla F^- = 0$.

**Proposition 3.2.** *Let $F^1$ and $F^2$ be respectively the $c_1$-self-dual and $c_2$-self-dual parts. Then for vectors $X, Y \in T_xM$, the quantity*

$$\sum_{j=1}^{4n} F^1_{e_j,X} \cdot F^2_{e_j,Y}$$

*is symmetric in $X$ and $Y$.*

Proof. Let $\{e_1, \dots, e_{4n}\}$ be an orthonormal frame field of $T_xM$. Substituting $X = e_{4k+1}$, $Y = e_{4k+2}$ into $\sum_{j=1}^{4n} F^1_{e_j,X} \cdot F^2_{e_j,Y}$ and using Proposition 2.1, we see that

$$\begin{aligned}\sum_{j=1}^{4n} F^1_{e_j,e_{4k+1}} \cdot F^2_{e_j,e_{4k+2}} &= F^1_{e_{4k+3},e_{4k+1}} \cdot F^2_{e_{4k+3},e_{4k+2}} + F^1_{e_{4k+4},e_{4k+1}} \cdot F^2_{e_{4k+4},e_{4k+2}} \\ &= F^1_{e_{4k+2},e_{4k+4}} \cdot F^2_{e_{4k+1},e_{4k+4}} + F^1_{e_{4k+3},e_{4k+2}} \cdot F^2_{e_{4k+3},e_{4k+1}} \\ &= \sum_{j=1}^{4n} F^1_{e_j,e_{4k+2}} \cdot F^2_{e_j,e_{4k+1}}\end{aligned}$$

for each $0 \le k \le n-1$. This completes the proof of Proposition 3.2. □

The following is the key of the proofs of the theorems.

**Proposition 3.3.** *Let $F^1$, $F^2$ and $F^3$ be respectively the $c_1$-self-dual, $c_2$-self-dual and $c_3$-self-dual parts. Then*
(1) $[F^1, F^2]_{X,Y} = 0$,
(2) $[F^2, F^3]_{X,Y} \in (S^2\mathbb{H}_x \oplus S^2\mathbb{E}_x)^\perp \otimes \mathfrak{g}$,
(3) $[F^1, F^3]_{X,Y} \in (S^2\mathbb{H}_x \oplus S^2\mathbb{E}_x)^\perp \otimes \mathfrak{g}$,
*where* $[F^\alpha, F^\beta]_{X,Y} = \sum_{j=1}^{4n}\{[F^\alpha_{e_j,X}, F^\beta_{e_j,Y}] - [F^\alpha_{e_j,Y}, F^\beta_{e_j,X}]\}$ *for all $X, Y \in T_xM$, $\alpha$, $\beta = 1, 2, 3$.*

Proof. (1) From Proposition 3.2, $X$ and $Y$ are symmetric. Hence, $[F^1, F^2]_{X,Y} = 0$.
(2) From the properties of the Killing form, we have

$$\langle [A,B], C\rangle = \langle A, [B,C]\rangle \tag{3.7}$$

for any $A, B, C \in \wedge^2 T^*_xM \otimes \mathfrak{g}$. Using Proposition 2.1, we see that $[F^1, F^1]_{X,Y} \in S^2\mathbb{H}_x \otimes \mathfrak{g}$, $[F^2, F^2]_{X,Y} \in S^2\mathbb{E}_x \otimes \mathfrak{g}$ and $[F^3, F^3]_{X,Y} \in \wedge^2 T^*_xM \otimes \mathfrak{g}$ and note that

$[F^\alpha, F^\beta] = [F^\beta, F^\alpha]$. Putting $A = F^1$, $B = F^2$ and $C = F^3$ in (3.7) and using $[F^1, F^2]_{X,Y} = 0$, we get

$$\langle F^1, [F^2, F^3]\rangle = 0. \tag{3.8}$$

Putting $A = F^1$, $B = F^3$ and $C = F^2$ in (3.7), we have

$$\langle [F^1, F^3], F^2\rangle = \langle F^1, [F^3, F^2]\rangle. \tag{3.9}$$

Putting $A = F^1$, $B = F^1$ and $C = F^3$ in (3.7), we get $\langle [F^1, F^1], F^3\rangle = \langle F^1, [F^1, F^3]\rangle$. From $[F^1, F^1]_{X,Y} \in S^2\mathbb{H}_x \otimes \mathfrak{g}$, we have

