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ORLICZ CLASSES OF HARDY-LITTLEWOOD MAXIMAL FUNCTIONS

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0. Introduction

It is well-known that for a locally integrable function $f : \mathbb{R}^n \to \mathbb{R}$ the Hardy-Littlewood maximal function Mf is defined by

$$Mf(x) = \sup\left\{\frac{1}{|Q|}\int_Q |f(t)|dt : x \in Q\right\} \qquad (x \in \mathbb{R}^n),$$

where the supremum is taken over all bounded cubic intervals Q of $\mathbb{R}^n$ containing x and $|Q|$ denotes the Lebesgue measure of Q.

Assuming that f has support in the unit ball B a well-known theorem of Hardy and Littlewood [8] states that if $f \in L(\log^+ L)$ then $Mf \in L^1(B)$. Later E.M. Stein [11] proved that the converse of the above mentioned theorem also holds. Also N.A. Fava has proved that if f is again a function with support in the unit ball B and α is a non negative number then the function $(Mf)(\log^+ Mf)^\alpha$ is integrable on B if and only if $f \in L(\log^+ L)^{\alpha+1}$.

M. Delgado and P. Jiménez Guerra [3] have defined the Hardy-Littlewood maximal functions $M_k f$ of order $k \in \mathbb{N}$ (for a measurable function $f : \mathbb{R}^n \to \mathbb{R}$) by iteration of the Hardy-Littlewood maximal operator, proving, among other results, that $M_k f \in L^1$ if and only if $f \in L(1 + \log^+ L)^k$.

The main object of this paper is to establish different characterizations of the Orlicz classes of the Hardy-Littlewood maximal functions of arbitrary orders and to state some relations between these classes according to its orders, and also between these classes for a given Young function and the corresponding ones for its associated function and its truncated function (also of different orders). Thus, many of the known results about these questions are improved. In particular, Corollary 13, which states that if Φ is a Young function satisfying the Δ_2-condition and $f \in L(\Phi)$ then $M_k f \in L(\Phi^{(\delta)})$ if and only if $f \in L(\Phi_\delta^{*k*})$ for every $\delta > 0$ and $k \in \mathbb{N}$, contains the above mentioned results of Stein, Fava and Delgado-Jiménez Guerra.

For every measurable function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, we will denote by Mf the maximal function obtained from f applying the Hardy-Littlewood maximal operator associated to the differentiation basis formed by the collection of all the bounded cubic intervals Q of $\mathbb{R}^n$, i.e.,

$$Mf(x) = \sup\left\{\frac{1}{|Q|}\int_Q |f(t)|dt : x \in Q\right\} \qquad (x \in \mathbb{R}^n),$$

where the supremum is taken over all the bounded cubic intervals Q of $\mathbb{R}^n$ containing x and $|Q|$ denotes the Lebesgue measure of Q.

Following [9] we will say that Φ is a Young function if

$$\Phi(t) = \int_0^t \varphi(s)ds \qquad (t \geq 0),$$

with the function $\varphi : [0, +\infty) \to \mathbb{R}$ having the following properties:
i) $\varphi(0) = 0$,
ii) $\varphi(s) > 0$ if $s > 0$,
iii) φ is right continuous at every point $s \geq 0$,
iv) φ is nondecreasing on $(0, +\infty)$,
v) $\lim_{s\to+\infty} \varphi(s) = +\infty$.

The function φ is called a density function of Φ and $L(\Phi)$ will denote the corresponding Orlicz space of the real measurable functions defined on $\mathbb{R}^n$.

If $\varphi : [0, +\infty) \to \mathbb{R}$ is a function verifying i), iv), v),
ii$'$) there exists $\alpha > 0$ such that $\varphi(s) > 0$ for every $s > \alpha$,
and
iii$'$) φ is right continuous at every point $s > 0$,
then the function

$$\Phi(t) = \int_0^t \varphi(s)ds \qquad (t \geq 0)$$

will be called a Young function in wide sense.

As it is well known, for every Young function Φ, there exist δ_0, $\delta_1 > 0$ such that $\Phi^{-1}(t) > t$ for every $t \in (0, \delta_0)$ and $\Phi(s) > s$ for every $s > \delta_1$. From now on we will refer to δ_0 and δ_1 in this sense.