$$\langle F^1, [F^1, F^3]\rangle = 0. \tag{3.10}$$

Putting $A = F^2$, $B = F^2$ and $C = F^3$ in (3.7), we get $\langle [F^2, F^2], F^3\rangle = \langle F^2, [F^2, F^3]\rangle$. From $[F^2, F^2]_{X,Y} \in S^2\mathbb{E}_x \otimes \mathfrak{g}$, we have

$$\langle F^2, [F^2, F^3]\rangle = 0 \tag{3.11}$$

From (3.8) and (3.11), we conclude that

$$[F^2, F^3]_{X,Y} \in (S^2\mathbb{H}_x \oplus S^2\mathbb{E}_x)^\perp \otimes \mathfrak{g}.$$

(3) From (3.8) and (3.9), we get

$$\langle [F^1, F^3], F^2\rangle = 0. \tag{3.12}$$

From (3.10) and (3.12), we conclude that

$$[F^1, F^3]_{X,Y} \in (S^2\mathbb{H}_x \oplus S^2\mathbb{E}_x)^\perp \otimes \mathfrak{g}$$

These complete the proof of Proposition 3.3. □

The proof of the following Proposition 3.4 is analogous to that of Proposition (5.6) in [1].

**Proposition 3.4.** *Let* $F_x^\nabla : \mathfrak{so}(4n) \longrightarrow \mathfrak{so}(N)$ *be a linear map and* $\lambda$ *be a positive real number.*
(I) *If* $\|F^\nabla\|^2 \leq (n(4n-1)\lambda^2)/(16(2n-1)^2)$, *then*

$$(F_x^{\nabla *}\Phi_{\mathfrak{g}}, \Phi_{\wedge^2 T_x M}) \leq \lambda \|F^\nabla\|^2. \tag{3.13}$$

(II) *Putting* $\lambda = 4(2n-1)$, *we have the following*:
*If* $\|F^\nabla\|^2 \leq n(4n-1)$, *then*

(3.14) $$(F_x^{\nabla *}\Phi_{\mathfrak{g}}, \Phi_{\wedge^2 T_x M}) \leq 4(2n-1)\|F^\nabla\|^2.$$

*The equality holds if and only if there is an orthogonal splitting* $\mathbb{R}^N = S_0 \oplus S_1$ $(\dim S_1 = 4)$ *with respect to which* $F_x^\nabla = 0 \oplus \sigma$ *where* $\sigma$ *is a representation* $\sigma : \mathfrak{sp}(1) \longrightarrow \mathfrak{so}(4)$.

Proof. We shall prove the inequality for $\|F^\nabla\|^2 = (n(4n-1)\lambda^2)/(16(2n-1)^2)$. Let $\{e_i \wedge e_j\}_{i<j}$ be the orthonormal basis of $\mathfrak{so}(4n) \cong \wedge^2 T_x M$. Then $\|F^\nabla\|^2 = \sum_{i<j}\|F_x^\nabla(e_i \wedge e_j)\|^2$ and $(F_x^{\nabla *}\Phi_{\mathfrak{g}}, \Phi_{\wedge^2 T_x M}) = \sum_{i,j,k=1}^{4n}\langle [F_x^\nabla(e_i \wedge e_j), F_x^\nabla(e_j \wedge e_k)], F_x^\nabla(e_k \wedge e_i)\rangle$. We now denote $F_x^\nabla(e_i \wedge e_j)$ by $F_{ij}$. We introduce the $4n \times 4n$-symmetric matrix $S = ((s_{ij}))$ with non-negative entries $s_{ij} = \sqrt{2}\|F_{ij}\|$. By the assumption, $\mathrm{tr}S^2 = \sum_{i,j=1}^{4n} s_{ij}^2 = 4\sum_{i<j}\|F_{ij}\|^2 = (4n(4n-1)\lambda^2)/((4n-2)^2 2^2)$. By Lemma 2.1 we have

$$\mathrm{tr}S^3 = \sum_{i,j,k=1}^{4n} s_{ij}s_{jk}s_{ki} \leq \frac{4n(4n-1)\lambda^3}{(4n-2)^2 2^3}.$$

Therefore, using (2.9), we see that

$$\begin{aligned}
(F_x^{\nabla *}\Phi_{\mathfrak{g}}, \Phi_{\wedge^2 T_x M}) &\leq \sum_{i,j,k=1}^{4n} |\langle [F_{ij}, F_{jk}], F_{ki}\rangle| \\
&\leq \sum_{i,j,k=1}^{4n} \|[F_{ij}, F_{jk}]\| \cdot \|F_{ki}\| \\
&\leq \sum_{i,j,k=1}^{4n} \sqrt{2}\|F_{ij}\| \cdot \|F_{jk}\| \cdot \|F_{ki}\| \\
&= \frac{1}{2}\sum_{i,j,k=1}^{4n} s_{ij}s_{jk}s_{ki} \leq \lambda\|F^\nabla\|^2.
\end{aligned}$$

Hence, we complete the proof of (I).

We next prove (II). Putting $\lambda = 4(2n-1)$ in (I), we see that

$$\begin{aligned}
(F_x^{\nabla *}\Phi_{\mathfrak{g}}, \Phi_{\wedge^2 T_x M}) &\leq \sum_{i,j,k=1}^{4n} |\langle [F_{ij}, F_{jk}], F_{ki}\rangle| \\
&\leq \sum_{i,j,k=1}^{4n} \|[F_{ij}, F_{jk}]\| \cdot \|F_{ki}\| \\
&\leq \sum_{i,j,k=1}^{4n} \sqrt{2}\|F_{ij}\| \cdot \|F_{jk}\| \cdot \|F_{ki}\|
\end{aligned}$$

$$= \frac{1}{2} \sum_{i,j,k=1}^{4n} s_{ij} s_{jk} s_{ki} \leq 4(2n-1)\|F^{\nabla}\|^2.$$

Suppose now that we have the equality in each line. From the last line we see that $s_{ij} = 1$, and so $\|F_{ij}\| = (1/\sqrt{2})$ for $i \neq j$. From the first and second lines we conclude that, when $i$, $j$, $k$ are mutually distinct, $[F_{ij}, F_{jk}] = tF_{ki}$ where $t > 0$. Taking the inner product with $F_{ki}$ and using the equality in each line we see that $t = 1$. Hence, we have

$$[F_{ij}, F_{jk}] = F_{ki} \tag{3.15}$$

for all $i$, $j$, $k$ distinct. This equation has a number of consequences. Setting $\alpha_{ijkl} = [F_{ij}, F_{kl}]$, we have $\alpha_{ijkl} = -\alpha_{jikl}$, $\alpha_{ijkl} = -\alpha_{ijlk}$, $\alpha_{ijkl} + \alpha_{iklj} + \alpha_{iljk} = 0$, $\alpha_{ijkl} = \alpha_{klij}$. However, from the definition we see $\alpha_{ijkl} = -\alpha_{klij}$, and so we conclude that

$$[F_{ij}, F_{kl}] = 0 \tag{3.16}$$

for $i$, $j$, $k$, $l$ distinct. Comparing (3.15) and (3.16) with (2.5) we conclude that $F_x^{\nabla} : \mathfrak{so}(4n) \longrightarrow \mathfrak{so}(N)$ is a Lie algebra homomorphism. Finally, we observe that by (2.9) each pair $(F_{ij}, F_{jk})$ for $i, j, k$ distinct is conjugate to a pair of matrices of (2.10). In particular, each of the endomorphisms $F_{ij}$ is supported in the same 4-dimensional subspace. Therefore, we conclude that $F_{ij} : \mathfrak{so}(4n) \longrightarrow \mathfrak{so}(4)$ is also a Lie algebra homomorphism. This homomorphism is injective. To see this directly we note that if $i$, $j$, $k$, $l$ are mutually distinct, then it is easy to see that $\langle F_{ij}, F_{kl}\rangle = 0$. The matrices $\{F_{ij}\}_{i<j}$ are orthogonal. Hence $F_{ij}$ is injective. Therefore, $F_{ij} : \mathfrak{so}(4n) \longrightarrow \mathfrak{so}(4)$ reduce the Lie algebra homomorphism $F_{ij} : \mathfrak{sp}(1) \longrightarrow \mathfrak{so}(4)$. Note that $\mathfrak{so}(4n) = \mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \oplus (\mathfrak{sp}(1) \oplus \mathfrak{sp}(n))^{\perp}$. This completes the proof of Proposition 3.4. □

## 4. Proof of theorems

In this section, we shall give the proofs of theorems stated in Introduction.