DEFINITION 1. Let Φ be a Young function with a density function φ verifying the following conditions:
1.1. φ satisfies the Δ_2-condition.
1.2. $\int_0^t (\varphi(s)/s)ds < +\infty$ for every $t > 0$.
Then, the associated function to Φ will be the function $\Phi_a : [0, +\infty) \to [0, +\infty)$ defined by

$$\Phi_a(u) = \begin{cases} 0 & \text{if} \quad u = 0 \\ u\displaystyle\int_0^u \frac{\varphi(s)}{s}ds & \text{if} \quad u > 0. \end{cases}$$

If $\Psi : [0, +\infty) \to [0, +\infty)$ is a Young function with a density function Ψ and $\delta > 0$ then the truncated function of Ψ in δ will be the function $\Psi^{(\delta)} : [0, +\infty) \to \mathbb{R}$ defined in the following way:

$$\Psi^{(\delta)}(u) = \begin{cases} 0 & \text{if} \quad 0 \leq u \leq \delta \\ \Psi(u) - \Psi(\delta) & \text{if} \quad u > \delta. \end{cases}$$

Clearly the function $\Psi^{(\delta)}$ is a Young function in wide sense since the function

$$\Psi^{(\delta)}(s) = \begin{cases} 0 & \text{if} \quad 0 \leq s \leq \delta \\ \Psi(s) & \text{if} \quad s > \delta \end{cases}$$

verifies the conditions i), ii′), iii′), iv) and v), and it is a density function of $\Psi^{(\delta)}$ Moreover, for every measurable function $f : \mathbb{R}^n \to \mathbb{R}$ we have that

$$\int_{|f|>\delta} \Psi(|f(t)|)dt < +\infty$$

if and only if

$$\int_{\mathbb{R}^n} \Psi^{(\delta)}(|f(t)|)dt < +\infty,$$

and

$$\Psi_a^{(\delta)}(u) = u \int_0^u \frac{\Psi^{(\delta)}(s)}{(s)} ds$$

for every $u \geq 0$.

Proposition 2. *If Φ is a Young function with a density function φ verifying* 1.1 *and* 1.2, *then the associated function Φ_a is also a Young function having a density function satisfying the Δ_2-condition.*

Proof. Clearly if the function $\varphi_a : [0, +\infty) \to \mathbb{R}$ is such that

$$\varphi_a(u) = \begin{cases} 0 & \text{if} \quad u = 0 \\ \displaystyle\int_0^u \frac{\varphi(s)}{s} ds + \varphi(u) & \text{if} \quad u > 0 \end{cases}$$

then

$$\Phi_a(u) = \int_0^u \varphi_a(s)ds$$

for every $u \geq 0$ and Φ_a is a Young function having φ_a as a density function. Let us prove now that φ_a satisfies the Δ_2-condition. In fact, since the function φ satisfies the Δ_2-condition then there exist $k_0 > 0$ and $M_0 \geq 0$ such that $\varphi(2u) \leq k_0\varphi(u)$ for every $u \in (M_0, +\infty)$ and therefore,

$$\begin{aligned}\int_0^{2u} \frac{\varphi(s)}{s} ds &= \int_0^{2M_0} \frac{\varphi(s)}{s} ds + \int_{M_0}^{u} \frac{\varphi(2t)}{t} dt \\ &\leq k_0 \int_{M_0}^{u} \frac{\varphi(t)}{t} dt + \int_0^{2M_0} \frac{\varphi(s)}{s} ds \\ &\leq k_0 \int_0^{u} \frac{\varphi(t)}{t} dt + \int_0^{2M_0} \frac{\varphi(s)}{s} ds\end{aligned}$$

and $\varphi_a(2u) \leq k_0\varphi_a(u) + c$ for every $u > M_0$ with

$$c = \int_0^{2M_0} \frac{\varphi(s)}{s} ds.$$

Since $\lim_{t\to+\infty} \varphi_a(t) = +\infty$ then there exists $M \geq M_0$ such that $\varphi_a(u) > c$ and $\varphi_a(2u) \leq (k_0 + 1)\varphi_a(u)$ for every $u > M$. □