Proof of Theorem 1.1. We shall rewrite the Bochner-Weitzenböck formula (2.4).

$$\langle \Delta^{\nabla}\varphi, \varphi\rangle - \langle \nabla^*\nabla\varphi, \varphi\rangle = \left\langle \varphi \circ \left(\frac{s}{2n} I - 2R\right), \varphi \right\rangle - \rho(\varphi), \tag{4.1}$$

where $\rho(\varphi) = \langle [F^{\nabla}, \varphi], \varphi\rangle$ for any $\varphi \in A^2(\mathfrak{g}_E)$. We put $A = (c_1 - c_2)/(c_1)$ and $B = (c_3 - c_2)/(c_3)$. Substituting $\varphi = AF^1 + BF^3$ into (4.1) and using Proposition 3.1, we have

$$-\|\nabla(AF^1 + BF^3)\|^2 = A^2\lambda_1\|F^1\|^2 + B^2\lambda_3\|F^3\|^2 - \rho(AF^1 + BF^3), \tag{4.2}$$

where $\lambda_i = ((s/(2n))I - 2R_i)_{X,Y} = s/(2n) - 2\mu_i$, $X, Y \in T_xM$. We see that

$$\begin{aligned}\rho(AF^1 + BF^3) &= \langle [F^\nabla, AF^1 + BF^3], AF^1 + BF^3\rangle \\ &= A^2\{\langle [F^1, F^1], F^1\rangle + \langle [F^2, F^1], F^1\rangle + \langle [F^3, F^1], F^1\rangle\} \\ &\quad + AB\{\langle [F^1, F^1], F^3\rangle + \langle [F^2, F^1], F^3\rangle + \langle [F^3, F^1], F^3\rangle\} \\ &\quad + AB\{\langle [F^1, F^3], F^1\rangle + \langle [F^2, F^3], F^1\rangle + \langle [F^3, F^3], F^1\rangle\} \\ &\quad + B^2\{\langle [F^1, F^3], F^3\rangle + \langle [F^2, F^3], F^3\rangle + \langle [F^3, F^3], F^3\rangle\}.\end{aligned}$$

Using Proposition 3.3, we get

$$\begin{aligned}\rho(AF^1 + BF^3) &= A^2\langle [F^1, F^1], F^1\rangle + (2AB + B^2)\langle [F^3, F^1], F^3\rangle \\ &\quad + B^2\langle [F^2, F^3], F^3\rangle + B^2\langle [F^3, F^3], F^3\rangle.\end{aligned} \tag{4.3}$$

Since we assume $F^3 = 0$, (4.2) implies that

$$-\|\nabla F^1\|^2 = \lambda_1\|F^1\|^2 - \langle [F^1, F^1], F^1\rangle. \tag{4.4}$$

By Proposition 3.4 (I), if $\|F^1\|^2 < (n(4n-1)\lambda_1^2)/(16(2n-1)^2)$, then $\langle [F^1, F^1], F^1\rangle < \lambda_1\|F^1\|^2$. Hence, the right hand side of (4.4) is non-negative. On the other hand, the left hand side of (4.4) is non-positive. This is a contradiction. This implies $F^1 = 0$. The same statement is true for $F^2$. □