Proposition 3. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function and Φ a Young function. If there exists $\delta > 0$ such that*

$$\int_{|f|>\delta} \Phi(|f(t)|)dt < +\infty \tag{3.1}$$

then

$$\int_{|f|>\delta} \Phi(|f(t)|)dt < +\infty. \tag{3.2}$$

Proof. Clearly it can be assumed that $\delta < \delta_1$ since in the other case the result is trivial. It follows from the properties of the Young functions that there exists $h > 0$ such that $\Phi(u) > h$ for every $u \in (\delta, \delta_1)$. Therefore

$$\int_{\delta<|f|<\delta_1} \Phi(|f(t)|)dt \geq h|\{t \in \mathbb{R}^n : \delta < |f(t)| < \delta_1\}|$$

and it is deduced from (3.1) that

$$\begin{aligned}\int_{|f|>\delta} |f(t)|dt &\leq \delta_1|\{x \in \mathbb{R}^n : \delta < |f(t)| < \delta_1\}| + \int_{|f|\geq\delta_1} \Phi(|f(t)|)\, dt \\ &\leq \frac{1}{h} \int_{\delta<|f|<\delta_1} \Phi(|f(t)|)dt + \int_{|f|\geq\delta_1} \Phi(|f(t)|)dt \\ &< +\infty.\end{aligned}$$

□

Lemma 4. *For every $n \in \mathbb{N}$, there exist c_n, $c_n^* > 0$ such that the inequalities*

$$|\{x \in \mathbb{R}^n : Mf(x) > \lambda\}| \leq \frac{c_n}{(1-\rho)\lambda} \int_{|f| \geq \rho\lambda} |f(t)| dt \tag{4.1}$$

and

$$|\{x \in \mathbb{R}^n : Mf(x) > \mu\}| \geq \frac{c_n^*}{\mu} \int_{Mf \geq \mu} |f(t)| dt \tag{4.2}$$

hold for every measurable function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and all the real numbers λ, $\mu > 0$ and $0 < \rho < 1$.

Proof. It is enough to proceed as in the proof of the similar results stated in [7] for integrable functions. □

Theorem 5. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function, Φ a Young function with a density function φ verifying* 1.1 *and* 1.2, *and* $0 < \delta_2 < \delta_0$ *such that*

$$\int_{|f| > \delta_2} \Phi(|f(t)|) dt < +\infty. \tag{5.1}$$

Then for every $\delta \in (\delta_2, \delta_0)$ the following assertions are equivalent:

5.1. $\int_{Mf > \delta} \Phi(Mf(t)) dt < +\infty$.
5.2. $\int_{|f| > \delta} \Phi_a(|f(t)|) dt < +\infty$.

Proof. Let us consider $\delta \in (\delta_2, \delta_0)$. Clearly

$$\int_{\Phi(Mf) > \Phi(\delta)} \Phi(Mf(t)) dt = -\int_{\Phi(\delta)}^{+\infty} \lambda d\omega(\lambda)$$

with $\omega(\lambda) = |\{t \in \mathbb{R}^n : \Phi(Mf(t)) > \lambda\}|$, and so integrating by parts we obtain that

$$\begin{aligned} \int_{Mf > \delta} \Phi(Mf(t)) dt &\leq [-\lambda\omega(\lambda)]_\delta^{+\infty} + \int_\delta^{+\infty} \omega(\lambda) d\lambda \\ &= \delta\omega(\delta) - \lim_{\lambda \to +\infty} \lambda\omega(\lambda) + \int_\delta^{+\infty} \omega(\lambda) d\lambda. \end{aligned}$$

Since $\omega(\lambda) = |\{x \in \mathbb{R}^n : Mf(x) > \Phi^{-1}(\lambda)\}|$, it follows from Lemma 4, Proposition 3 and (5.1) that

$$\delta\omega(\delta) \leq \frac{\delta c_n}{\Phi^{-1}(\delta) - \delta} \int_{|f| > \delta} |f(t)|\, dt < +\infty.$$

In a similar way,

$$\begin{aligned}\lim_{\lambda\to+\infty}\lambda\omega(\lambda) &\le 2c_n \lim_{\lambda\to+\infty}\frac{\lambda}{\Phi^{-1}(\lambda)}\int_{|f|>\frac{1}{2}\Phi^{-1}(\lambda)}|f(t)|\,dt\\ &= 2c_n\lim_{\mu\to+\infty}\frac{\Phi(2\mu)}{2\mu}\int_{|f|>\mu}|f(t)|\,dt\end{aligned}$$

and since the function φ satisfies the Δ_2-condition we can find $k>0$ and $K\geq 0$ such that $\Phi(2\mu)<k\Phi(\mu)$ is verified for every $\mu>K$, and considering the nondecreasing function $\eta(\mu)=\Phi(\mu)/\mu$ we have that

$$\frac{\eta(2\mu)}{\eta(\mu)}=\frac{1}{2}\frac{\Phi(2\mu)}{\Phi(\mu)}\leq\frac{k}{2}$$

for every $\mu>K$ and then it is deduced from (5.1) that

$$\begin{aligned}\lim_{\lambda\to+\infty}\lambda\omega(\lambda) &\le c_n\lim_{\mu\to+\infty}k\eta(\mu)\int_{|f|>\mu}|f(t)|\,dt\\ &\le c_nk\lim_{\mu\to+\infty}\int_{|f|>\mu}|f(t)|\,\eta(|f(t)|)dt\\ &= c_nk\lim_{\mu\to+\infty}\int_{|f|>\mu}\Phi\,|f(t)|\,dt\\ &=0\end{aligned}$$

and therefore 5.1 holds if and only if

$$\int_{\delta}^{+\infty}\omega(\lambda)d\lambda<+\infty.$$

Let us suppose that 5.1 holds, then

$$\begin{aligned}&\int_{\delta}^{+\infty}\omega(\lambda)d\lambda\\ &\geq c_n^*\int_{\delta}^{+\infty}\frac{1}{\Phi^{-1}(\delta)}\int_{Mf>\Phi^{-1}(\delta)}|f(t)|\,dtd\lambda\\ &\geq c_n^*\int_{\delta}^{+\infty}\frac{1}{\Phi^{-1}(\delta)}\int_{|f|>\Phi^{-1}(\delta)}|f(t)|\,dtd\lambda\\ &\geq c_n^*\int_{\Phi^{-1}(\delta)}^{+\infty}\frac{1}{\mu}\int_{|f|>\mu}\varphi(\mu)\,|f(t)|\,dtd\mu\\ &= c_n^*\int_{|f|>\Phi^{-1}(\delta)}|f(t)|\int_{\Phi^{-1}(\delta)}^{|f(t)|}\frac{\varphi(\mu)}{\mu}d\mu dt\end{aligned}$$

$$
\begin{aligned}
&= c_n^* \int_{|f|>\Phi^{-1}(\delta)} |f(t)| \left[\frac{\Phi_a(\mu)}{\mu}\right]_{\Phi^{-1}(\delta)}^{|f(t)|} dt \\
&= c_n^* \int_{|f|>\Phi^{-1}(\delta)} \Phi_a(|f(t)|)dt - \frac{c_n^* \Phi_a(\Phi^{-1}(\delta))}{\Phi^{-1}(\delta)} \int_{|f|>\Phi^{-1}(\delta)} |f(t)|\, dt,
\end{aligned}
$$

and since it follows from Proposition 3 and (5.1) that

$$
\int_{|f|>\Phi^{-1}(\delta)} |f(t)|\, dt < +\infty,
$$

we have that

$$
\int_{|f|>\Phi^{-1}(\delta)} \Phi_a(|f(t)|)dt < +\infty
$$

and 5.2 holds because

$$
\begin{aligned}
\int_{|f|>\delta} \Phi_a(|f(t)|)dt &= \int_{\delta<|f|<\Phi^{-1}(\delta)} \Phi_a(|f(t)|)dt + \int_{|f|>\Phi^{-1}(\delta)} \Phi_a(|f(t)|)dt \\
&\leq h_a \left|\{x \in \mathbb{R}^n : \delta < |f(x)| < \Phi^{-1}(\delta)\}\right| + \int_{|f|>\Phi^{-1}(\delta)} \Phi_a(|f(t)|)dt \\
&\leq \frac{h_a}{\rho} \int_{\delta<|f(t)|<\Phi^{-1}(\delta)} \Phi(|f(t)|)dt + \int_{|f|>\Phi^{-1}(\delta)} \Phi_a(|f(t)|)dt \\
&< +\infty,
\end{aligned}
$$

where h_a, $\rho > 0$ are respectively an upper bound of Φ_a and a lower bound of Φ_a on $[\delta, \Phi^{-1}(\delta)]$.