Proof of Theorem 1.4. From (4.2), (4.3) and using (2.7), we obtain

$$\begin{aligned}&-\|\nabla(AF^1 + BF^3)\|^2 \\ &\quad = A^2\{\lambda_1\|F^1\|^2 - \langle [F^1, F^1], F^1\rangle\} \\ &\qquad + B^2\{\lambda_3\|F^3\|^2 - (n+3)\langle [F^3, F^1], F^3\rangle \\ &\qquad - \langle [F^2, F^3], F^3\rangle - \langle [F^3, F^3], F^3\rangle\} \\ &\quad \geq A^2\{\lambda_1\|F^1\|^2 - \sqrt{2}\|F^1\|^3\} \\ &\qquad + B^2\{\lambda_3\|F^3\|^2 - \sqrt{2}(n+3)\|F^1\|\|F^3\|^2 \\ &\qquad - \sqrt{2}\|F^2\|\|F^3\|^2 - \sqrt{2}\|F^3\|^3\} \\ &\quad = A^2\{(\lambda_1 - \sqrt{2}\|F^1\|)\|F^1\|^2\} \\ &\qquad + B^2\{(\lambda_3 - \sqrt{2}(n+3)\|F^1\| - \sqrt{2}\|F^2\| - \sqrt{2}\|F^3\|)\|F^3\|^2\}.\end{aligned}$$

Hence, if

$$\lambda_1 - \sqrt{2}\|F^1\| > 0 \quad \text{and} \quad \lambda_3 - \sqrt{2}(n+3)\|F^1\| - \sqrt{2}\|F^2\| - \sqrt{2}\|F^3\| > 0,$$

we see that $F^1 = F^3 = 0$. When $F^1 = F^3 = 0$, moreover, from the second inequality stated above, we have $\|F^2\| < \lambda_3/\sqrt{2}$. On the other hand, from the Bochner-

Weitzenböck formula for $F^\nabla = F^2$ and using Proposition 3.3 and (2.7), we get

$$\begin{aligned}\langle \varDelta^\nabla F^2, F^2\rangle - \|\nabla F^2\|^2 &= \lambda_2\|F^2\|^2 - \langle [F^2, F^2], F^2\rangle \\ &\geq (\lambda_2 - \sqrt{2}\|F^2\|)\|F^2\|^2.\end{aligned}$$

Since $\varDelta^\nabla F^2 = 0$, we have

$$-\|\nabla F^2\|^2 \geq (\lambda_2 - \sqrt{2}\|F^2\|)\|F^2\|^2.$$

If $\|F^2\| < \lambda_2/\sqrt{2}$, then $F^2 = 0$. Thus if $F^2 \neq 0$, then $\lambda_2/\sqrt{2} \leq \|F^2\|$. Consequently, if $\nabla$ is a non-flat, then the $c_2$-self-dual part $F^2$ satisfies $\lambda_2/\sqrt{2} \leq \|F^2\| < \lambda_3/\sqrt{2}$, where $\lambda_1$, $\lambda_2$ and $\lambda_3$ always satisfy $\lambda_1 < \lambda_2 < \lambda_3$ on $(M,\ g)$. The same argument can be applied to (2) of Theorem 1.4. □

Proof of Theorem 1.2. We put $A = (c_1 - c_3)/c_1$ and $B = (c_2 - c_3)/c_2$. Substituting $\varphi = AF^1 + BF^2$ into the Bochner-Weitzenböck formula (4.1) and using Proposition 3.1, we have

$$\begin{aligned}-\|\nabla(AF^1 + BF^2)\|^2 &= A^2\{\lambda_1\|F^1\|^2 - \langle [F^1, F^1], F^1\rangle\} \\ &\quad + B^2\{\lambda_2\|F^2\|^2 - \langle [F^2, F^2], F^2\rangle\}.\end{aligned} \tag{4.5}$$

By Proposition 3.4 (I), if

$$\|F^1\|^2 < \frac{n(4n-1)\lambda_1^2}{16(2n-1)^2} \quad \text{and} \quad \|F^2\|^2 < \frac{n(4n-1)\lambda_2^2}{16(2n-1)^2}$$

then

$$\langle [F^1, F^1], F^1\rangle < \lambda_1\|F^1\|^2 \quad \text{and} \quad \langle [F^2, F^2], F^2\rangle < \lambda_2\|F^2\|^2.$$

Hence, the right hand side of (4.5) is non-negative. Meanwhile, the left hand side of (4.5) is non-positive. This is a contradiction. This implies $F^1 = F^2 = 0$. □