Let us suppose now that 5.2 holds. It follows from Lemma 4 that for every $0 < \gamma < 1$ we have that

$$
\begin{aligned}
\int_\delta^{+\infty} \omega(\lambda)d\lambda &\leq \frac{c_n}{1-\gamma} \int_\delta^{+\infty} \frac{1}{\Phi^{-1}(\lambda)} \int_{|f|>\gamma\Phi^{-1}(\lambda)} |f(t)|\, dt d\lambda \\
&= \frac{c_n}{1-\gamma} \int_{\gamma\Phi^{-1}(\delta)}^{+\infty} \frac{1}{\mu} \int_{|f|>\mu} \varphi\left(\frac{\mu}{\gamma}\right) |f(t)|\, dt d\mu \\
&= \frac{c_n}{1-\gamma} \int_{|f|>\gamma\Phi^{-1}(\delta)} |f(t)| \int_{\gamma\Phi^{-1}(\delta)}^{|f(t)|} \frac{\varphi\left(\frac{\mu}{\gamma}\right)}{\mu} d\mu dt \\
&= \frac{c_n\gamma}{1-\gamma} \int_{|f|>\gamma\Phi^{-1}(\delta)} \Phi_a\left(\frac{|f(t)|}{\gamma}\right) dt \\
&\quad - \frac{c_n}{1-\gamma} \frac{\Phi_a(\Phi^{-1}(\delta))}{\Phi^{-1}(\delta)} \int_{|f|>\gamma\Phi^{-1}(\delta)} |f(t)|\, dt.
\end{aligned}
$$

If $\gamma \in (1/2, 1)$ and $\gamma\Phi^{-1}(\delta) > \delta$, then it follows from Proposition 3 and (5.1) that

$$\int_{|f|>\gamma\Phi^{-1}(\delta)} |f(t)|\, dt < +\infty. \tag{5.2}$$

Moreover we have that

$$\begin{aligned}
\frac{\Phi_a\left(\dfrac{\lambda}{\gamma}\right)}{\Phi_a(\lambda)} &= \frac{1}{\gamma}\frac{\displaystyle\int_0^{\frac{\lambda}{\gamma}} \frac{\varphi(s)}{s} ds}{\displaystyle\int_0^{\lambda} \frac{\varphi(s)}{s} ds} \le \frac{1}{\gamma}\frac{\displaystyle\int_0^{2\lambda} \frac{\varphi(s)}{s} ds}{\displaystyle\int_0^{\lambda} \frac{\varphi(s)}{s} ds} \\
&= \frac{1}{\gamma}\left[1 + \frac{\displaystyle\int_\lambda^{2\lambda} \frac{\varphi(s)}{s} ds}{\displaystyle\int_0^{\lambda} \frac{\varphi(s)}{s} ds}\right] \\
&< \frac{1}{\gamma}\left[1 + \frac{\displaystyle\frac{1}{\lambda}\int_\lambda^{2\lambda} \varphi(s) ds}{\displaystyle\frac{1}{\lambda}\int_0^{\lambda} \varphi(s) ds}\right] \\
&= \frac{1}{\gamma}\left[1 + \frac{\displaystyle\int_\lambda^{2\lambda} \varphi(s) ds}{\displaystyle\int_0^{\lambda} \varphi(s) ds}\right] \\
&= \frac{1}{\gamma}\frac{\displaystyle\int_0^{2\lambda} \varphi(s) ds}{\displaystyle\int_0^{\lambda} \varphi(s) ds} \\
&= \frac{1}{\gamma}\frac{\Phi(2\lambda)}{\Phi(\lambda)} < k
\end{aligned}$$

for every $\lambda > K$, where k and K are the constants obtained before from the fact that φ satisfies the Δ_2-condition. Therefore,

$$\begin{aligned}
\int_\delta^{+\infty} \omega(\lambda) d\lambda &\le \frac{c_n\gamma}{1-\gamma}\int_{\gamma\Phi^{-1}(\delta)<|f|<K} \Phi_a\left(\frac{|f(t)|}{\gamma}\right) dt + \frac{c_n\gamma}{1-\gamma}\int_{|f|>K} \Phi_a\left(\frac{|f(t)|}{\gamma}\right) dt \\
&\le h_a \left|\{x \in \mathbb{R}^n : \gamma\Phi^{-1}(\delta) < |f| < K\}\right| + \frac{c_n\gamma k}{1-\gamma}\int_{|f|>K} \Phi_a |f(t)|\, dt \\
&\le h_a \frac{1}{\gamma\Phi^{-1}(\delta)}\int_{\gamma\Phi^{-1}(\delta)<|f|<K} |f(t)|\, dt + \frac{c_n\gamma k}{1-\gamma}\int_{|f|>K} \Phi_a |f(t)|\, dt,
\end{aligned}$$

with h_a an appropriate bound, and 5.1 holds since it follows from (5.1) and

Proposition 3 that

$$\int_{\gamma\Phi^{-1}(\delta)<|f|<K} |f(t)|\,dt < +\infty. \qquad \square$$

Corollary 6. *With the notations of Theorem* 5, *if* (5.1) *is verified and* $\delta'' > \delta_2$, *then the following assertions are equivalent*:

6.1. $\int_{Mf>\delta''} \Phi(Mf(t))dt < +\infty$.