Proof of Corollary 1.1. Let $\mathbb{H}P^n = Sp(n+1)/Sp(n) \times Sp(1)$ be the quaternionic projective space. Let $sp(n+1) = sp(n) + sp(1) + \mathfrak{m}$ be the orthogonal decomposition of $sp(n+1)$ with respect to Killing form $B$. We identity $\mathfrak{m}$ with the tangent space of $\mathbb{H}P^n$ at the origin in a natural manner. Let $g_0$ denote the invariant Riemannian metric on $\mathbb{H}P^n$ defined by $-2(2n-1)(n+2)B|\mathfrak{m}$. The Ricci tensor of $(\mathbb{H}P^n, g_0)$ is given by $Ric(X,Y) = (2n-1)(n+2)g_0(X,Y)$ for $X, Y \in \mathfrak{m}$. Accordingly, the scalar curvature is given by $s = 4n(2n-1)(n+2)$. Corresponding to the decomposition (2.2), we can write the Riemannian curvature operator $R$ as $R = R_1 + R_2 + R_3$. From [4] and for this metric $g_0$ we know

$$R_1 = n(2n-1)I, \quad R_2 = (2n-1)I, \quad R_3 = 0.$$

We have $\lambda_1 = (s/(2n)I - 2R_1)_{X,Y} = (s/(2n)I - 2n(2n-1)I)_{X,Y} = 4(2n-1)$. In the same way, we have $\lambda_2 = 2(2n-1)(n+1)$. Substituting $\lambda_1$ and $\lambda_2$ into Theorem 1.1, we get Corollary 1.1. $\square$

Proof of Theorem 1.3. When $\|F^\nabla\|^2 \leq n(4n-1)$, by the Bochnor-Weitzenböck formula (4.1) and Proposition 3.4 (II), we conclude that $F^\nabla = 0$. Hence $E$ is flat bundle. When $\|F^\nabla\|^2 \equiv n(4n-1)$, we get $\nabla F^\nabla = 0$. Proposition 3.4 (II) implies that there is an orthogonal splitting $E = E_0 \oplus S$ where $E_0$ is flat, where $S$ is a 4-dimensional bundle. By Corollary 1.1 (2) and $\nabla F^\nabla = 0$, $F^\nabla : \wedge^2 TM \longrightarrow P \times_{Ad} \mathfrak{so}(4)$ reduces to $F^\nabla : S^2\mathbb{H} \longrightarrow P \times_{Ad} \mathfrak{so}(4)$. This implies that the connection $\nabla$ is a $c_1$-self-dual connection. The vector bundle $\mathbb{H}$ on any simply-connected quaternionic Kähler manifold with non-zero scalar curvature admits a unique $c_1$-self-dual connection ([5]). The vector bundle $\mathbb{H}$, only when $M = \mathbb{H}P^n$, is globally defined ([8]). Consequently $S \cong \mathbb{H}$, hence $E = E_0 \oplus \mathbb{H}$. $\square$

## References

[1] J.P. Bourguignon and H.B. Lawson: *Stability and isolation phenomena for Yang-Mills fields*, Commun. Math. Phys. **79** (1981), 189–230.

[2] M. Mamone Capria and S.M. Salamon: *Yang-Mills fields on quaternionic spaces*, Nonlinearity **1** (1988), 517–530.

[3] K. Galicki and Y.S. Poon: *Duality and Yang-Mills fields on quaternionic Kähler manifold*, J. Math. Phys. **32** (1991), 1263–1268.

[4] S. Kobayashi, Y. Ohnita and M. Takeuchi: *On instability of Yang-Mills connections*, Math. Z. **193** (1986), 165–189.

[5] Y. Nagatomo: *Rigidity of $c_1$-self-dual connections on quaternionic Kähler manifolds*, J. Math. Phys. **33** (1992), 4020–4025.

[6] T. Nitta: *Vector bundles over quaternionic Kähler manifolds*, Tohoku. Math. J. **40** (1988), 425–440.

[7] Y. Ohnita and Y. Pan: *On weakly stable Yang-Mills fields over positively pinched manifolds and certain symmetric spaces*, Kodai. Math. J. **13** (1990), 317–332.

[8] S. Salamon: *Quaternionic Kähler manifolds*, Invent. Math. **67** (1982), 143–171.

[9] J.A. Wolf: *Complex homogeneous contact manifolds and quaternionic symmetric spaces*, J. Math. Mech. **14** (1965), 1033–1047.

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