6.2. $\int_{|f|>\delta''} \Phi_a(|f(t)|)dt < +\infty$.

Proof. Let $\delta'' > \delta_0$ be and suppose first that 6.1 is verified. Then for every $\delta \in (\delta_2, \delta_0)$ we have that

$$\begin{aligned}\int_{Mf>\delta} \Phi(Mf(t))dt &= \int_{\delta''>Mf>\delta} \Phi(Mf(t))dt + \int_{Mf>\delta''} \Phi(Mf(t))dt \\ &\leq \Phi(\delta'')\,|\{x \in \mathbb{R}^n : \delta < Mf(x)\}| + \int_{Mf>\delta''} \Phi(Mf(t))dt\end{aligned}$$

and it follows from Lemma 4 and Proposition 3 that

$$\begin{aligned}\int_{Mf>\delta} \Phi(Mf(t))dt &\leq \Phi(\delta'')\frac{c_n}{\left(1-\dfrac{\delta_2}{\delta}\right)\delta}\int_{|f|>\delta_2} |f(t)|\,dt + \int_{Mf>\delta''} \Phi(Mf(t))dt \\ &< +\infty.\end{aligned}$$

Now it is deduced from Theorem 5 that 5.2 holds and so 6.2 is verified.

If it is assumed that 6.2 holds, then for every $\delta \in (\delta_2, \delta_0)$ we have that

$$\begin{aligned}\int_{|f|>\delta} \Phi_a(|f(t)|)dt &\leq \Phi_a(\delta'')\,|\{x \in \mathbb{R}^n : \delta < |f(x)| < \delta''\}| + \int_{|f|>\delta''} \Phi_a(|f(t)|)dt \\ &\leq \frac{\Phi_a(\delta'')}{\Phi(\delta)}\int_{|f|>\delta_2} \Phi(|f(t)|)dt + \int_{|f|>\delta''} \Phi_a(|f(t)|)dt \\ &< +\infty\end{aligned}$$

and it follows now from Theorem 5 that 5.1 is verified, and therefore 6.1 also holds. $\square$

Corollary 7. *Let* $\Phi : [0, +\infty) \to [0, +\infty)$ *be a Young function satisfying the* Δ_2*-condition,* $f : \mathbb{R}^n \to \mathbb{R}$ *a measurable function and* $\delta, \delta_2 > 0$ *such that* $0 < \delta_2 < \delta$ *and*

$$\int_{\mathbb{R}^n} \Phi^{(\delta_2)}(|f(t)|)dt < +\infty. \tag{7.1}$$

Then

$$\int_{\mathbb{R}^n} \Phi^{(\delta)}(Mf(t))dt < +\infty. \tag{7.2}$$

if and only if

$$\int_{\mathbb{R}^n} \Phi_a^{(\delta)}(|f(t)|)dt < +\infty. \tag{7.3}$$

Corollary 8. *Let $\Phi : [0, +\infty) \to [0, +\infty)$ be a Young function satisfying the Δ_2-condition and $\delta > 0$. Then for every $f \in L(\Phi)$ we have that $Mf \in L(\Phi^{(\delta)})$ if and only if $f \in L(\Phi_a^{(\delta)})$.*

DEFINITION 9. From the Hardy-Littlewood maximal operator, the k-iterated maximal function $M_k f$ $(k = 2, 3, \ldots)$ is defined in the following way:

$$M_k f(x) = \sup\left\{\frac{1}{|Q|}\int_Q M_{k-1}f(t)dt : x \in Q\right\} \qquad (x \in \mathbb{R}^n). \tag{9.1}$$

Sometimes we will write $M_1 f$ instead of Mf.

In a similar way, for a Young function Φ with a density function φ, $\delta > 0$ and $k = 2, 3, \ldots$, we will use the following notations:

$$\varphi_\delta^{*1*}(t) = \begin{cases} \varphi(t) & \text{if} \quad 0 \le t \le \delta \\ \varphi(t) + \displaystyle\int_\delta^t \frac{\varphi(s)}{s} ds & \text{if} \quad t > \delta, \end{cases}$$

$$\Phi_\delta^{*1*}(t) = \begin{cases} \Phi(t) & \text{if} \quad 0 \le t \le \delta \\ t\displaystyle\int_\delta^t \frac{\varphi(s)}{s} ds & \text{if} \quad t > \delta, \end{cases}$$

$$\varphi_\delta^{*k*}(t) = \begin{cases} \varphi_\delta^{*k-1*}(t) & \text{if} \quad 0 \le t \le \delta \\ \varphi_\delta^{*k-1*}(t) + \displaystyle\int_\delta^t \frac{\varphi_\delta^{*k-1*}(s)}{s} ds & \text{if} \quad t > \delta \end{cases}$$

and

$$\Phi_\delta^{*k*}(t) = \begin{cases} \Phi_\delta^{*k-1*}(t) & \text{if} \quad 0 \le t \le \delta \\ t\displaystyle\int_\delta^t \frac{\varphi_\delta^{*k-1*}(s)}{s} ds & \text{if} \quad t > \delta \end{cases}$$

Proposition 10. *Let $\Phi : [0, +\infty) \to [0, +\infty)$ be a Young function, $f : \mathbb{R}^n \to \mathbb{R}$*

a measurable function, $k \in \mathbb{N}$ *and* $\delta, \delta_2 \in \mathbb{R}$ *such that* $k \geq 2$, $0 < \delta_2 < \delta$,

$$\int_{|f|>\delta_2} |f(t)|dt < +\infty \tag{10.1}$$

and

$$\int_{M_k f>\delta} \Phi(M_k f(t))dt < +\infty \tag{10.2}$$

Then

$$\int_{M_i f>\delta'} \Phi(M_i f(t))dt < +\infty \tag{10.3}$$

for every $i \in \mathbb{N}$ *and* $\delta' \in \mathbb{R}$ *such that* $i < k$ *and* $\delta_2 < \delta' < \delta$.

Proof. Let us suppose first that $k = 2$. If $\delta_2 < \delta' < \delta$ then, since $Mf \leq M_2 f$ a.e. and the function Φ is nondecreasing, it follows from Lemma 4, (10.1) and (10.2) that

$$\begin{aligned}\int_{Mf>\delta'} \Phi(Mf(t))dt &= \int_{\delta'<Mf<\delta} \Phi(Mf(t))dt + \int_{Mf>\delta} \Phi(Mf(t))dt \\ &\leq \Phi(\delta)\,|\{t \in \mathbb{R}^n : Mf(t) > \delta'\}| + \int_{M_2 f>\delta} \Phi(M_2 f(t))dt \\ &\leq \frac{\Phi(\delta)c_n}{\delta' - \delta_2} \int_{|f|>\delta_2} |f(t)|\,dt + \int_{M_2 f>\delta} \Phi(M_2 f(t))dt \\ &< +\infty.\end{aligned}$$

Suppose now that the result holds for $2 \leq k$ and let us prove it for $k+1$. Then, we have that

$$\int_{M_k f>\delta} \Phi(M_k f(t))dt < \int_{M_{k+1} f>\delta} \Phi(M_{k+1} f(t))dt < +\infty \tag{10.4}$$

and therefore, (10.3) is verified for every $i \in \mathbb{N}$ and $\delta' \in \mathbb{R}$ such that $i < k$ and $\delta_2 < \delta' < \delta$. In particular, we also have that

$$\int_{M_{k-1} f>\delta''} \Phi(M_{k-1} f(t))dt < +\infty \tag{10.5}$$

for every $\delta'' \in (\delta_2, \delta')$.

Therefore, it follows now from (10.4), Lemma 4, (10.5) and Proposition 3 that

$$\int_{M_k f>\delta'} \Phi(M_k f(t))dt \leq \int_{\delta'<M_k f<\delta} \Phi(M_k f(t))dt + \int_{M_k f>\delta} \Phi(M_k f(t))dt$$

$$\leq \Phi(\delta)\,|\{t \in \mathbb{R}^n : M_k f(t) > \delta'\}| + \int_{M_k f > \delta} \Phi(M_k f(t))dt$$
$$\leq \Phi(\delta)\frac{c_n}{\delta' - \delta''}\int_{M_{k-1} f > \delta''} M_{k-1} f(t)dt + \int_{M_k f > \delta} \Phi(M_k f(t))dt$$
$$< +\infty,$$

and the proof is finished. □

Let us remark that Proposition 3 ensures that (10.1) holds in particular if

$$\int_{|f|>\delta_2} \Phi(|f(t)|)dt < +\infty.$$

Proposition 11. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function, Φ a Young function, $k \in N$ and $\delta > \delta_2 > 0$ such that*

$$\int_{|f|>\delta_2} \Phi(|f(t)|)dt < +\infty \tag{11.1}$$

and

$$\int_{|f|>\delta_2} \Phi_\delta^{*k*}(|f(t)|)dt < +\infty. \tag{11.2}$$

Then

$$\int_{|f|>\delta'} \Phi_\delta^{*i*}(|f(t)|)dt < +\infty \tag{11.3}$$

for every $i \leq k$ and $\delta' \in (\delta_2, \delta)$.

Proof. In fact, if $\delta' \in (\delta_2, \delta)$ and $i \leq k$ we have that

$$\int_{|f|>\delta'} \Phi_\delta^{*i*}(|f(t)|)dt \leq \Phi_\delta^{*i*}(\delta)\,|\{t \in \mathbb{R}^n : \delta' < |f(t)| < \delta\}| + \int_{|f|>\delta} \Phi_\delta^{*k*}(|f(t)|)dt$$
$$\leq \frac{\Phi_\delta^{*i*}(\delta)}{\Phi(\delta')}\int_{\delta'<|f|<\delta} \Phi(|f(t)|)dt + \int_{|f|>\delta} \Phi_\delta^{*k*}(|f(t)|)dt$$
$$< +\infty.$$
□

Theorem 12. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function, Φ a Young function verifying the conditions* 1.1 *and* 1.2[1] *and $\delta_2 \in (0, \delta_0)$ such that*

$$\int_{|f|>\delta_2} \Phi(|f(t)|)dt < +\infty. \tag{12.1}$$

[1]This condition can be avoided using Young functions in the wide sense.

Then for every $k \in \mathbb{N}$ and $\delta > \delta_2$ the following conditions are equivalent:
12.1. $\int_{M_k f > \delta} \Phi(M_k f(t))dt < +\infty$.
12.2. $\int_{|f| > \delta} \Phi_\delta^{*k*}(|f(t)|)dt < +\infty$.

Proof. It follows trivially from Corollary 6 that it is enough to prove the result for $k(\geq 2)$ assuming that it holds for $k-1$, thus let us make this assumption.

If 12.1 is verified then it follows from Proposition 10 that

$$\int_{M_{k-1} f > \delta'} \Phi(M_{k-1} f(t))dt < +\infty \tag{12.2}$$

for every $\delta' \in (\delta_2, \delta)$ and so it is deduced from Corollary 6 that

$$\int_{M_{k-1} f > \delta} \Phi_\delta^{*1*}(M_{k-1} f(t))dt < +\infty. \tag{12.3}$$

Moreover, it follows from (12.2) that

$$\int_{|f| > \delta'} \Phi_\delta^{*k-1*}(|f(t)|)dt < +\infty$$

for every $\delta' \in (\delta_2, \delta)$, and then Proposition 11 assures that

$$\int_{|f| > \delta'} \Phi_\delta^{*1*}(|f(t)|)dt < +\infty \tag{12.4}$$

for every $\delta' \in (\delta_2, \delta)$. Now 12.2 is deduced from (12.4) and (12.3) (applying the induction hypothesis to Φ_δ^{*1*}).

If 12.2 holds then it follows from Proposition 11 that

$$\int_{|f| > \delta'} \Phi_\delta^{*k-1*}(|f(t)|)dt < +\infty$$

for every $\delta' \in (\delta_2, \delta)$ and so

$$\int_{M_{k-1} f > \delta'} \Phi(M_{k-1} f(t))dt < +\infty$$

for every $\delta' \in (\delta_2, \delta)$ and 12.1 follows immediately from Corollary 6. □

Corollary 13. *Let Φ be a Young function satisfying the Δ_2-condition, $f \in L(\Phi)$, $\delta > 0$ and $k \in N$, then $M_k f \in L(\Phi^{(\delta)})$ if and only if $f \in L(\Phi_\delta^{*k*})$.*

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